A propositional logic with binary metric operators

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Abstract

The aim of this paper is to combine distance functions and Boolean propositions by developing a formalism suitable for speaking about distances between Boolean formulas. We introduce and investigate a formal language that is an extension of classical propositional language obtained by adding new binary (modal-like) operators of the form $D_{\leqslant s}$ and $D_{\geqslant s}$, $s \in \mathbb{Q}_0^+$. Our language allows making formulas such as $D_{\leqslant s}(\alpha, \beta)$ with the intended meaning 'distance between formulas α and β is less than or equal to s'. The semantics of the proposed language consists of possible worlds with a distance function defined between sets of worlds. Our main concern is a complete axiomatization that is sound and strongly complete with respect to the given semantics.

Keywords: metric operators, soundness, completeness

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1 Introduction

Formalisms for representing uncertain, incomplete or vague data, information and knowledge, as well as reasoning about them are the subject of increasing interest in many scientific fields closely related to many applications (such as technology development). Besides many mathematical concepts that are useful in these fields, we emphasize two of them: *distance functions* and *Boolean propositions*.

In general, distance functions are fundamental for many areas of mathematics and computer science. Roughly speaking, distance functions express the degree of similarity (or dissimilarity) between two objects: matrices (in algebra), graphs (in discrete mathematics, combinatorics), strategies (in game theory), probability distributions (in probability theory), knowledge (in artificial intelligence), messages (in coding theory), strings (in information theory, linguistics), etc.

Boolean propositions (Boolean functions or propositional formulas, what the reader prefers) are important in many of the above mentioned areas. The language of Boolean propositions is very suited for a representation of different discrete systems. Some recent applications include circuit design, social choice theory, learning theory etc.

The aim of this paper is to combine distance functions and Boolean propositions by developing a formalism suitable for speaking about distances between Boolean formulas. More precisely, we introduce and investigate a formal language that is an extension of classical propositional language obtained by adding new binary (modal-like) operators of the form $D_{\leqslant s}$ and $D_{\geqslant s}$, $s \in \mathbb{Q}_0^+$ (where $\mathbb{Q}_0^+$ is the set of non negative rational numbers). The language allows making formulas such as $D_{\leqslant s}(\alpha, \beta)$ with the intended meaning 'distance between formulas α and β is less than or equal to s'. Thus, our formalism is substantially related to distance functions between Boolean propositions, and it enables us to infer consistent conclusions from propositional and metric statements. In the next section, Example 3 gives an illustrative sketch of possible applications.

The idea of constructing logical formalisms that include the notion of distance is not new ([2], [3], [6], [9], [10], [11], [13], [20], [22] and [25]). More attention has been devoted to metric (or quantitative) temporal logics (see [1], [7] and [14]), which reflects the fact that temporal logic in general is more developed than spatial logic.

In this paper, we adopt an approach similar to the development of probabilistic propositional logic (see [5], [15], [16], [17], [18] and [19]). The semantics of our language consists of possible worlds with a distance function defined between sets of words. Our main concern is a complete axiomatization that is sound and strongly complete with respect to the given semantics ('Every consistent set of formulas has a model' in contrast to the weak completeness 'every consistent formula has a model'). Finitary axiomatizations could not be expected, because of the inherent non-compactness: in the proposed language it is possible to define an inconsistent set of formulas such that all its finite subsets are consistent ($T = \{\neg D_{=0}(\alpha, \beta)\} \cup \{D_{<\frac{1}{n}}(\alpha, \beta) \mid n \text{ is a positive integer}\}$). The lack of compactness forces us to consider an infinitary axiomatization.

The rest of the paper is organized as follows. Section 2 presents some preliminaries and gives several examples that motivate our investigation and indicate possible applications. Syntax and semantics of our logic are introduced in Section 3. In Section 4 we examine a sound and (strongly) complete axiomatization, and discuss several modifications of the proposed logic. This Section contains the main results of this paper. Concluding remarks are given in Section 5.

2 Preliminaries

The term *distance* generally refers to a function satisfying some properties of the (most common) distance between two points in Euclidean space. In this paper, we focus on metrics and pseudometrics.

Recall that a metric space is a pair (X, d), where X is a nonempty set and $d : X \times X \to [0, +\infty)$ is a metric, i.e., a function satisfying the following constraints:

(D1) $d(x, y) = 0$ iff $x = y$ (identity of indiscernibles),

(D2) $d(x, z) \leqslant d(x, y) + d(y, z)$ (triangle inequality),

(D3) $d(x, y) = d(y, x)$ (symmetry),

for all $x, y, z \in X$. The value $d(x, y)$ is called the distance from x to y. Although acceptable in many cases, the requirements (D1), (D2) and (D3) all together are too strong in many real contexts. This is especially true for the condition: $d(x, y) = 0$ implies $x = y$. In a pseudometric space, the distance between two distinct points can be zero; $d : X \times X \to [0, +\infty)$ is a pseudometric if it satisfies (D2), (D3) and the following constraint less strong than (D1):

(D1-) $d(x, x) = 0$, for all $x \in X$.

There are many ways of relaxing the constraints on metrics. For instance, it is reasonable to omit the symmetry of distance (e.g., $d(x, y) =$ 'work required to get from x to y in a mountainous region'.) A quasimetric is defined as a function that satisfies (D1) and (D2). Any quasimetric traditionally can be symmetrized, e.g. by one of the procedures: $(d(x, y) + d(y, x))/2$ or $\max\{d(x, y), d(y, x)\}$. A pseudoquasimetric (also called hemimetric) satisfies (D1-) and (D2).

In many naturally arising examples, distance functions are bounded. If it is not bounded, there are well-known procedures that normalize a metric space (X, d) into an 1-bounded topologically-equivalent metric space such as $(X, d/(1 + d))$ or $(X, \min\{1, d\})$. Especially, if d is bounded by M, $(X, d/M)$ is a straightforward conversion.

The rest of this section brings some motivating examples that highlight possible applications of our logic.

Example 1. Given a bounded pseudometric space (X, d), one can define a pseudo-quasimetric on the subsets of X, called the one-sided Hausdorff distance:

$$d_H(A, B) = \sup_{a\in A} \inf_{b\in B} d(a, b),\ A, B \subseteq X.$$

The (bidirectional) Hausdorff distance is defined as:

$$D_H(A, B) = \max\{d_H(A, B), d_H(B, A)\},\ A, B \subseteq X.$$

D_H is a pseudometric on $\mathcal{P}(X)$ (the power set of X) that is of a great importance in many applications (e.g. [21]).

Besides the Hausdorff metric, many other distance functions between sets are important for applications (see for instance [4] and the references given there). Any such distance function is closely related to our semantics (specified in the next section) that is based on a distance function defined on a Boolean algebra (more precisely, on the Lindenbaum-Tarski algebra of a classical propositional theory).

Example 2. Given a probability space $(W, \mathcal{F}, P)$, a natural example of distance between two sets (events) is the probability of their symmetric difference, $d_P(A, B) = P(A\triangle B)$. It is well-known that d_P is a pseudometric. This example could inspire development of logics that extend probabilistic propositional logics by enriching their languages with distance operators. Some interesting ideas in that direction are given in [12].

Example 3. Let For_n be the set of classical propositional formulas over the propositional variables $p_1, \ldots, p_n$. Let $a_1, \ldots, a_N$, where $N = 2^n$, run through the 2^n conjunctive clauses of the form $p_1^{e_1} \wedge \cdots \wedge p_n^{e_n}$, where $e_1, \ldots, e_n \in \{0, 1\}$ ($p^1 = p$ and $p^0 = \neg p$). We call a_i's atoms, and denote the set of these atoms by A. It is obvious that for a given atom a there is a unique valuation $v_a : \{p_1, \ldots, p_n\} \to \{0, 1\}$ such that $v_a(a) = 1$, and vice versa. Moreover, each formula can be regarded as a set of atoms. For each α there is $S_\alpha \subseteq A$ such that α is classically equivalent to $\bigvee S_\alpha$:

$$S_\alpha = \{a \in A \mid a \models \alpha\} = \{a \in A \mid v_a(\alpha) = 1\}.$$

Identifying the atoms from A with the binary strings of the length n, any distance function between strings can be transferred into the logical context. For instance, the Hamming distance could be very useful (the Hamming distance between two strings of equal length is the number of positions at which the corresponding symbols are different). Note that this distance can be derived from the notion of logical consequence, or more precisely from the mapping:

$$v(a, \alpha) = \begin{cases} 1, & a \models \alpha, \\ 0, & a \not\models \alpha. \end{cases}$$

Any distance between atoms can be lifted to a distance between formulas in a manner analogous to the way in which one obtains the Hausdorff metrics (see Example 1). The Hamming distance between atoms defines $\overline{D}$: For$\times$For $\to [0, +\infty)$,

$$\overline{D}(\alpha, \beta) = \sum_{i=1}^{N} |v(a_i, \alpha) - v(a_i, \beta)|,$$

which is a metric. Normalizing the metric $\overline{D}$, we obtain another metric

$$D(\alpha, \beta) = \left(\sum_{i=1}^{N} |v(a_i, \alpha) - v(a_i, \beta)| \right) /N.$$

Using the metric D, we sketch out an idea for more serious applications. Suppose that the patient can use four different types of medicines A, B, C, D for medical treatment (taking only one type or mixing two or more types simultaneously). We use a, b, c, d to denote the (propositional) statements: the patient takes A, B, C, D, respectively. Let p denote 'the patient is cured'. Three experimentally approved facts can be expressed by the following formulas: $a \wedge b \to p$, $a \vee b \vee c \to p$, $(a \wedge d) \vee (b \wedge c) \to p$. In order to identify the most efficient medicine, i.e. to prescribe only one medicine to a new patient, the doctor could consider the distances from $a \to p$, $b \to p$, $c \to p$ or $d \to p$ to each of the approved facts. The following table is obtained by easy calculations. For instance, $D(a \wedge b \to p, a \to p) = 4/32 = 0.125$. The table shows that $a \to p$ or $b \to p$ are the closest to one of the approved statements.

D	$a \to p$	$b \to p$	$c \to p$	$d \to p$
$(a \wedge b) \to p$	0.125	0.125	0.25	0.25
$(a \vee c \vee d) \to p$	0.1875	0.25	0.1875	0.1875
$((a \wedge d) \vee (b \wedge c)) \to p$	0.15625	0.15625	0.15625	0.15625

Note that D has very interesting properties:

1. $D(\alpha, \neg\alpha) = 1$,

2. $D(\alpha, \neg\beta) = 1 - D(\alpha, \beta)$,

3. $D(\alpha, \beta \vee \gamma) = D(\alpha, \beta) + D(\alpha, \gamma) - D(\alpha, \beta \wedge \gamma)$,

for all formulas α, β and γ. Since the metric D shares some properties with conditional probabilities, it would be interesting to investigate some deeper connections between our logic and the appropriate probabilistic logics (see [5], [8], [15], [16], [17], [18] and [19]).

3 Syntax and Semantics

Syntax. The language of distance logics consists of a countable set $P = \{p_1, p_2, \ldots\}$ of propositional letters, classical connectives $\wedge$ and $\neg$, and a list of binary metric operators $D_{\leqslant s}$ i $D_{\geqslant s}$ for every $s \in \mathbb{Q}_0^+$. The set For_C of all classical propositional formulas over the set P is defined as usual. The formulas from the set For_C will be denoted by $\alpha, \beta, \gamma, \ldots$ If $\alpha, \beta \in \mathrm{For}_C$ and $s \in \mathbb{Q}_0^+$, then $D_{\leqslant s}(\alpha, \beta)$ and $D_{\geqslant s}(\alpha, \beta)$ are basic metric formulas. The set For_M of all metric formulas is the smallest set:

- containing all basic metric formulas, and
- closed under formation rules: if $A, B \in \mathrm{For}_M$, then $\neg A, A \wedge B \in \mathrm{For}_M$.

The formulas from the set For_M will be denoted by $A, B, C, \ldots$ Let $\mathrm{For} = \mathrm{For}_C \cup \mathrm{For}_M$. The formulas from For will be denoted by $\Phi, \Psi \ldots$ For example, the following is a formula: $D_{\leqslant 0.4}(\alpha, \beta) \wedge \neg D_{\geqslant 0.1}(\alpha \wedge \gamma, \neg\beta)$.

We use the usual abbreviations for the other classical connectives $\vee$, $\rightarrow$ and $\leftrightarrow$, and the standard conventions for the omission of parentheses. We also abbreviate:

- $\neg D_{\leqslant s}(\alpha, \beta)$ to $D_{>s}(\alpha, \beta)$,
- $\neg D_{\geqslant s}(\alpha, \beta)$ to $D_{<s}(\alpha, \beta)$,
- $D_{\leqslant s}(\alpha, \beta) \wedge D_{\geqslant s}(\alpha, \beta)$ to $D_{=s}(\alpha, \beta)$, and
- $\neg D_{=s}(\alpha, \beta)$ to $D_{\neq s}(\alpha, \beta)$.

Both $\alpha \wedge \neg\alpha$ and $A \wedge \neg A$ are denoted by $\perp$, for arbitrary formulas $\alpha \in \mathrm{For}_C$ and $A \in \mathrm{For}_M$. Note that neither mixing of pure propositional formulas and metric formulas, nor nested metric operators are allowed. Thus, $\alpha \vee D_{\geqslant 0.2}(\gamma, \beta)$ and $D_{\geqslant 0.4}(D_{<0.9}(\alpha, \beta), D_{=0.6}(\gamma, \beta))$ do not belong to the set For.

Semantics. The semantics of our logic is essentially based on a distance function $D : \mathrm{For}_C \times \mathrm{For}_C \rightarrow [0, +\infty)$ satisfying the corresponding constraints for every $\alpha, \beta, \gamma \in \mathrm{For}_C$:

(D1-) If $\alpha \leftrightarrow \beta$ is a tautology, then $D(\alpha, \beta) = 0$;

(D2) $D(\alpha, \beta) \leqslant D(\alpha, \gamma) + D(\gamma, \beta)$;

(D3) $D(\alpha, \beta) = D(\beta, \alpha)$;

possibly with the stronger version of (D1-):

(D1) $\alpha \leftrightarrow \beta$ is a tautology iff $D(\alpha, \beta) = 0$.

In the other words, the semantics is based on a (pseudo)metric on the Lindenbaum–Tarski algebra of a classical propositional theory, i.e., on the quotient algebra obtained by factoring For_C by the equivalence relation which identifies any two formulas provably equivalent in the theory. However, we propose a slightly more general semantics based on the possible-world approach. A possible-world interpretation of propositional language is specified by a nonempty set W of worlds and the truth values of all propositional letters at each world, $v : W \times P \to \{0, 1\}$. Given a world w, the truth values of all propositional formulas is defined in the standard recursive way, and we write $v(w, \alpha)$ for the truth value of α determined by the valuation assigned to w. Given a possible-world interpretation (W, v), each formula $\alpha \in \mathrm{For}_C$ defines a set of worlds $[\alpha] = \{w : v(w, \alpha) = 1\}$. Let $\mathcal{F} = \{[\alpha] \mid \alpha \in \mathrm{For}_C\}$.

Definition 3.1. An LPM-model is a structure $\mathbf{M} = \langle W, v, d\rangle$ where:

- W is a nonempty set of objects called worlds and $v : W \times P \to \{0, 1\}$ provides for each world $w \in W$ a two-valued evaluation of the propositional letters;
- $d : \mathcal{F} \times \mathcal{F} \to [0, +\infty)$ is a pseudometric.

An LPM-model $\mathbf{M} = \langle W, v, d\rangle$ is an LM-model if d is a metric.

Note that our semantics is completely analogous to the semantics for some probabilistic propositional logics ([5]).

Definition 3.2. The satisfiability relation fulfills the following conditions for every LPM-model or LM-model $\mathbf{M} = \langle W, v, d\rangle$:

- if $\alpha \in \mathrm{For}_C$, $\mathbf{M} \models \alpha$ iff for every $w \in W$, $v(w, \alpha) = true$,
- if $\alpha, \beta \in \mathrm{For}_C$, $\mathbf{M} \models D_{\leqslant s}(\alpha, \beta)$ iff $d([\alpha], [\beta]) \leqslant s$,
- if $\alpha, \beta \in \mathrm{For}_C$, $\mathbf{M} \models D_{\geqslant s}(\alpha, \beta)$ iff $d([\alpha], [\beta]) \geqslant s$,
- if $A \in \mathrm{For}_M$, $\mathbf{M} \models \neg A$ iff $\mathbf{M} \not\models A$,

- if $A, B \in \mathrm{For}_M$, $\mathbf{M} \models A \wedge B$ iff $\mathbf{M} \models A$ and $\mathbf{M} \models B$.

A formula $\Phi \in$ For is (LPM-satisfiable) LM-satisfiable if there is an (LPM-model) LM-model $\mathbf{M}$ such that $\mathbf{M} \models \Phi$. A set T of formulas is (LPM-satisfiable) LM-satisfiable if there is an (LPM-model) LM-model $\mathbf{M}$ such that $\mathbf{M} \models \Phi$ for every $\Phi \in T$.

Φ is (LPM-valid) LM-valid if for every (LPM-model) LM-model $\mathbf{M}$, $\mathbf{M} \models \Phi$.

We could further restrict the class of LM-models to those models whose distance function fulfills some additional conditions. For instance, the following conditions are motivated by the examples 2 and 3:

(D4) $d([\alpha], [\neg\alpha]) = 1$;

(D5) if $[\beta] \cap [\gamma] = \emptyset$ then $d([\alpha], [\beta]) + d([\alpha], [\gamma]) \leqslant 1$.

Such model will be called LM^+-model. The equality $d([\alpha], [\beta]) + d([\alpha], [\neg\beta]) = 1$ is an easy consequence of (D2), (D3), (D4) and (D5): $1 = d([\beta], [\neg\beta]) \leqslant d([\alpha], [\beta]) + d([\alpha], [\neg\beta]) \leqslant 1$. Note that in general $D_{\leqslant s}$ and $D_{\geqslant s}$ are not interdefinable. But if we consider LM^+-models, then one type of our operators can be defined by the other: e.g., $D_{\leqslant s}(\alpha, \beta) = D_{\geqslant 1-s}(\alpha, \neg\beta)$. At the end of the next section, we will briefly discuss a complete axiomatization with respect to LM^+-models.

4 Sound and complete axiomatization

The set of all LPM-valid formulas can be characterized by the following set of axiom schemata:

(A1) all For_C-instances of classical propositional tautologies,

(A2) all For_M-instances of classical propositional tautologies,

(A3) $D_{\geqslant 0}(\alpha, \beta)$,

(A4) $D_{\leqslant s}(\alpha, \beta) \rightarrow D_{<r}(\alpha, \beta)$, $r > s$,

(A5) $D_{<s}(\alpha, \beta) \rightarrow D_{\leqslant s}(\alpha, \beta)$,

(A6) $D_{\leqslant s}(\alpha, \beta) \wedge D_{\leqslant r}(\beta, \gamma) \rightarrow D_{\leqslant s+r}(\alpha, \gamma)$,

(A7) $D_{\leqslant s}(\alpha, \beta) \rightarrow D_{\leqslant s}(\beta, \alpha)$;

and inference rules:

(R1) From Φ and $\Phi \to \Psi$ infer Ψ,

(R2) From $\alpha \leftrightarrow \beta$ infer $D_{=0}(\alpha, \beta)$,

(R3) From $A \to D_{<s+\frac{1}{k}}(\alpha, \beta)$, for every positive integer k, infer $A \to D_{\leqslant s}(\alpha, \beta)$,

(R4) From $A \to D_{>s-\frac{1}{k}}(\alpha, \beta)$, for every positive integer $k > \frac{1}{s}$, infer $A \to D_{\geqslant s}(\alpha, \beta)$ ($s \neq 0$).

We denote this axiomatic system by Ax.

Let us shortly discuss the above axioms and rules. The classical propositional logic is a sublogic of our logics, because of the axioms (A1), (A2) and the rule (R1).

The axioms (A3), (A4), (A5) and the rules (R3), (R4) force the range of (pseudo)-metrics to be the set of non negative reals, $[0, +\infty)$. The rules (R3) and (R4) are the infinitary inference rules. Each of them has a countable set of assumptions and one conclusion. The rules correspond to the Archimedean axiom for real numbers.

The axioms (A6), (A7) and the rule (R2) describe the conditions (D1-), (D2) and (D3). The rule (R2) can be considered as the rule of necessitation in modal logics, but it can be applied on the classical propositional formulas only.

Definition 4.1. A formula Φ is deducible from a set T of formulas ($T \vdash \Phi$) if there is an at most countable sequence of formulas $\Phi_0, \Phi_1, \dots, \Phi$ such that every Φ_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule.

A formula Φ is a theorem ($\vdash \Phi$) if it is deducible from the empty set, and a proof for Φ is the corresponding sequence of formulas.

A set T of formulas is consistent if there is at least one formula from For_C, and at least one formula from For_M that are not deducible from T, otherwise T is inconsistent.

A consistent set T of formulas is said to be maximal consistent if for every $A \in \mathrm{For}_M$, either $A \in T$ or $\neg A \in T$.

A set T is deductively closed if for every $\Phi \in \mathrm{For}$, if $T \vdash \Phi$, then $\Phi \in T$.

Lemma 4.1. *Let T^* be a maximal consistent set of formulas, and $\alpha, \beta \in \mathrm{For}_C$. If $T \vdash \alpha \leftrightarrow \beta$, then $D_{=0}(\alpha, \beta) \in T$.*

Proof. If $T \vdash \alpha \leftrightarrow \beta$, then $T \vdash D_{=0}(\alpha, \beta)$ by the rule (R2). If $D_{=0}(\alpha, \beta) \notin T$, then $\neg D_{=0}(\alpha, \beta) \in T$ (since T is maximal) which contradicts the consistency of T. $\square$

Theorem 4.1. *(Deduction theorem). If T is a set of formulas, Φ is a formula, and $T \cup \{\Phi\} \vdash \Psi$, then $T \vdash \Phi \to \Psi$, where Φ and Ψ are either both classical or both metric formulas.*

Proof. We use the transfinite induction on the length of the proof of Ψ from $T \cup \{\Phi\}$. The classical cases follow as usual.

Suppose that $\Psi = D_{=0}(\alpha, \beta)$ is obtained from $T \cup \{\Phi\}$ by an application of the inference rule (R2) and $\Phi \in \mathrm{For}_M$. In that case:

$T \cup \{\Phi\} \vdash \alpha \leftrightarrow \beta$

$T \cup \{\Phi\} \vdash D_{=0}(\alpha, \beta)$, by (R2).

However, since $\alpha \leftrightarrow \beta \in \mathrm{For}_C$, and $\Phi \in \mathrm{For}_M$, Φ does not affect the proof of $\alpha \leftrightarrow \beta$ from $T \cup \{\Phi\}$. Note that a classical propositional formula can be inferred only by the rule (R1) applied on classical formulas. Thus, we have:

$T \vdash \alpha \leftrightarrow \beta$

$T \vdash D_{=0}(\alpha, \beta)$, by (R2)

$T \vdash D_{=0}(\alpha, \beta) \to (\Phi \to D_{=0}(\alpha, \beta))$, by (A2), since $p \to (q \to p)$ is a tautology,

$T \vdash \Phi \to D_{=0}(\alpha, \beta)$, by (R1)

$T \vdash \Phi \to \Psi$

Next, let us consider the case where $\Psi = A \to D_{\leqslant s}(\alpha, \beta)$ is obtained from $T \cup \{\Phi\}$ by an application of (R3), and $\Phi \in \mathrm{For}_M$. Then:

$T \cup \{\Phi\} \vdash A \to D_{<s+\frac{1}{k}}(\alpha, \beta)$, for every positive integer k

$T \vdash \Phi \to (A \to D_{<s+\frac{1}{k}}(\alpha, \beta))$, every positive integer k, by the induction hypothesis

$T \vdash (\Phi \wedge A) \to D_{<s+\frac{1}{k}}(\alpha, \beta)$, every positive integer k

$T \vdash (\Phi \wedge A) \to D_{\leqslant s}(\alpha, \beta)$, by (R3)

$T \vdash \Phi \to (A \to D_{\leqslant s}(\alpha, \beta))$

$T \vdash \Phi \to \Psi$

The case concerning formulas obtained by (R4) can be proved in the same way. □

The perceptive reader might think that it is a bit strange having a deduction theorem in the presence of an analogue of the necessary rule. However, we assure the reader that this is a common situation in probabilistic logics. Please see [18], and the references therein.

Theorem 4.2. *(Soundness theorem). The axiomatic system Ax is sound with respect to the LPM-models (and therefore to the LM-models).*

Proof. Soundness of the axiomatic system Ax follows from the soundness of propositional classical logics and from the properties of pseudometrics. We can show that every instance of an axiom schemata holds in every LPM-model, while the inference rules preserve validity.

It is easy to see that if α is an instance of a classical propositional tautologies, then for every model $\mathbf{M} = \langle W, v, d\rangle$, $\mathbf{M} \models \alpha$. The axioms (A3-7) concern the properties of the ordering on $\mathbb{Q}_0^+$ and the conditions (D2) and (D3), and these axioms obviously hold in every LPM-model.

The rule (R1) is validity-preserving for the same reason as in classical logic. Consider Rule (R2) and suppose that a formula $\alpha \leftrightarrow \beta$ is valid. Then $[\alpha] = [\beta]$ holds in any LPM-model and hence $D_{=0}(\alpha, \beta)$ must be true in that LPM-model. The rules (R3) and (R4) preserves validity because of the Archimedean property. $\square$

Theorem 4.3. *Every consistent set of formulas can be extended to a maximal consistent set.*

Proof. Let T be a consistent set of formulas and let $A_0, A_1, A_2, \ldots$ be an enumeration of all formulas from For_M. We define a sequence of sets T_i, $i = 0, 1, 2, \ldots$ as follows:

(1) $T_0 = T \cup \mathrm{Con}_C(T) \cup \{D_{=0}(\alpha, \beta) : \alpha \leftrightarrow \beta \in \mathrm{Con}_C(T)\}$, where $\mathrm{Con}_C(T)$ is the set of all classical consequences of T ($\mathrm{Con}_C(T) \subset \mathrm{For}_C$);

For every $i \geqslant 0$,

(2) if $T_i \cup \{A_i\}$ is consistent, then $T_{i+1} = T_i \cup \{A_i\}$;

(3) otherwise, if $T_i \cup \{A_i\}$ is inconsistent, we have:

  (a) if A_i is of the form $B \to D_{\leqslant s}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\neg A_i, B \to D_{\geqslant s+\frac{1}{k}}(\alpha, \beta)\}$, where k is a positive integer chosen so that T_{n+1} is consistent;

  (b) if A_i is of the form $B \to D_{\geqslant s}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\neg A_i, B \to D_{\leqslant s-\frac{1}{k}}(\alpha, \beta)\}$, where k is a positive integer chosen so that T_{n+1} is consistent;

  (c) otherwise, $T_{i+1} = T_i \cup \{\neg A_i\}$.

Note that at each stage we extend the previous set of formulas by finitely many formulas.

Let $T^* = \cup_{i=0}^{\infty} T_i$. The rest of the proof is divided into tree parts.

Claim 1. T_i is consistent for each $i \geqslant 0$.

Proof of Claim 1. The sets obtained by the steps (1) and (2) are obviously consistent. The sets obtained by the step (3c) are consistent by classical arguments:

if $T_i \cup \{A_i\} \vdash \bot$, by the deduction theorem we have $T_i \vdash \neg A_i$, and since T_i is consistent, so it is $T_i \cup \{\neg A_i\}$.

Let us consider the step (3a).

If $T_i \cup \{B \to D_{\leqslant s}(\alpha,\beta)\}$ is not consistent, then the set T_i can be consistently extended as it is described above. Suppose that it is not the case. Then

$T_i, \neg(B \to D_{\leqslant s}(\alpha,\beta)), B \to \neg D_{<s+\frac{1}{k}}(\alpha,\beta) \vdash \bot$, for every positive integer k

$T_i, \neg(B \to D_{\leqslant s}(\alpha,\beta)) \vdash \neg(B \to \neg D_{<s+\frac{1}{k}}(\alpha,\beta))$, for every positive integer k, by the deduction theorem

$T_i, \neg(B \to D_{\leqslant s}(\alpha,\beta)) \vdash B \to D_{<s+\frac{1}{k}}(\alpha,\beta)$, for every positive integer k, by the classical tautology $\neg(p \to q) \to (p \to \neg q)$

$T_i, \neg(B \to D_{\leqslant s}(\alpha,\beta)) \vdash B \to D_{\leqslant s}(\alpha,\beta)$, by (R3)

The last line contradicts the consistency of T_i. In the same manner we prove that the step (3b) produces consistent sets. Thus, the proof of the Claim 1 is completed.

Claim 2. T^* is deductively closed.

Proof of Claim 2. We can show that T^* is a deductively closed set.

Let Φ be a formula from For. It can be proved by induction on the length of the inference that if $T^* \vdash \Phi$, then $\Phi \in T^*$. Note that if $T_i \vdash A$ and $A = A_n$, it must be $A \in T^*$ because $T_{\max\{n,i\}+1}$ is consistent.

Suppose that the sequence $\Phi_1, \Phi_2, \dots, \Phi$ is a formal inference of φ from T^*.

If the sequence is finite, there must be a set T_i such that $T_i \vdash \Phi$, and $\Phi \in T^*$. Thus, suppose that the sequence is countable infinite. We can show that for every i, if Φ_i is obtained by an application of an inference rule, and all the premises belong to T^*, then it must be $\Phi_i \in T^*$.

If the rule is a finitary one (either (R1) or (R2)), then we conclude $\Phi_i \in T^*$ by reasoning as above. Next we consider the infinitary rule (R3). Let $\Phi_i = B \to D_{\leqslant s}(\alpha,\beta)$ be obtained by (R3) from the premises $\Phi_i^j = B \to D_{<s+\frac{1}{j}}(\alpha,\beta) \in T^*$, for every positive integer j. Assume $\Phi_i \notin T^*$. The step (3a) of the construction of T^* provides a positive integer k, such that $B \to \neg D_{<s+\frac{1}{k}}(\alpha,\beta) \in T^*$. Thus, there is m, such that T_m contains both $B \to D_{<s+\frac{1}{k}}(\alpha,\beta)$ and $B \to \neg D_{<s+\frac{1}{k}}(\alpha,\beta)$. It follows that $T_m \cup \{B\}$ is not consistent. $T_m \vdash B \to \bot$ implies $T_m \vdash B \to D_{\leqslant s}(\alpha,\beta)$, and hence $B \to D_{\leqslant s}(\alpha,\beta) \in T^*$, i.e. $\Phi_i \in T^*$ which contradicts the assumption $\Phi_i \notin T^*$. The case when $\Phi_i = B \to D_{\geqslant s}(\alpha,\beta)$ is obtained by (R4) follows similarly.

Henceforth, the set T^* is deductively closed.

Claim 3. T^* is maximal consistent.

Proof of Claim 3. It is easy to see that T^* does not contain all formulas. If $\alpha \in \mathrm{For}_C$, by the definition of T_0, α and $\neg\alpha$ cannot be simultaneously in T_0. If for some A, both A and $\neg A$ belong to T^*, then there is a set T_i such that $A, \neg A \in T_i$, contrary to the consistency of T_i. In summary, for a formula Φ, either $\Phi \in T^*$

or $\neg\Phi \in T^*$, and the set T^* does not contain both. Thus, T^* is consistent. The construction guarantees that it is maximal. Note that T^* could not be complete for classical formulas, in the sense that T^* may contain neither α nor $\neg\alpha$. □

The next lemma gives some auxiliary statements which will be needed for the proof of the completeness theorem.

Lemma 4.2. *Let T^* be a maximal consistent set of formulas as in the proof of the previous theorem. Then the following hold:*

(i) *T^* is deductively closed, and consequently contains all valid formulas.*

(ii) *T^* contains either A or $\neg A$ (and certainly not both), for each $A \in \mathrm{For}_M$.*

(iii) *$A, B \in T^*$ iff $A \wedge B \in T^*$, for every $A, B \in \mathrm{For}_M$.*

(iv) $\inf\{s \in \mathbb{Q}_0^+ \mid D_{\leqslant s}(\alpha,\beta) \in T^*\} \leqslant r$ *iff* $D_{\leqslant r}(\alpha,\beta) \in T^*$*, for every nonnegative rational number r.*

(v) $\inf\{s \in \mathbb{Q}_0^+ \mid D_{\leqslant s}(\alpha,\beta) \in T^*\} < r$ *iff* $D_{<r}(\alpha,\beta) \in T^*$*, for every positive rational number r.*

Proof. The statements (i) and (ii) were already proved. The proof of the statement (iii) is standard. Assume $\inf\{s \mid D_{\leqslant s}(\alpha,\beta) \in T^*\} \leqslant r$ in order to prove the nontrivial part of the statement (iv). If $D_{\leqslant r}(\alpha,\beta) \notin T^*$, then $\neg D_{\leqslant r}(\alpha,\beta) \in T^*$, and by the step (3a) there is a positive integer k such that $D_{\geqslant r+\frac{1}{k}}(\alpha,\beta) \in T^*$. Because of the consistency of T^*, there is no rational $s < r + \frac{1}{k}$ such that $D_{\leqslant s}(\alpha,\beta) \in T^*$, but that is in contradiction with the assumption. Finally, let us prove the nontrivial part of (v). If $D_{<r}(\alpha,\beta) \in T^*$, then $D_{\leqslant r}(\alpha,\beta) \in T^*$, by (A5) and (i), and hence $\inf\{s \mid D_{\leqslant s}(\alpha,\beta) \in T^*\} \leqslant r$, by (iv). The equality $\inf\{s \mid D_{\leqslant s}(\alpha,\beta) \in T^*\} = r$ implies $D_{\leqslant r-\frac{1}{n}}(\alpha,\beta) \notin T^*$, and therefore $D_{>r-\frac{1}{n}}(\alpha,\beta) \in T^*$, for every integer $n > \frac{1}{r}$. By the rule (R4), we obtain $D_{\geqslant r}(\alpha,\beta) \in T^*$, a contradiction. We thus get $\inf\{s \mid D_{\leqslant s}(\alpha,\beta) \in T^*\} < r$. □

Theorem 4.4. *(Completeness theorem for LPM-models) Every consistent set T of formulas has an LPM-model.*

Proof. Let T be a consistent set of formulas, and T^* its maximal consistent extension as in the proof of Theorem 4.3. Using T^*, we define a tuple $\mathbf{M} = \langle W, v, d\rangle$, where:

- W contains all classical propositional interpretations (valuations of propositional letters) that satisfy the set $\mathrm{Con}_C(T)$ of all classical consequences of T;

- $v : W \times P \to \{0,1\}$ is an assignment such that for every world $w \in W$ and every propositional letter $p \in P$, $v(w,p) = 1$ iff $w \models p$,
- $d : \mathcal{F} \times \mathcal{F} \to [0,+\infty)$, such that $d([\alpha],[\beta]) = \inf\{s : D_{\leqslant s}(\alpha,\beta) \in T^*\}$.

Remember $[\alpha] = \{w \in W : w \models \alpha\}$ and $\mathcal{F} = \{[\alpha] : \alpha \in \mathrm{For}_C\}$.

Claim 1. $\mathbf{M}$ is an LPM-model.

Proof of Claim 1. For every formulas $\alpha, \beta \in \mathrm{For}_C$,

$$d([\alpha],[\beta]) = \inf\{s : D_{\leqslant s}(\alpha,\beta) \in T^*\} \geqslant 0,$$

because $D_{\geqslant 0}(\alpha,\beta)$ is an axiom, and $D_{\geqslant 0}(\alpha,\beta) \in T^*$ by the statement (i) in the previous lemma. Therefore, d fulfills the non-negativity constraint.

(D1-) Assume $[\alpha] = [\beta]$. Then, for every $w \in W$, $w \models \alpha$ iff $w \models \beta$, and consequently $\mathrm{Con}_C(T) \vdash \alpha \leftrightarrow \beta$, by the Completeness theorem for the classical propositional logic. Thus, $\alpha \leftrightarrow \beta \in \mathrm{Con}_C(T)$ and $D_{=0}(\alpha,\beta) \in T_0 \subseteq T^*$. It follows that

$$d([\alpha],[\beta]) = \inf\{s : D_{\leqslant s}(\alpha,\beta) \in T^*\} = 0.$$

(D2) Let

$$d([\alpha],[\gamma]) = \inf\{s : D_{\leqslant s}(\alpha,\gamma) \in T^*\} = s_1$$

and

$$d([\gamma],[\beta]) = \inf\{s : D_{\leqslant s}(\gamma,\beta) \in T^*\} = s_2.$$

According to the statement (iv) of the previous lemma, for every rationals $r \geqslant s_1$ and $t \geqslant s_2$, $D_{\leqslant r}(\alpha,\gamma) \in T^*$ and $D_{\leqslant t}(\gamma,\beta) \in T^*$. The axiom (A6) and (i) in the previous lemma imply that $D_{\leqslant r+t}(\alpha,\beta) \in T^*$, i.e.

$$d([\alpha],[\beta]) = \inf\{s : D_{\leqslant s}(\alpha,\beta) \in T^*\} \leqslant r+t, \text{ for all rationals } r \geqslant s_1, t \geqslant s_2.$$

Therefore, $d([\alpha],[\beta]) \leqslant s_1 + s_2 = d([\alpha],[\gamma]) + d([\gamma],[\beta])$.

(D3) In this case we omit details which are the same as above. If $d([\alpha],[\beta]) = \inf\{s : D_{\leqslant s}(\alpha,\beta) \in T^*\} = s_0$, then for any rational $r \geqslant s_0$, $D_{\leqslant r}(\alpha,\beta) \in T^*$, and so $D_{\leqslant r}(\beta,\alpha) \in T^*$, by the axiom (A7). It follows that $d([\beta],[\alpha]) = \inf\{s : D_{\leqslant s}(\beta,\alpha) \in T^*\} \leqslant s_0$. If there were a rational $t < s_0$, such that $D_{\leqslant t}(\beta,\alpha) \in T^*$, we would have $D_{\leqslant t}(\alpha,\beta) \in T^*$, a contradiction. This gives $d([\beta],[\alpha]) = s_0 = d([\alpha],[\beta])$.

Claim 2. For every formula Φ, $\mathbf{M} \models \Phi$ iff $\Phi \in T^*$.

Proof of Claim 2. For every $\alpha \in \mathrm{For}_C$:

$$\begin{aligned} \mathbf{M} \models \alpha \text{ iff } & w \models \alpha, \text{ for every } w \in W \\ & \text{iff } \mathrm{Con}_C(T) \vdash \alpha, \text{ by the definition of } W \\ & \text{iff } \alpha \in T_0 \\ & \text{iff } \alpha \in T^*. \end{aligned}$$

For every $\alpha, \beta \in \mathrm{For}_C$, and $r \in \mathbb{Q}_0^+$:

$$
\begin{aligned}
\mathbf{M} \models D_{\leqslant r}(\alpha,\beta) &\text{ iff } d([\alpha],[\beta]) \leqslant r \\
&\text{ iff } \inf\{s : D_{\leqslant s}(\alpha,\beta) \in T^*\} \leqslant r, \text{ by the definition of } d, \\
&\text{ iff } D_{\leqslant r}(\alpha,\beta) \in T^*, \text{ by the statement (iv) of Lemma 4.2;} \\
\mathbf{M} \models D_{\geqslant r}(\alpha,\beta) &\text{ iff } d([\alpha],[\beta]) \geqslant r \\
&\text{ iff } d([\alpha],[\beta]) < r \text{ does not hold} \\
&\text{ iff } \inf\{s : D_{\leqslant s}(\alpha,\beta) \in T^*\} < r \text{ does not hold} \\
&\text{ iff } D_{<r}(\alpha,\beta) \notin T^*, \text{ by the statement (v) of Lemma 4.2} \\
&\text{ iff } D_{\geqslant r}(\alpha,\beta) \in T^*
\end{aligned}
$$

For every $A, B \in \mathrm{For}_M$:

$$
\begin{aligned}
\mathbf{M} \models A \wedge B &\text{ iff } \mathbf{M} \models A \text{ and } \mathbf{M} \models B \\
&\text{ iff } A \in T^* \text{ and } B \in T^*, \text{ by the induction hypothesis} \\
&\text{ iff } A \wedge B \in T^*, \text{ by the statement (iii) of Lemma 4.2;}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{M} \models \neg A &\text{ iff } \mathbf{M} \not\models A \\
&\text{ iff } A \notin T^*, \text{ by the induction hypothesis} \\
&\text{ iff } \neg A \in T^*, \text{ by the statement (ii) of Lemma 4.2.}
\end{aligned}
$$

□

Theorem 4.5. *(Completeness theorem for LM-models) Every consistent set T of formulas has an LM-model.*

Proof. The main points in this proof are the same as in the proof of Theorem 4.4. We first extend T to a maximal consistent set. But, the extension given in the proof of Theorem 4.3 will be slightly changed. The sequence of sets T_i, $i = 0, 1, 2, \ldots$ is now defined as follows:

(1) $T_0 = T \cup \mathrm{Con}_C(T) \cup \{D_{=0}(\alpha,\beta) : \alpha \leftrightarrow \beta \in \mathrm{Con}_C(T)\}$, where $\mathrm{Con}_C(T)$ is the set of all classical consequences of T ($\mathrm{Con}_C(T) \subset \mathrm{For}_C$);

For every $i \geqslant 0$,

(2) if $T_i \cup \{A_i\}$ is consistent, then $T_{i+1} = T_i \cup \{A_i\}$;

(3) otherwise, if $T_i \cup \{A_i\}$ is inconsistent, we have:

(a) if A_i is of the form $B \to D_{\leqslant s}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\neg A_i, B \to D_{\geqslant s+\frac{1}{k}}(\alpha, \beta)\}$, where k is a positive integer chosen so that T_{n+1} is consistent;

(b) if A_i is of the form $B \to D_{\geqslant s}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\neg A_i, B \to D_{\leqslant s-\frac{1}{k}}(\alpha, \beta)\}$, where k is a positive integer chosen so that T_{n+1} is consistent;

(c) if A_i is of the form $D_{=0}(\alpha, \beta)$, then $T_{i+1} = T_i \cup \{\neg A_i, \neg(\alpha \leftrightarrow \beta)\}$;

(d) otherwise, $T_{i+1} = T_i \cup \{\neg A_i\}$.

We show that the step (3c) produces consistent sets.
Suppose $T_i \cup \{\neg A_i, \neg(\alpha \leftrightarrow \beta)\} \vdash \bot$, i.e., $T_i \cup \{\neg A_i\} \vdash \alpha \leftrightarrow \beta$. Since $\alpha \leftrightarrow \beta \in \mathrm{For}_C$, $\alpha \leftrightarrow \beta$ belongs to Con_C, and consequently $D_{=0}(\alpha, \beta) \in T_0$, which contradicts the consistency of T_i.

The rest of the proof is the same as for Theorem 4.4. □

The fact that the axiomatic system Ax is sound and complete with respect to two different classes of models is quite similar to the one for probabilistic logics (see [18]), or from the modal framework where, for instance, the modal system K is characterized by the class of all models, but also by the class of all irreflexive models. Consequently, our syntax cannot expresses differences between the mentioned classes of distance models, LPM-models and LM-models.

Note that with the LPM-semantics, as well as LM-semantics the set formulas $\{D_{\neq s}(\alpha, \beta) : s \in \mathbb{Q}_0^+\}$ is satisfiable (in a model where $d([\alpha], [\beta])$ is an irrational number). Although there are no formal reasons why this would be problematic, it is possible to determine, at syntax level, a countable range of distance functions. If we want the range to be $\mathbb{Q}_0^+$, the rule (R3) should be replaced with the following rule:

(R) From $A \to D_{\neq s}(\alpha, \beta)$, for every $s \in \mathbb{Q}_0^+$, infer $\neg A$.

Following the ideas given in the previous theorems, one could prove the completeness theorem for LPM-models (LM-models) with $\mathbb{Q}_0^+$-valued pseudometrics (metrics).

If we extended Ax with the following axioms:

(A8) $D_{\leqslant 1}(\alpha, \beta)$,

(A9) $D_{=1}(\alpha, \neg\alpha)$,

(A10) $D_{=0}(\beta \wedge \gamma, \bot) \wedge D_{\geqslant s}(\alpha, \beta) \to D_{\leqslant 1-s}(\alpha, \gamma)$, $s \leqslant 1$,

we would be able to prove the completeness theorem with respect to the class of LM^+-models.

5 Conclusion

In this paper we have introduced propositional metric logics with binary metric operators and provided strongly complete axiomatizations. One of interesting problems for further investigation might be to find axiomatization of a logic that allows the iterations of metric operators and mixing of classical and metric formulas. Namely, allowing iterations of the metric operators can help us formalize many things. Another direction for research might be extending our logic to corresponding first order logics. All these formalizations could be useful tool in modelling and understanding real-world problems.

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