Frames for weighted shift-invariant spaces

Stevan Pilipović[1], Suzana Simić[2]

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Abstract. In this paper we prove the equivalence of the frame property and the closedness for a weighted shift-invariant space

$$V_\mu^p(\Phi) = \Big\{ \sum_{i=1}^{r} \sum_{j\in\mathbb{Z}^d} c_i(j)\phi_i(\cdot - j) \;\Big|\; \{c_i(j)\}_{j\in\mathbb{Z}^d} \in \ell_\mu^p \Big\}, \quad p \in [1,\infty],$$

which corresponds to $\Phi = \Phi^r = (\phi_1, \phi_2, \ldots, \phi_r)^T \in (W_\omega^1)^r$. We, also, construct a sequence Φ^{2k+1} and the sequence of spaces $V_\mu^p(\Phi^{2k+1})$, $k \in \mathbb{N}$, on $\mathbb{R}$, with the useful properties in sampling, approximations and stability.

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1 Introduction

In this paper, we investigate weighted shift-invariant spaces quoted in the abstract by following the methods from [2] and [25]. Such spaces figure in several areas of applied mathematics, notably in wavelet theory and approximation theory ([2], [8]). In recent years, they have been extensively studied by many authors (see [1]-[8], [14]-[16], [19], [20], [25], [26]). Sampling with non-bandlimited functions in shift-invariant spaces is a suitable and realistic model for many applications, such as modeling signals with the spectrum that is smoother then in the case of bandlimited functions, or for the numerical implementation (see [6], [9], [10], [12], [13], [17]). These requirements can often be met by choosing appropriate functions in Φ. This means that the functions in Φ have a shape corresponding to a particular impulse response of a device, or that they are compactly supported or that they have a Fourier transform decaying smoothly to zero as $|\xi| \to \infty$.

Weighted shift-invariant spaces $V_\mu^p(\Phi)$, $p \in [1,\infty]$, where μ is a weight, were introduced for the non uniform sampling as a direct generalization of the space $V^p(\Phi)$ ([1], [26]). The determination of p and the signal smoothness are used for optimal compression and coding signals and images (see [9]).

The first aim of this paper is to show that the main result of [2] holds in the case of weighted shift-invariant spaces which correspond to L^p_μ and ℓ^p_μ, i.e., weighted L^p and ℓ^p spaces, respectively. Namely, we follow [2] and [25] and prove assertions which need additional arguments depending on the weights. We show that under the appropriate conditions on the frame vectors, there is an equivalence between the concept of p-frames, Banach frames with respect to ℓ^p_μ and closedness of the space which they generate. A weighted analog of Corollary 3.2 from [25] simplifies a part of the proof of our main result. Although another part of the proof follows, step by step, the proof of the corresponding theorem in [2], we think that it is not simple at all, and that it is worth to be done.

The second aim of this paper is to construct $V^p_\mu(\Phi^{2k+1})$ spaces with specially chosen functions, ϕ_0, ϕ_1, ..., ϕ_{2k}, that generate a Banach frame for the shift-invariant space $V^p_\mu(\Phi^{2k+1})$. Actually, we take functions from a sequence $\{\phi_i\}_{i\in\mathbb{Z}}$ so that the sequence of Fourier transforms $\widehat{\phi}_i = \theta(\cdot + i\pi)$, $i \in \mathbb{Z}$, $\theta \in C_0^\infty(\mathbb{R})$, makes a partition of unity in the frequency domain ($\mathbb{Z} = \mathbb{N}_0 \cup -\mathbb{N}, \mathbb{N}$ is the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). We note that properties of the constructed frame guarantee the feasibility of a stable and continuous reconstruction algorithm in $V^p_\mu(\Phi)$ ([26]). Also, we note that $\{\phi_i(\cdot - k) \mid k \in \mathbb{Z},\ i = 1, \ldots, r\}$ forms a Riesz basis for $V^p_\mu(\Phi)$ when the spectrum of the Gram matrix $[\widehat{\Phi}, \widehat{\Phi}](\xi)$ is bounded and bounded away from zero (see [8]). The d-dimensional case, $d > 1$, is technically more complicated and because of that it is not considered in this paper.

The paper is organized as follows. In Section 2 we quote basic properties of subspaces of weighted L^p and ℓ^p spaces. The weighted shift-invariant spaces are investigated in Section 3, where we presented our first result quoted in the abstract, Theorem 3.10. In Section 4 we show relations between the dual of the Fréchet space $\bigcap_{s\in\mathbb{N}_0} V^p_{(1+|x|^2)^{s/2}}(\Phi)$ and the space of periodic distributions. The case of periodic ultradistributions is obtain by using subexponential growth functions. In Section 5, we use a special sequence of functions $\{\phi_k \mid k \in \mathbb{N}\}$ to construct a sequence of p-frames. Our construction shows that the sampling and reconstruction problem in the shift-invariant spaces is robust. In the final remark of Section 5, we list good properties of these frames.

2 Basic spaces

Denote by $L^1_{loc}(\mathbb{R}^d)$ the space of measurable functions integrable over compact subsets of $\mathbb{R}^d$. For a nonnegative function $\omega \in L^1_{loc}(\mathbb{R}^d)$ we say that is submultiplicative if $\omega(x+y) \le \omega(x)\omega(y)$, x, $y \in \mathbb{R}^d$ and a function μ on $\mathbb{R}^d$ is ω-moderate if $\mu(x+y) \le C\omega(x)\mu(y)$, x, $y \in \mathbb{R}^d$. We assume that ω is continuous and symmetric and both μ and ω call weights, as usual. The standard class of weights on $\mathbb{R}^d$ are of the polynomial type $\omega_s(x) = (1+|x|)^s$, $s \ge 0$. To quantify faster decay of functions we use the subexponential weights $\omega(x) = \mathrm{e}^{\alpha|x|^\beta}$, for some $\alpha > 0$ and $0 < \beta < 1$. Weighted L^p spaces with moderate weights

are translation-invariant spaces (see [1]). We, also, consider weighted sequence spaces $\ell^p_\mu(\mathbb{Z}^d)$ with ω-moderate weight μ. Recall, a sequence c belongs to $\ell^p_\mu(\mathbb{Z}^d)$ if $c\mu$ belongs to $\ell^p(\mathbb{Z}^d)$.

In the sequel ω is a submultipicative weight, continuous and symmetric and μ is ω-moderate. Let $p \in [1,\infty)$. Then (with obvious modification for $p=\infty$)

$$\mathcal{L}^p_\omega = \Big\{ f \mid \|f\|_{\mathcal{L}^p_\omega} = \Big(\int\limits_{[0,1]^d} \Big(\sum_{j\in\mathbb{Z}^d} |f(x+j)|\omega(x+j)\Big)^p \mathrm{d}x\Big)^{1/p} < +\infty \Big\},$$

$$W^p_\omega := \Big\{ f \mid \|f\|_{W^p_\omega} = \Big(\sum_{j\in\mathbb{Z}^d} \sup_{x\in[0,1]^d} |f(x+j)|^p \omega(j)^p \Big)^{1/p} < +\infty \Big\}.$$

Obviously, we have $W^p_\omega \subset W^q_\omega \subset \mathcal{L}^\infty_\omega \subset \mathcal{L}^q_\omega \subset \mathcal{L}^p_\omega \subset L^p_\omega$, $W^p_\omega \subset W^p_\mu \subset W^q_\mu \subset L^q_\mu$ and $L^p_\omega \subset L^p_\mu$, where $1 < p < q \le +\infty$. For $p=1$ and $\omega = 1$ we have $\mathcal{L}^1 = L^1$. We also have $\ell^1_\omega \subset \ell^p_\omega \subset \ell^q_\omega \subset \ell^q_\mu$, for $1 < p < q \le +\infty$. From [1] we have the following properties.

1) If $f \in L^p_\mu$, $g \in L^1_\omega$ and $p \in [1,\infty]$, then $\|f * g\|_{L^p_\mu} \le \|f\|_{L^p_\mu}\|g\|_{L^1_\omega}$.

2) If $f \in L^p_\mu$, $g \in W^1_\omega$ and $p \in [1,\infty]$, then $\|f * g\|_{W^p_\mu} \le \|f\|_{L^p_\mu}\|g\|_{W^1_\omega}$.

3) If $c \in \ell^p_\mu$ and $d \in \ell^1_\omega$, then holds the inequality $\|c * d\|_{\ell^p_\mu} \le \|c\|_{\ell^p_\mu}\|d\|_{\ell^1_\omega}$.

Denote by $\mathcal{WC}^p_\mu$, $p \in [1,\infty]$, a space of all 2π-periodic functions with their sequences of Fourier coefficients in ℓ^p_μ. Let D_1 and D_2 be the sequences of Fourier coefficients of 2π-periodic functions K_1 and K_2, respectively. If $D_1 * D_2 \in \ell^p_\mu$, then $D_1 * D_2$ is the sequence of Fourier coefficients of the product K_1K_2. For $K = (K_1,\dots,K_r)^T \in (WC^p_\mu)^r$, ($T$ means transpose) define $\|K\|_{\ell^p_{\mu,*}}$ to be the ℓ^p_μ norm of its sequence of Fourier coefficients.

In the sequel we use the notation $\Phi = (\phi_1,\dots,\phi_r)^T$. Define $\|\Phi\|_{\mathcal{H}} = \sum\limits_{i=1}^r \|\phi_i\|_{\mathcal{H}}$, where $\mathcal{H} = L^p_\omega$, $\mathcal{L}^p_\omega$ or W^p_ω, $p \in [1,\infty]$.

We list several lemmas needed to prove our results. Their proofs are analogous to the proof of the corresponding lemmas in [2].

Lemma 2.1. *Let $f \in L^p_\mu$ and $g \in W^1_\omega$, $p \in [1,\infty]$. Then the sequence*

$$\Big\{ \int\limits_{\mathbb{R}^d} f(x)g(x-j)\mathrm{d}x \Big\}_{j\in\mathbb{Z}^d} \in \ell^p_\mu$$

and $\Big\| \Big\{ \int\limits_{\mathbb{R}^d} f(x)g(x-j)\mathrm{d}x \Big\}_{j\in\mathbb{Z}^d} \Big\|_{\ell^p_\mu} \le \|f\|_{L^p_\mu}\|g\|_{W^1_\omega}$.

Let $c = \{c_i\}_{i\in\mathbb{N}} \in \ell^p_\mu$ and $f \in L^p_\omega$, $p \in [1,\infty]$. We define, as in [2], their semi-convolution $f *' c$ by $(f *' c)(x) = \sum\limits_{j\in\mathbb{Z}^d} c_j f(x-j)$, $x \in \mathbb{R}^d$.

Lemma 2.2. *a) If $f \in L^p_\omega$ and $c \in \ell^p_\mu$, $p \in [1,\infty]$, then $f *' c \in L^p_\mu$ and $\|f *' c\|_{L^p_\mu} \le \|c\|_{\ell^p_\mu}\|f\|_{L^p_\omega}$.*

b) If $f \in \mathcal{L}^p_\omega$, $p \in [1,\infty]$, and $c \in \ell^1_\mu$, then $\|f *' c\|_{\mathcal{L}^p_\mu} \le \|c\|_{\ell^1_\mu}\|f\|_{\mathcal{L}^p_\omega}$.

c) If $f \in W^p_\omega$, $p \in [1,\infty]$, and $c \in \ell^1_\mu$, then $\|f *' c\|_{W^p_\mu} \le \|c\|_{\ell^1_\mu}\|f\|_{W^p_\omega}$,

d) If $f \in W^1_\omega$ and $c \in \ell^p_\mu$, $p \in [1,\infty]$, then $\|f *' c\|_{W^p_\mu} \le \|c\|_{\ell^p_\mu}\|f\|_{W^1_\omega}$.

3 Characterization of $V^p_\mu(\Phi)$

In [11] Feichtinger and Gröchening extended the notation of atomic decomposition to Banach spaces ([10], [12]), while Gröchening [18] introduced a more general concept of decomposition through Banach frames. We recall the definition.

Let X be a Banach space and Θ be an associated Banach space of scalar valued sequences, indexed by $I = \mathbb{N}$ or $I = \mathbb{Z}$. Let $\{f_n\} \subset X^*$ and $S : \Theta \to X$ be given. The pair $(\{f_n\}_{n\in I}, S)$ is called a Banach frame for E with respect to Θ if

(1) $\{f_n(x)\}_{n\in I} \in \Theta$ for each $x \in X$,

(2) there exist positive constants A and B with $0 < A \le B < +\infty$ such that $A\|x\|_X \le \|\{f_n(x)_{n\in I}\}\|_\theta \le B\|x\|_X$, $x \in X$,

(3) S is a bounded linear operator such that $S(\{f_n(x)\}_{n\in I}) = x$, $x \in X$.

It is said that a collection $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, 1 \le i \le r\}$ is a p-frame for $V^p_\mu(\Phi)$ if there exists a positive constant C (depending on Φ, p and ω)

$$C^{-1}\|f\|_{L^p_\mu} \le \sum_{i=1}^{r} \Big\| \Big\{ \int_{\mathbb{R}^d} f(x)\phi_i(x-j)\mathrm{d}x \Big\}_{j\in\mathbb{Z}^d} \Big\|_{\ell^p_\mu} \le C\|f\|_{L^p_\mu}, \quad f \in V^p_\mu(\Phi). \tag{3.1}$$

A typical application is the problem of finding a shift-invariant space model that describes a given class of signals or images (e.g. the class of chest X-rays). The observation set of r signals or images $f_1, \ldots, f_r$ may be theoretical samples, or experimental data.

Recall [1], the shift-invariant spaces are defined by

$$V^p_\mu(\Phi) := \Big\{ f \in L^p_\mu \ \mid \ f(\cdot) = \sum_{i=1}^{r} \sum_{j\in\mathbb{Z}^d} c^i_j\, \phi_i(\cdot - j), \ \ \{c^i_j\}_{j\in\mathbb{Z}^d} \in \ell^p_\mu, \ 1 \le i \le r \Big\}.$$

Remark 3.1. If $\Phi \in W^1_\omega$ and μ is ω-moderate, then $V^p_\mu(\Phi)$ is a subspace (not necessarily closed) of L^p_μ and W^p_μ for any $p \in [1,\infty]$. If $r = 1$ and $\{\phi(\cdot - j) \mid j \in \mathbb{Z}^d\}$ is a p-frame for $V^p_\mu(\phi)$, then $V^p_\mu(\phi)$ is a closed subspace of L^p_μ and W^p_μ for $p \in [1,\infty]$ (see [23]).

Let $[\widehat{\Phi},\widehat{\Phi}](\xi)=\Big[\sum_{k\in\mathbb{Z}^d}\widehat{\phi}_i(\xi+2k\pi)\overline{\widehat{\phi}_j(\xi+2k\pi)}\Big]_{1\le i\le r,\ 1\le j\le r}$ where we assume that $\widehat{\phi}_i(\xi)\overline{\widehat{\phi}_j(\xi)}$ is integrable for any $1\le i,j\le r$. Let $A=(a(j))_{j\in\mathbb{Z}^d}$ be an $r\times\infty$ matrix and $A\overline{A^T}=\Big[\sum_{j\in\mathbb{Z}^d}a_i(j)\overline{a_{i'}(j)}\Big]_{1\le i,i'\le r}$. Then $\operatorname{rank}A=\operatorname{rank}A\overline{A^T}$.

Also, since $[\widehat{\Phi},\widehat{\Phi}](\xi)$ is continuous (as a function with r^2 components) for any $\Phi\in(\mathcal{L}^2_\omega)^r$, it follows that $\big\{\xi\in\mathbb{R}^d\mid \operatorname{rank}\big[\widehat{\Phi}(\xi+2k\pi)_{k\in\mathbb{Z}^d}\big]>k_0\big\}$ is an open set for any $k_0>0$ and $\Phi\in(\mathcal{L}^2_\omega)^r$.

Denote by Σ^μ_α the family of all α-slant matrices $A=[a(j,k)_{j\in\mathbb{Z}^d,k\in\mathbb{Z}^d}]$ with

$$\|A\|_{\Sigma^\omega_\alpha}=\sum_{k\in\mathbb{Z}^d}\sup_{j,k\in\mathbb{Z}^d}|a(k,j)|\chi_{k+[0,1)^d}(k-\alpha j)<\infty,$$

where μ is a weight on $\mathbb{R}^d$ and α is a positive number. The slanted matrices appear in wavelet theory, signal processing and sampling theory (see [25]). Note $\Sigma^\mu_\alpha\subset\Sigma^{\mu_0}_\alpha$ for any weight μ where $\mu_0\equiv 1$ is the trivial weight.

We assume in this subsection that $\Phi=(\phi_1,\dots,\phi_r)^T\in(\mathcal{L}^p_\omega)^r$ for $p\in[1,\infty)$.

To prove Theorem 3.10, we need several lemmas. First we recall a result from [2].

Lemma 3.2 ([2])**.** *The following statements are equivalent.*

1) $\operatorname{rank}\big[\widehat{\Phi}(\xi+2j\pi)_{j\in\mathbb{Z}^d}\big]$ *is a constant function on* $\mathbb{R}^d$.

2) $\operatorname{rank}[\widehat{\Phi},\widehat{\Phi}](\xi)$ *is a constant function on* $\mathbb{R}^d$.

3) *There exists a positive constant* C *independent of* ξ *such that*

$$C^{-1}[\widehat{\Phi},\widehat{\Phi}](\xi)\le[\widehat{\Phi},\widehat{\Phi}](\xi)\,\overline{[\widehat{\Phi},\widehat{\Phi}](\xi)^T}\le C\,[\widehat{\Phi},\widehat{\Phi}](\xi),\quad \xi\in[-\pi,\pi]^d.$$

The proofs of the following two lemmas are similar to proofs of the corresponding lemmas from [2]; hence we will not include them here. The second one provides a localization technique in Fourier domain. It allows us to replace locally the generator $\widehat{\Phi}$ of size r by $\widehat{\Psi}_{1,\lambda}$ of size k_0.

Lemma 3.3. *All the entries of* $[\widehat{\Phi},\widehat{\Phi}](\xi)$ *belong to* $\mathcal{WC}^1_\omega$ *and are continuous.*

Lemma 3.4. *Let the* $\operatorname{rank}[\widehat{\Phi}(\xi+2j\pi)_{j\in\mathbb{Z}^d}]=k_0\ge 1$ *for all* $\xi\in\mathbb{R}^d$. *Then there exist a finite index set* Λ, *points* $\eta_\lambda\in[-\pi,\pi]^d$, $0\le\delta_\lambda<1/4$, *a nonsingular* 2π*-periodic* $r\times r$ *matrix* $P_\lambda(\xi)$ *with all entries in the class* $\mathcal{WC}^1_\omega$ *and* $K_\lambda\subset\mathbb{Z}^d$ *with cardinality* k_0 *for all* $\lambda\in\Lambda$, *such that:*

(i) $\bigcup_{\lambda\in\Lambda}B(\eta_\lambda,\delta_\lambda/2)\supset[-\pi,\pi]^d$, *where* $B(x_0,\delta)$ *denotes the open ball in* $\mathbb{R}^d$ *with center* x_0 *and radius* δ;

(ii) $P_\lambda(\xi)\widehat{\Phi}(\xi)=\begin{bmatrix}\widehat{\Psi}_{1,\lambda}(\xi)\\ \widehat{\Psi}_{2,\lambda}(\xi)\end{bmatrix}$, $\xi\in\mathbb{R}^d$, $\lambda\in\Lambda$, *where* $\Psi_{1,\lambda}$ *and* $\Psi_{2,\lambda}$ *are functions from* $\mathbb{R}^d$ *to* C^{k_0} *and* C^{r-k_0}, *respectively, satisfying*

$$\operatorname{rank}\big[\widehat{\Psi}_{1,\lambda}(\xi+2k\pi)_{k\in K_\lambda}\big]=k_0,\quad \xi\in B(\eta_\lambda,2\delta_\lambda),$$

$$\widehat{\Psi}_{2,\lambda}(\xi) = 0, \quad \xi \in B(\eta_\lambda, 8\delta_\lambda/5) + 2\pi\mathbb{Z}^d.$$

Furthermore, there exist 2π-periodic C^∞ functions h_λ, $\lambda \in \Lambda$, on $\mathbb{R}^d$ such that $\sum_{\lambda\in\Lambda} h_\lambda(\xi) = 1$, $\xi \in \mathbb{R}^d$, and $\operatorname{supp} h_\lambda \subset B(\eta_\lambda, \delta_\lambda/2) + 2\pi\mathbb{Z}^d$.

The next lemma is needed for the proof of Theorem 3.10. Although the formulation is not the same as [2, Lemma 3], the proof is based on the same procedure, and we omit it.

Lemma 3.5. *(a) Let $\phi \in \mathcal{L}^p_{\omega_s}$ if $p \in [1,\infty)$ and $\phi \in W^1_{\omega_s}$ if $p = +\infty$. Assume that $\sum_{j\in\mathbb{Z}^d} \phi(\cdot + j) = 0$. Then for any function h on $\mathbb{R}^d$ which satisfies*

$$|h(x)| \le C(1+|x|)^{-s-d-1}, \quad |h(x)-h(y)| \le C\frac{|x-y|}{(1+\min\{|x|,|y|\})^{s+d+1}},$$

we have

$$\lim_{n\to+\infty} 2^{-nd}\Big\| \sum_{j\in\mathbb{Z}^d} h(2^{-n}j)\phi(\cdot - j)\Big\|_{\mathcal{L}^p_{\omega_s}} = 0.$$

(b) Let $\mu(x) = \mathrm{e}^{\alpha|x|^\beta}$. Let $\phi \in \mathcal{L}^p_\mu$ if $p \in [1,\infty)$ and $\phi \in W^1_\mu$ if $p = +\infty$. Assume that $\sum_{j\in\mathbb{Z}^d} \phi(\cdot + j) = 0$. Then for any function h on $\mathbb{R}^d$ which satisfies

$$|h(x)| \le C\mathrm{e}^{-(\alpha+d+1)|x|^\beta}, \quad |h(x)-h(y)| \le C|x-y|\mathrm{e}^{-(\alpha+d+1)(1+\min\{|x|^\beta,|y|^\beta\})},$$

we have

$$\lim_{n\to+\infty} 2^{-nd}\Big\| \sum_{j\in\mathbb{Z}^d} h(2^{-n}j)\phi(\cdot - j)\Big\|_{\mathcal{L}^p_\mu} = 0.$$

Next, we give a result on the equivalence of ℓ^p_μ-stability of the synthesis operator S_Φ for a different $p \in [1,\infty]$ (see [25]; here we have $\Lambda = \{1,2,\dots,r\}$).

Proposition 3.6. *[25, Corollary 3.2] Let $\Phi = (\phi_1,\dots,\phi_r)^T \in (W^1_\omega)^r$, $p \in [1,\infty]$ and μ is ω-moderate. Define the synthesis operator $S_\Phi : (\ell^p_\mu(\mathbb{Z}^d))^r \mapsto V^p_\mu(\Phi)$ by*

$$S_\Phi : c = \{c^i_j\}_{j\in\mathbb{Z}^d, 1\le i\le r} \mapsto \sum_{i=1}^r \sum_{j\in\mathbb{Z}^d} c^i_j \phi_i(\cdot - j).$$

If the synthesis operator S_Φ has ℓ^p_μ-stability for some $p \in [1,\infty]$, i.e., there exists a positive constant C such that

$$C^{-1}\|c\|_{(\ell^p_\mu(\mathbb{Z}^d))^r} \le \|S_\Phi c\|_{L^p_\mu} \le C\|c\|_{(\ell^p_\mu(\mathbb{Z}^d))^r}, \tag{3.2}$$

for all $c \in (\ell^p_\mu(\mathbb{Z}^d))^r$, then the synthesis operator S_Φ has ℓ^q_μ-stability for any $q \in [1,\infty]$.

As a consequence of the previous proposition, we have the next result.

Proposition 3.7. *[25, Corollary 3.3] Let $p \in [1,\infty]$ and $\Phi = (\phi_1, \dots, \phi_r)^T \in (W^1_\omega)^r$, and μ ω-moderate. If the synthesis operator S_Φ has ℓ^p_μ-stability, then there exists another family $\Psi = (\psi_1, \dots, \psi_r)^T \in (W^1_\omega)^r$ such that the inverse of the synthesis operator S_Φ is given by*

$$(S_\Phi)^{-1}(f) = \Big\{ \int\limits_{\mathbb{R}^d} f(x)\psi_i(x-j)\mathrm{d}x \Big\}_{1\le i\le r, j\in\mathbb{Z}^d}, \ f \in V^p_\mu.$$

Proposition 3.6 and 3.7 imply:

Theorem 3.8. *Let $\Phi = (\phi_1, \dots, \phi_r)^T \in (W^1_\omega)^r$, $p_0 \in [1,\infty]$, and μ is ω-moderate. Then the following three statements are equivalent.*

a) The synthesis operator S_Φ has $\ell^{p_0}_\mu$-stability.

b) $V^{p_0}_\mu(\Phi)$ is closed in $L^{p_0}_\mu$.

c) There exists $\Psi = (\psi_1, \dots, \psi_r)^T \in (W^1_\omega)^r$, such that

$$f = \sum_{i=1}^r \sum_{j\in\mathbb{Z}^d} \langle f, \psi_i(\cdot - j)\rangle \phi_i(\cdot - j), \ \ f \in V^{p_0}_\mu(\Phi).$$

Also we have the next assertion.

d) If the synthesis operator S_Φ has $\ell^{p_0}_\mu$-stability, then the collection $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, 1 \le i \le r\}$ is a p_0-frame for $V^{p_0}_\mu(\Phi)$.

Proof. The implication $a) \Rightarrow c)$ is a consequence of Proposition 3.6 (see Proposition 3.7).

$c) \Rightarrow a)$: Let $f = \sum\limits_{i=1}^r \sum\limits_{j\in\mathbb{Z}^d} \langle f, \psi_i(\cdot - j)\rangle \phi_i(\cdot - j)$ and

$$c^i = \{\langle f, \psi_i(\cdot - j)\rangle\}_{j\in\mathbb{Z}^d}, \ \ 1 \le i \le r.$$

Then

$$\|c\|_{(\ell^p_\mu)^r} = \sum_{i=1}^r \Big\| \Big\{ \int\limits_{\mathbb{R}^d} f(x)\psi_i(x-j)\mathrm{d}x \Big\}_{j\in\mathbb{Z}^d} \le C\|f\|_{L^{p_0}_\mu},$$

where $C = \sum\limits_{i=1}^r \|\psi_i\|_{W^1_\omega}$. Using Lemma 2.1 and the inequality (2.2), we obtain the right-hand side of (3.2).

The equivalence $a) \Leftrightarrow b)$ follows from standard functional analytic arguments (see [2, Theorem 2, Lemma 4]).

$d)$ Lemma 2.1 implies that $\{\langle f, \phi_i(\cdot - j)\rangle\} \in \ell^{p_0}_\mu$, $1 \le i \le r$, and

$$\sum_{i=1}^r \Big\| \Big\{ \int\limits_{\mathbb{R}^d} f(x)\phi_i(x-j)\mathrm{d}x \Big\}_{j\in\mathbb{Z}^d} \le \|f\|_{L^{p_0}_\mu} \sum_{i=1}^r \|\phi_i\|_{W^1_\omega}.$$

Now, ℓ^p_μ-stability implies

$$\|f\|_{L^{p_0}_\mu} \leq C \sum_{i=1}^{r} \Big\| \Big\{ \int_{\mathbb{R}^d} f(x)\phi_i(x-j)\mathrm{d}x \Big\}_{j\in\mathbb{Z}^d} \Big\|_{\ell^{p_0}_\mu}.$$

□

Remark 3.9. Note that ℓ^p_μ-stability of the synthesis operator implies ℓ^q_μ-stability, for any $q \in [1,\infty]$ ([25]), so the statements *b*), *c*) and *d*), do not depend on $p \in [1,\infty]$.

Now, we give our main result.

Theorem 3.10. *Let $\Phi = (\phi_1,\dots,\phi_r)^T \in (W^1_\omega)^r$, $p_0 \in [1,\infty]$, and μ is ω-moderate. Then the following statements are equivalent.*

i) *$V^{p_0}_\mu(\Phi)$ is closed in $L^{p_0}_\mu$.*

ii) *$\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, 1 \leq i \leq r\}$ is a p_0-frame for $V^{p_0}_\mu(\Phi)$.*

iii) *There exists a positive constant C such that*

$$C^{-1}[\widehat{\Phi},\widehat{\Phi}](\xi) \leq [\widehat{\Phi},\widehat{\Phi}](\xi)\overline{[\widehat{\Phi},\widehat{\Phi}](\xi)^T} \leq C[\widehat{\Phi},\widehat{\Phi}](\xi), \quad \xi \in [-\pi,\pi]^d.$$

iv) *There exist positive constants C_1 and C_2 (depending on Φ and ω) such that*

$$C_1\|f\|_{L^{p_0}_\mu} \leq \inf_{f=\sum_{i=1}^{r}\phi_i *' c^i} \sum_{i=1}^{r} \|\{c^i_j\}_{j\in\mathbb{Z}^d}\|_{\ell^{p_0}_\mu} \leq C_2\|f\|_{L^{p_0}_\mu}, \quad f \in V^{p_0}_\mu(\Phi). \tag{3.3}$$

v) *There exists $\Psi = (\psi_1,\dots,\psi_r)^T \in (W^1_\omega)^r$, such that*

$$f = \sum_{i=1}^{r}\sum_{j\in\mathbb{Z}^d} \langle f, \psi_i(\cdot - j)\rangle \phi_i(\cdot - j) = \sum_{i=1}^{r}\sum_{j\in\mathbb{Z}^d} \langle f, \phi_i(\cdot - j)\rangle \psi_i(\cdot - j), \quad f \in V^{p_0}_\mu(\Phi).$$

Proof. If the synthesis operator has ℓ^p_μ-stability, then the statement *iv*) is satisfied. Conversely, if the statement *iv*) is satisfied, then the right-hand side of (3.2) (with $p = p_0$) immediately follows. Using *c*) from Theorem 3.8, we obtain the left-hand side of (3.2). Hence, by Theorem 3.8, we have *i*) $\Leftrightarrow$ *iv*) and *iv*) $\Rightarrow$ *ii*). The equivalence *iv*) $\Leftrightarrow$ *v*) follows from Lemma 2.1.

We follow [2] to prove *iii*) $\Rightarrow$ *v*) and *ii*) $\Rightarrow$ *iii*), and carefully check the use of weights.

iii) $\Rightarrow$ *v*). Let $B_\lambda(\xi) = H_\lambda(\xi)\overline{P_\lambda(\xi)^T} \begin{pmatrix} \left[\widehat{\Psi}_{1,\lambda}, \widehat{\Psi}_{1,\lambda}\right](\xi)^{-1} & 0 \\ 0 & \mathrm{I} \end{pmatrix} P_\lambda(\xi)$, for $h_\lambda(\xi)$, $P_\lambda(\xi)$ and $\widehat{\Psi}_{1,\lambda}$ as in Lemma 3.4. We have $B_\lambda(\xi) \in \mathcal{WC}^p_\omega$, for all

$p \in [1,+\infty]$. Define $\widehat{\Psi}(\xi) = \sum_{\lambda\in\Lambda} h_\lambda(\xi)B_\lambda(\xi)\widehat{\Phi}(\xi)$. One has $\Psi \in W^1_\omega$. For any $f \in V^p_\mu(\Phi)$, define $g(x) = \sum_{i=1}^{r}\sum_{j\in\mathbb{Z}^d} \langle f, \psi_i(x-j)\rangle \phi_i(x-j)$, $x \in \mathbb{R}^d$. Since $f \in V^p_\mu(\Phi)$, there exists a 2π-periodic distribution $A(\xi) \in \mathcal{WC}^p_\mu$ such that $\widehat{f}(\xi) = A(\xi)^T\widehat{\Phi}(\xi)$. By Lemma 3.4, we have $\widehat{g}(\xi) = \widehat{f}(\xi)$.

Since $\widehat{\Psi}(\xi) = \sum_{\lambda\in\Lambda} h_\lambda(\xi)B_\lambda(\xi)\widehat{\Phi}(\xi)$, for $f = \sum_{i=1}^{r}\sum_{j\in\mathbb{Z}^d} \langle f, \phi_i(\cdot-j)\rangle \psi(\cdot-j)$ the proof is similar.

ii) $\Rightarrow$ *iii)*. Let $k_0 = \min_{\xi\in\mathbb{R}^d} \operatorname{rank}\left[\widehat{\Phi}(\xi+2k\pi)_{k\in\mathbb{Z}^d}\right]$ and let

$$\Omega_{k_0} = \left\{\xi \in \mathbb{R}^d \mid \operatorname{rank}\left[\widehat{\Phi}(\xi+2k\pi)_{k\in\mathbb{Z}^d}\right] > k_0\right\}.$$

Then $\Omega_{k_0} \neq \mathbb{R}^d$. It is sufficient to prove that $\Omega_{k_0} = \emptyset$ (see Lemma 3.2). Suppose that $\Omega_{k_0} \neq \emptyset$. Since Ω_{k_0} is open set, then $\partial\Omega_{k_0} \neq \emptyset$ and $\operatorname{rank}\left[\widehat{\Phi}(\xi_0+2k\pi)\right]_{k\in\mathbb{Z}^d} = k_0$, for any $\xi_0 \in \partial\Omega_{k_0}$, and $\max_{\xi\in B(\xi_0,\delta)} \operatorname{rank}\left[\widehat{\Phi}(\xi+2k\pi)\right]_{k\in\mathbb{Z}^d} > k_0$, $\delta > 0$. By Lemma 3.4, there exist a nonsingular 2π-periodic $r\times r$ matrix $P_{\xi_0}(\xi)$ with all entries in the class $\mathcal{WC}^1_\omega$, $\delta_0 > 0$ and $K_0 \subset \mathbb{Z}^d$ with cardinality k_0. Define Ψ_{ξ_0}, $\widehat{\Psi}_{\xi_0}(\xi)$ as in Lemma 3.4. The construction of Ψ_{ξ_0} and (2.2) imply $\Psi_{\xi_0} \in W^1_\omega$. Choose n_0 such that $2^{-n_0} < \delta_0$ and define $\alpha_n(\xi)$, $H_{n,\xi_0}(\xi)$ and $\widetilde{H}_{n,\xi_0}(\xi)$ as in [2]. For any 2π-periodic distribution $F \in \mathcal{WC}^{p_0}_\mu$ define, g_n, for $n \geq n_0+1$, as in [2]. Note that $g_n \in V^{p_0}_\mu(\Phi)$ and $[\widehat{g}_n, \widehat{\Psi}_{1,\xi_0}](\xi) = 0$. This leads to

$$\|[\widehat{g}_n, \widehat{\Phi}](\xi)\|_{\ell^{p_0}_{\mu,*}} \leq C\|g_n\|_{L^{p_0}_\mu}\|\mathcal{F}^{-1}(H_{n,\xi_0}(\xi)\widehat{\Psi}_{2,\xi_0}(\xi))\|_{\mathcal{L}^\infty_\omega}.$$

Using Lemma 3.5, we obtain $\lim_{n\to+\infty}\|\mathcal{F}^{-1}(H_{n,\xi_0}(\xi)\widehat{\Psi}_{2,\xi0}(\xi))\|_{\mathcal{L}^\infty_\omega} = 0$. There exists a sequence ρ_n, $n \geq n_0$, such that $\|[\widehat{g}_n, \widehat{\Phi}](\xi)\|_{\ell^{p_0}_{\mu,*}} \leq \rho_n\|g_n\|_{L^{p_0}_\mu}$ and $\lim_{n\to+\infty}\rho_n = 0$. This, together with the assumption *ii)* and

$$\|[\widehat{g}_n, \widehat{\Phi}](\xi)\|_{\ell^{p_0}_{\mu,*}} = \left\|\left\{\int_{\mathbb{R}^d} g_n(\xi)\overline{\Phi(\xi-j)}dx\right\}_{j\in\mathbb{Z}^d}\right\|_{\ell^{p_0}_\mu} \geq C\|g_n\|_{L^{p_0}_\mu},$$

leads to $g_n = 0$, $n \geq n_0+1$. Then

$$\widetilde{H}_{n,\xi_0}(\xi)[\widehat{\Psi}_{1,\xi_0}, \widehat{\Psi}_{1,\xi_0}](\xi)(\alpha_n(\xi))^{-1}\widehat{\Psi}_{1,\xi_0}(\xi) = \widetilde{H}_{n,\xi_0}(\xi)\widehat{\Psi}_{2,\xi_0}(\xi), \tag{3.4}$$

for any 2π-periodic distribution $F \in \mathcal{WC}^{p_0}_\mu$ and $n \geq n_0+1$. We, also, get

$$\widetilde{H}_{n,\xi_0}(\xi)[\widehat{\Psi}_{1,\xi_0}, \widehat{\Psi}_{1,\xi_0}](\xi)(\alpha_n(\xi))^{-1}\widehat{\Psi}_{1,\xi_0}(\xi) = 0, \quad \xi \in B(\xi_0, 2^{-n_0-1}) + 2\pi\mathbb{Z}^d.$$

So, from (3.4) and the fact that it is valid for all $n \geq n_0+1$, we have $\widehat{\Psi}_{2,\xi_0}(\xi) = 0$, $\xi \in B(\xi_0, 2^{-n_0-3}) + 2\pi\mathbb{Z}^d$. This contradicts the fact that $\widehat{\Psi}_{2,\xi_0}(\xi) \neq 0$, $\forall\xi \in B(\xi_0,\delta) + 2\pi\mathbb{Z}^d$, $0 < \delta < 2\delta_0$.

With this we complete the proof *ii)* $\Rightarrow$ *iii)* and the proof of the theorem. $\square$

Remark 3.11. Note that conditions in Theorem 3.8 and Theorem 3.10 do not depend on $p \in [1, \infty]$, so we obtain the next corollary.

Corollary 3.12. *Let $\Phi \in (W^1_\omega)^r$ and $p_0 \in [1, \infty]$.*

i) If $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, 1 \le i \le r\}$ is a p_0-frame for $V^{p_0}_\mu(\Phi)$, then $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d, 1 \le i \le r\}$ is a p-frame for $V^p_\mu(\Phi)$, for any $p \in [1, \infty]$.

ii) If $V^{p_0}_\mu(\Phi)$ is closed in $L^{p_0}_\mu$ and $W^{p_0}_\mu$, then $V^p_\mu(\Phi)$ is closed in L^p_μ and W^p_μ, for any $p \in [1, \infty]$.

Remark 3.13. $(v) \Rightarrow (ii)$ implies that $\{\psi_i(\cdot - j) \mid 1 \le i \le r, j \in \mathbb{Z}^d\}$ is a dual p-frame of $\{\phi_i(\cdot - j) \mid 1 \le i \le r, j \in \mathbb{Z}^d\}$. So, the p-frame for $V^p_\mu(\Phi)$ is a Banach frame (with respect to ℓ^p_μ).

4 Connections with periodic distributions

We will use the notation V^p_s instead of $V^p_{(1+|x|^2)^{s/2}}$ (similarly for ℓ^p_s). Since ℓ^p_s and V^p_s are isomorphic Banach spaces for all $s \ge 0$ and $p \in [1, \infty]$, we have $V^p_{s_1}(\Phi) \subset V^p_{s_2}(\Phi)$ for $0 \le s_2 \le s_1$, $p \in [1, \infty]$. We define Fréchet spaces $X_{F,p}$, $p \in [1, \infty]$, as $X_{F,p} = \bigcap_{s \in \mathbb{N}_0} V^p_s(\Phi)$. Clearly, $X_{F,p}$ is dense in $V^p_s(\Phi)$ for all $s \in \mathbb{N}_0$. The corresponding sequence space is $Q_{F,p} = \bigcap_{s \in \mathbb{N}_0} \ell^p_s$, $p \in [1, \infty]$, which is the space of rapidly decreasing sequences s. By Corollary 3.12 it follows that the definition of $X_{F,p}$ does not depend on $p \in [1, \infty]$. So we use notation X_F, Q_F instead of $X_{F,p}$, $Q_{F,p}$. The set $\{\Phi(\cdot - k) \mid k \in \mathbb{Z}^d\}$ forms a F-frame for X_F since it forms a Banach frame for every space in the intersection (see [23] for the definition).

Since the corresponding function space for s is the space of rapidly decreasing functions $\mathcal{S} = \{f \mid \|f\|_m = \sup_{n \le m} (1 + |x|^2)^{m/2} |f^{(n)}(x)| < +\infty\}$, and its dual is $\mathcal{S}'$- the space of tempered distributions, we obtain that the dual space X'_F is isomorphic to (a complemented subspace of) $\mathcal{S}'$.

Denote by $\mathcal{P}(-\pi, \pi)$ the space of smooth 2π- periodic functions on $\mathbb{R}^d$ with the family of norms $|\theta|_k = \sup\{|\theta^{(k)}(t)|\,; t \in (-\pi, \pi)\}$, $k \in \mathbb{N}_0$. It is a Fréchet space and its dual is the space of 2π-periodic tempered distributions. We say that T is a 2π-periodic distribution if it is a tempered distribution on $\mathbb{R}^d$ and $T = T(\cdot + 2j\pi)$, for all $j \in \mathbb{Z}^d$. Denote by $\mathcal{P}'(-\pi, \pi)$ the space of periodic tempered distributions (see [24]). Recall that $\mathcal{F}(h) = \hat{h} = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi\sqrt{-1}t\cdot} h(t) \mathrm{d}t$ for $h \in L^1$.

Theorem 4.1. *Let $\Phi = (\phi_1, \dots, \phi_r)^T \in \bigcap_{s \ge 0} (W^1_s)^r$ and $\Psi = (\psi_1, \dots, \psi_r)^T$ be its dual frame (according to v) of Theorem 3.10). Then*

$$X_F = \mathcal{F}^{-1}\Big(\sum_{i=1}^r \hat{\phi}_i \cdot \mathcal{P}(-\pi, \pi)\Big), \qquad X'_F = \mathcal{F}^{-1}\Big(\sum_{i=1}^r \hat{\psi}_i \cdot \mathcal{P}'(-\pi, \pi)\Big)$$

in the topological sense. Let

$$f=\sum_{k=1}^{r}\sum_{p\in\mathbb{Z}^d}c_p^k\phi_k(\cdot-p)\in X_F \ \ \textit{and}\ \ F=\sum_{i=1}^{r}\sum_{j\in\mathbb{Z}^d}d_j^i\psi_i(\cdot-j)\in X_F'.$$

The dual pairing is given by

$$\langle F,f\rangle=\sum_{i=1}^{r}\sum_{k=1}^{r}\Big\langle\widehat{\psi}_i(\xi)\widehat{\phi}_k(-\xi)\sum_{j\in\mathbb{Z}^d}d_j^i\mathrm{e}^{2\pi j\xi\sqrt{-1}},\sum_{p\in\mathbb{Z}^d}c_p^k\mathrm{e}^{-2\pi p\xi\sqrt{-1}}\Big\rangle, \tag{4.1}$$

where $f=\sum\limits_{k=1}^{r}\sum\limits_{p\in\mathbb{Z}^d}c_p^k\phi_k(\cdot-p)\in X_F$ *and* $F=\sum\limits_{i=1}^{r}\sum\limits_{j\in\mathbb{Z}^d}d_j^i\psi_i(\cdot-j)\in X_F'$.
In particular, we have $\int\limits_{\mathbb{R}^d}\varphi_i\psi_k\mathrm{d}t=\int\limits_{\mathbb{R}^d}\widehat{\varphi}_i(\xi)\widehat{\psi_k}(-\xi)\mathrm{d}\xi=\delta_{ik}$, $1\le i,k\le r$.

Proof. Since $\sum\limits_{p\in\mathbb{Z}^d}c_p^k\mathrm{e}^{2\pi\sqrt{-1}p\xi}\in\mathcal{P}(-\pi,\pi)$, we obtain the structure of $f\in X_F$ as in the theorem. The same explanation works for X_F'.

By the fact that $\langle F(x),f(x)\rangle=\langle\widehat{F}(\xi),\widehat{f}(-\xi)\rangle$, we have that (4.1) follows.

Let $d_0^i=\delta_{ik}$, $i=1,\dots,r$, and $d_j^i=0$, $j\neq 0$, and, also, let $c_0^k=\delta_{ik}$ for $k=1,\dots,r$ and $c_p^k=0$, $p\neq 0$. Using that, we obtain

$$\langle F(\xi),f(\xi)\rangle=\sum_{i=1}^{r}\sum_{k=1}^{r}\langle\widehat{\psi}_i(\xi),\widehat{\phi}_k(-\xi)d_0^i,c_0^k\rangle=\int\limits_{\mathbb{R}^d}\widehat{\psi}_{k_0}(\xi)\widehat{\phi}_{k_0}(-\xi)\mathrm{d}\xi,\ 1\le k_0\le r.$$

On the other hand $f(x)=\langle f(x),\psi_{k_0}(x)\rangle\phi_{k_0}(x)$ and $f=\phi_{k_0}$ for some $1\le k_0\le r$, so we obtain $\langle f,\psi_{k_0}\rangle=1$. Since $F=\psi_{k_0}$, we get $\langle F,f\rangle=\langle f,\psi_{k_0}\rangle=1$. Finally, we have $\int\limits_{\mathbb{R}^d}\widehat{\varphi}_i(\xi)\widehat{\psi_k}(-\xi)\mathrm{d}\xi=\delta_{ik}$, $1\le i,k\le r$. □

Let $\beta\in(0,1)$. Now, we consider weights $\mu_k=\mathrm{e}^{k|x|^\beta}$, $k\in\mathbb{N}$, and the corresponding spaces $V_{\mu_k}^p(\Phi)$ and their intersection $X_{F,p}^{(\beta)}=\bigcap\limits_{k\in\mathbb{N}}V_{\mu_k}^p(\Phi)$. It is a Fréchet space not depending on p, so we use notation $X_F^{(\beta)}$. The corresponding sequence space is $s^{(\beta)}=\bigcap\limits_{k\in\mathbb{N}}\ell_{\mu_k}^p$, i.e., the space of subexponentially rapidly decreasing sequences determining the space of periodic tempered ultradistributions via the mapping $s^{(\beta)}\ni(a_j)_{j\in\mathbb{Z}^d}\leftrightarrow\sum\limits_{j\in\mathbb{Z}^d}a_j\mathrm{e}^{j\xi\sqrt{-1}}\in\mathcal{P}(-\pi,\pi)$ (see [22]).

5 Construction of p-frames

Let θ be a smooth non negative function such that $\theta(x)=1$, $x\in[-\pi+\varepsilon,\pi-\varepsilon]$, for $0<\varepsilon<\frac{1}{4}$, and $\operatorname{supp}\theta\subseteq[-\pi,\pi]$. Let $\phi_k(x)=\mathcal{F}^{-1}(\theta(\cdot+k\pi))(x)$, $x\in\mathbb{R}$,

$k \in \mathbb{Z}$. We can divide every $\theta(\cdot + k\pi)$ with the sum $\sum_{k\in\mathbb{Z}} \theta(\cdot + k\pi)$ in order to obtain the partition of unity. By the Paley-Wiener theorem, we have that $\phi_k \in W^1_\mu(\mathbb{R})$, $k \in \mathbb{Z}$. We say that set $\{\phi_{i_1}, \phi_{i_2}, \dots, \phi_{i_r}\}$, $i_1 < i_2 < \cdots < i_r$, is a set of r successive functions if $i_n = i_1 + (n-1)$, $n = 2, \dots, r$. Note that for every $\xi \in \mathbb{R}$ there exist $\xi_0 \in (-\pi, \pi)$ and $k \in \mathbb{Z}$ such that $\xi = \xi_0 + k\pi$.

Now, we consider the following three cases.

1° The case of two successive functions.

If $\Phi = (\phi_i, \phi_{i+1})^T$, $i \in \mathbb{Z}$, then $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j\in\mathbb{Z}}]$, $\xi \in \mathbb{R}$, is not a constant function on $\mathbb{R}$. In this case, for the matrix $[\widehat{\Phi}(\xi + 2j\pi)_{j\in\mathbb{Z}}]$, we obtain the $2 \times \infty$ matrix

$$A(\xi_0) = \begin{bmatrix} \cdots & 0 & \alpha_0^{\xi_0} & 0 & 0 \cdots \\ \cdots & 0 & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 \cdots \end{bmatrix},$$

which depends on $\xi_0 \in (-\pi, \pi)$, where $\alpha_{-1}^{\xi_0} = \theta(\xi_0 - \pi)$, $\alpha_0^{\xi_0} = \theta(\xi_0)$ and $\alpha_1^{\xi_0} = \theta(\xi_0 + \pi)$.

For $\xi_0^1 = \frac{\pi}{2}$, we have $\alpha_0^{\xi_0^1} \neq 0$, $\alpha_{-1}^{\xi_0^1} \neq 0$, and for $\xi_0^2 = -\frac{\pi}{2}$, we have $\alpha_0^{\xi_0^2} \neq 0$, $\alpha_1^{\xi_0^2} \neq 0$. Since $\operatorname{rank} A(\xi_0^1) = 1$ and $\operatorname{rank} A(\xi_0^2) = 2$, we conclude that for successive functions ϕ_i, ϕ_{i+1}, $i \in \mathbb{Z}$, the rank of the matrix $[\widehat{\Phi}(\xi + 2j\pi)_{j\in\mathbb{Z}}]$ is not a constant function on $\mathbb{R}$.

2° The case of three successive functions.

If $\Phi = (\phi_i, \phi_{i+1}, \phi_{i+2})^T$, $i \in \mathbb{Z}$, then $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j\in\mathbb{Z}}]$ is a constant function on $\mathbb{R}$. We have that $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j\in\mathbb{Z}}] = 2$, for all $\xi \in \mathbb{R}$.

Indeed, the matrix $[\widehat{\Phi}(\xi + 2j\pi)_{j\in\mathbb{Z}}]$, $\xi \in \mathbb{R}$, is $3 \times \infty$ matrix

$$B(\xi_0) = \begin{bmatrix} \cdots & 0 & \alpha_0^{\xi_0} & 0 & 0 & \cdots \\ \cdots & 0 & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 & \cdots \\ \cdots & 0 & 0 & \alpha_0^{\xi_0} & 0 & \cdots \end{bmatrix},$$

which depends on $\xi_0 \in (-\pi, \pi)$, where $\alpha_{-1}^{\xi_0} = \theta(\xi_0 - \pi)$, $\alpha_0^{\xi_0} = \theta(\xi_0)$ and $\alpha_1^{\xi_0} = \theta(\xi_0 + \pi)$. Since, $\theta(\xi_0) \neq 0$ for all $\xi_0 \in (-\pi, \pi)$, the matrix $B(\xi_0)$ has 2 columns with non-zero elements for all $\xi_0 \in (-\pi, \pi)$. So, $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j\in\mathbb{Z}}]$ is a constant function on $\mathbb{R}$ and $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j\in\mathbb{Z}}] = 2$, for all $\xi \in \mathbb{R}$.

3° The case of $r > 3$ successive functions.

By taking $r + 1$ successive functions $\phi_i, \phi_{i+1}, \dots, \phi_{i+r}$, $r > 2$, we have different situations described in the next lemma.

Lemma 5.1. *a) If $\Phi = (\phi_i, \phi_{i+1}, \dots, \phi_{i+r})^T$, for $i \in \mathbb{Z}$, $r \in 2\mathbb{N} + 1$, then* $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j\in\mathbb{Z}}]$ *is not a constant function on $\mathbb{R}$.*

b) If $\Phi = (\phi_i, \phi_{i+1}, \dots, \phi_{i+r})^T$, $i \in \mathbb{Z}$, $r \in 2\mathbb{N}$, then $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j\in\mathbb{Z}}]$ *is a constant function on $\mathbb{R}$ and we have, for all $\xi \in \mathbb{R}$, and $r = 2n$, $n \in \mathbb{N}$,* $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)_{j\in\mathbb{Z}}] = n + 1$.

Proof. Since supports of products $\widehat{\phi}_{i_1}(\xi + 2j_1\pi)\widehat{\phi}_{i_2}(\xi + 2j_2\pi)$ are non-empty if the arguments are of the form $\xi - \pi$, ξ, $\xi + \pi$, modulo $2j\pi$, $j \in \mathbb{Z}$, we have that

only blocks with elements

$$\begin{bmatrix} \theta(\xi) & \theta(\xi+2\pi) \\ \theta(\xi-\pi) & \theta(\xi+\pi) \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \theta(\xi-\pi) & \theta(\xi+\pi) \\ \theta(\xi-2\pi) & \theta(\xi) \end{bmatrix},$$

can determine the rank of the matrix $[\widehat{\Phi}(\xi+2j\pi)_{j\in\mathbb{Z}}]$. For any other choice of 2×2 matrix, we get determinant equal 0.

(a) Let $\Phi=(\phi_i,\phi_{i+1},\dots,\phi_{i+(2n-1)})^T$, $n\in\mathbb{N}$.

For the matrix $[\widehat{\Phi}(\xi+2j\pi)_{j\in\mathbb{Z}}]$, we obtain the $r\times\infty$ matrix

$$A_r(\xi_0)=\begin{bmatrix} \cdots & \alpha_0^{\xi_0} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & \alpha_0^{\xi_0} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \alpha_0^{\xi_0} & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots & \alpha_0^{\xi_0} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & \cdots \end{bmatrix},$$

where $\alpha_{-1}^{\xi_0}=\theta(\xi_0-\pi)$, $\alpha_0^{\xi_0}=\theta(\xi_0)$ and $\alpha_1^{\xi_0}=\theta(\xi_0+\pi)$, $\xi_0\in(-\pi,\pi)$.

For $\xi_0^1=\frac{\pi}{2}$, we have $\alpha_0^{\xi_0^1}\neq 0$, $\alpha_{-1}^{\xi_0^1}\neq 0$, and for $\xi_0^2=-\frac{\pi}{2}$, we obtain $\alpha_0^{\xi_0^2}\neq 0$, $\alpha_1^{\xi_0^2}\neq 0$. Since $\operatorname{rank} A_r(\xi_0^1)=n$ and $\operatorname{rank} A_r(\xi_0^2)=n+1$, we conclude that for even number of successive functions $\phi_i,\phi_{i+1},\dots,\phi_{i+(2n-1)}$, $i\in\mathbb{Z}$, $n\in\mathbb{N}$, the rank of the matrix $[\widehat{\Phi}(\xi+2j\pi)_{j\in\mathbb{Z}}]$ is not a constant function on $\mathbb{R}$.

(b) Let $\Phi=(\phi_i,\phi_{i+1},\dots,\phi_{i+2n})^T$, $i\in\mathbb{Z}$, $n\in\mathbb{N}$. The matrix

$$[\widehat{\Phi}(\xi+2j\pi)_{j\in\mathbb{Z}}]=\begin{bmatrix} \cdots & 0 & \alpha_0^{\xi_0} & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \alpha_0^{\xi_0} & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \alpha_{-1}^{\xi_0} & \alpha_1^{\xi_0} & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_0^{\xi_0} & \cdots \end{bmatrix},$$

has the constant rank on $\mathbb{R}$. Indeed, since $\alpha_0^{\xi_0}\neq 0$ for all $\xi_0\in(-\pi,\pi)$, the matrix $[\widehat{\Phi}(\xi+2j\pi)_{j\in\mathbb{Z}}]$ has $n+1$ columns with non-zero elements for all $\xi\in\mathbb{R}$ and $\operatorname{rank}[\widehat{\Phi}(\xi+2j\pi)_{j\in\mathbb{Z}}]=n+1$, for all $\xi\in\mathbb{R}$.

□

As a consequence of Corollary 3.10 and Lemma 5.1, 1° we have the next result.

Theorem 5.2. *Let $\Phi=(\phi_i,\phi_{i+1},\dots,\phi_{i+2n})^T$, for $i\in\mathbb{Z}$, $n\in\mathbb{N}$. Then $V_\mu^p(\Phi)$ is closed in L_μ^p, for any $p\in[1,\infty]$, and $\{\phi_{i+s}(\cdot-j)\mid j\in\mathbb{Z},\ 0\le s\le 2n\}$ is a p-frame for $V_\mu^p(\Phi)$ for any $p\in[1,\infty]$.*

Remark 5.3. In this way we obtain the sequence of closed spaces $V_\mu^p(\phi_0,\phi_1,\phi_2)$, $V_\mu^p(\phi_0,\phi_1,\phi_2,\phi_3,\phi_4)$, $V_\mu^p(\phi_0,\phi_2,\dots,\phi_6)$, etc. We also conclude that spaces generated with even numbers of successive functions, for example $V_\mu^p(\phi_0,\phi_1)$, $V_\mu^p(\phi_0,\phi_1,\dots,\phi_5)$, are not closed subspaces of L_μ^p.

Theorem 5.4. *Let $\Phi=(\phi_{k_1},\phi_{k_2},\dots,\phi_{k_r})^T$, $k_1<k_2<\cdots<k_r$, $r\in\mathbb{N}$, $k_1,k_2,\dots,k_r\in\mathbb{Z}$, and $V^p_{\mu,k_1,k_2,\dots,k_r}=V_\mu^p(\Phi)$. We consider the following cases.*

i) $k_{i+1}-k_i>1$, $i=1,\dots,r-1$;

ii) *If for some $i_0\in\{1,2,\dots,r\}$ holds $k_{i_0+1}-k_{i_0}=1$, then there exists $n\in\mathbb{N}$, $2\le 2n\le r$, such that $k_{i_0}+2$, $k_{i_0}+3,\dots$, $k_{i_0}+2n$ are elements of the set $\{k_1,\dots,k_r\}$.*

In these cases the following statements hold.

1° $\operatorname{rank}[\widehat{\Phi}(\xi+2j\pi)_{j\in\mathbb{Z}}]$ *is a constant function for all $\xi\in\mathbb{R}$.*

2° $V_\mu^p(\Phi)$ *is closed in L_μ^p for any $p\in[1,\infty]$.*

3° $\{\phi_{k_i}(\cdot-j)\mid j\in\mathbb{Z},1\le i\le r\}$ *is a p-frame for $V_\mu^p(\Phi)$ for any $p\in[1,\infty]$.*

Remark 5.5. (1) We refer to [4] and [26] for the γ-dense set $X=\{x_j\mid j\in J\}$. Let $\phi_k(x)=\mathcal{F}^{-1}(\theta(\cdot-k\pi))(x)$, $x\in\mathbb{R}$. Following the notation of [26], we put $\psi_{x_j}=\phi_{x_j}$ where $\{x_j\mid j\in J\}$ is γ-dense set determined by $f\in V^2(\phi)=V^2(\mathcal{F}^{-1}(\theta))$. Checking the proofs of Theorems 3.1, 3.2 and 4.1 in [26], we obtain the same conclusions as in these theorems. These theorems show the conditions and explicit C_p and c_p such that the inequality

$$c_p\|f\|_{L_\mu^p}\le\Big(\sum_{j\in J}|\langle f,\psi_{x_j}\rangle\mu(x_j)|^p\Big)^{1/p}\le C_p\|f\|_{L_\mu^p}$$

holds. This inequality guarantee the feasibility of a stable and continuous reconstruction algorithm in the signal spaces $V_\mu^p(\Phi)$.

(2) Since the spectrum of the Gram matrix $[\widehat{\Phi},\widehat{\Phi}](\xi)$, for Φ defined in Theorem 5.4, is bounded and bounded away from zero (see [8]), then the family $\{\Phi(\cdot-j)\mid j\in\mathbb{Z}\}$ forms a p-Riesz basis for $V_\mu^p(\Phi)$.

(3) For the appropriate choice of function Φ, for example Φ defined in Theorem 5.4, the associated Gram matrix satisfies a suitable Munckenhoupt A_2 condition (see [21]), so the system $\{\Phi(\cdot-j)\mid j\in\mathbb{Z}\}$ is stable in $L_\mu^2(\mathbb{R})$.

(4) Frames of the above type may be useful in applications since they satisfy assumptions of Theorem 3.1 and Theorem 3.2 in [5]. They show that error analysis for sampling and reconstruction can be tolerated, or that the sampling and reconstruction problem in shift-invariant space is robust with respect to appropriate set of functions $\phi_{k_1},\dots,\phi_{k_r}$.

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[1] Department of Mathematics and Informatics,
Faculty of Science,
University of Novi Sad,
Trg Dositeja Obradovica 4,
21000 Novi Sad,
Serbia
E-mail:stevan.pilipovic@dmi.uns.ac.rs

[2] Department of Mathematics and Informatics,
Faculty of Science,
University of Kragujevac,
Radoja Domanovića 12,
34000 Kragujevac,
Serbia
E-mail: suzanasimic@kg.ac.rs