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ON THE INTRINSIC DESZCZ SYMMETRIES AND THE EXTRINSIC CHEN CHARACTER OF WINTGEN IDEAL SUBMANIFOLDS

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Abstract. In this paper it is shown that all *Wintgen ideal submanifolds* in ambient real space forms are *Chen submanifolds*. It is also shown that the Wintgen ideal submanifolds of dimension > 3 in real space forms do intrinsically enjoy some *curvature symmetries in the sense of Deszcz of their Riemann–Christoffel curvature tensor, of their Ricci curvature tensor and of their Weyl conformal curvature tensor.*

1. Wintgen ideal submanifolds

Let M^n be an $n-$dimensional Riemannian submanifold of an $(n+m)$–dimensional real space form $\tilde{M}^{n+m}(c)$ of curvature c, $(n \geq 2, m \geq 1)$. Let g and ∇, and, respectively, $\tilde{g}$ and $\tilde{\nabla}$, denote the *Riemannian metrics* and the corresponding *Levi–Civita connections* of M^n and of $\tilde{M}^{n+m}(c)$. The *formulae of Gauss and Weingarten* are then given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{1}$$

and

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2}$$

whereby h, A_ξ and $\nabla^\perp$ denote the *second fundamental form*, the *shape operator* or *Weingarten map* with respect to ξ and the *normal connection* of M^n in $\tilde{M}^{n+m}(c)$, respectively, (X, Y, etc. stand for *tangent vector fields* and ξ etc. for *normal vector fields* on M^n in $\tilde{M}^{n+m}(c)$). From (1) and (2) it follows that

$$\tilde{g}(h(X,Y),\xi) = g(A_\xi(X), Y), \tag{3}$$

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such that, for any *orthonormal local normal frame* $\{\xi_\alpha\}$ on M^n in $\tilde{M}^{n+m}(c)$, $(\alpha, \beta, \ldots \in \{1, 2, \ldots, m\})$:

$$h(X,Y) = \sum_\alpha g(A_\alpha(X), Y)\xi_\alpha, \tag{4}$$

whereby $A_\alpha = A_{\xi_\alpha}$. The *mean curvature vector field* $\overrightarrow{H}$ of M^n in $\tilde{M}^{n+m}(c)$ is defined as $\overrightarrow{H} = \frac{1}{n}\, tr\ h = \frac{1}{n}\sum_{i=1}^n h(E_i, E_i)$, for any *orthonormal local tangent frame* $\{E_i\}$ on M^n, $(i, j, \ldots \in \{1, 2, \ldots, n\})$, and its length $H = \|\overrightarrow{H}\|$ is the *mean curvature* of M^n in $\tilde{M}^{n+m}(c)$.

Let R denote the $(0,4)$ *Riemann–Christoffel curvature tensor* of (M^n, g). Then, according to the *equation of Gauss*,

$$\begin{aligned} R(X,Y,Z,W) = {} & \tilde{g}(h(Y,Z), h(X,W)) - \tilde{g}(h(X,Z), h(Y,W)) \\ & +c\ \{g(Y,Z)\ g(X,W) - g(X,Z)\ g(Y,W)\}. \end{aligned} \tag{5}$$

Denoting by τ the *scalar curvature function* of (M^n, g), we have

$$\tau(p) := \sum_{i<j} K(p, E_i(p) \wedge E_j(p)), \tag{6}$$

whereby $K(p, E_i(p) \wedge E_j(p))$ denotes the sectional curvature of (M^n, g) at a point p of M^n for the plane section $\pi = E_i(p) \wedge E_j(p)$ in T_pM^n. By K_{inf} we will further denote the function $K_{inf} : M \to R : p \mapsto K_{inf}(p) :=$ *the minimal value of all sectional curvatures* of M at p.

The *normalised scalar curvature* ρ of the Riemannian manifold M^n is defined to be

$$\rho = \frac{2}{n(n-1)} \sum_{i<j} R(E_i, E_j, E_j, E_i). \tag{7}$$

By the *equation of Ricci*, the *normal curvature tensor* $R^\perp$ of M in $\tilde{M}$ is given as follows:

$$R^\perp(X, Y; \xi, \eta) := \tilde{g}(R^\perp(X,Y)\xi, \eta) = g([A_\xi, A_\eta]X, Y), \tag{8}$$

whereby $R^\perp(X,Y) := \nabla^\perp_X \nabla^\perp_Y - \nabla^\perp_Y \nabla^\perp_X - \nabla^\perp_{[X,Y]}$ and $[A_\xi, A_\eta] := A_\xi A_\eta - A_\eta A_\xi$.

The *normalised scalar normal curvature* $\rho^\perp$ of M in $\tilde{M}$ is then defined to be

$$\rho^\perp = \frac{2}{n(n-1)}\{\sum_{i<j}\sum_{\alpha<\beta}[R^\perp(E_i, E_j; \xi_\alpha, \xi_\beta)]^2\}^{\frac{1}{2}}. \tag{9}$$

We remark that $\rho^\perp = 0$ if and only if *the normal connection is flat*, which, as follows from (8) and as already observed by Cartan [2], is equivalent to the simultaneous diagonalisability of all shape operators A_ξ of M^n in $\tilde{M}^{n+m}$.

For surfaces M^2 in E^3, the *Euler inequality* $K \leq H^2$, whereby K is the intrinsic *Gauss curvature* of M^2 and H^2 is the extrinsic *squared mean curvature* of M^2 in E^3, at once follows from the fact that $K = k_1k_2$ and $H = \frac{1}{2}(k_1 + k_2)$ whereby k_1 and k_2 denote the *principal curvatures* of M^2 in E^3. And, obviously, $K = H^2$ everywhere on M^2 if and only if the surface M^2 is *totally umbilical* in E^3, i.e. $k_1 = k_2$ at all points of M^2, or still, by a *theorem of Meusnier*, if and only if M^2 is a part of a *plane* E^2 or of a *round sphere* S^2 in E^3. In the late 19 seventies, Wintgen proved that the *Gauss curvature* K and the *squared mean curvature* H^2 and the *normal curvature* $K^\perp$ of any surface M^2 in E^4 always satisfy the inequality

$$K \leq H^2 - K^\perp, \tag{10}$$

and that actually *the equality holds* if and only if the curvature ellipse of M^2 in E^4 is a circle [24]. We recall that the *ellipse of curvature* at a point p of M is defined as $\mathcal{E}_p = \{h(X,X) | X \in T_pM, \|X\| = 1\}$. This Wintgen inequality between the most important intrinsic and extrinsic scalar valued curvatures of surfaces M^2 in E^4 was shown to hold more generally for all surfaces M^2 in arbitrary dimensional space forms $\tilde{M}^{2+m}(c)$, inclusive the above characterisation of the equality case, by Rouxel in 1981 [18] and by Guadalupe and Rodriguez in 1983 [11]. After these extensions of Wintgen inequality (10), in 1999 De Smet, Dillen, Vrancken and one of the authors [6] proved the *Wintgen inequality $\rho \leq H^2 - \rho^\perp + c$ for all submanifolds M^n of codimension 2 in all real space forms $\tilde{M}^{n+2}(c)$* and characterised the equality as follows in terms of the shape operators.

Theorem A. *For any submanifold M^n of arbitrary dimension n and codimension 2 in a real space form $\tilde{M}^{n+2}(c)$ of curvature c, at every point p of M^n:*

$$\rho \leq H^2 - \rho^\perp + c, \tag{11}$$

and equality holds if and only if there exist orthonormal bases of the tangent space T_pM and the normal space $T_p^\perp M$ with respect to which the corresponding Weingarten maps are given by

$$A_1 = \begin{pmatrix} \lambda & \mu & 0 & \dots & 0 \\ \mu & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 & 0 & \dots & 0 \\ 0 & -\mu & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

for some $\lambda, \mu \in R$.

We remark that, in the case of trivial normal connection, (11) reduces to Chen's inequality $\rho \leq H^2 + c$ established in [3]. *The Wintgen inequality* (11) *was conjectured to hold for all submanifolds M^n in all real space forms* $\tilde{M}^{n+m}(c)$ in the same paper [6], and this is called "the DDVV conjecture" or "the conjecture on Wintgen's inequality". Recently, Choi and Lu [5] proved that this conjecture is true for all 3-dimensional submanifolds of arbitrary dimensional real space forms $\tilde{M}^{3+m}(c), (m \geq 2)$, and very recently, and independently, Lu [16] and Ge and Tang [10], settled this conjecture in general. From [10] we recall the final result in this respect.

Theorem B. *For any submanifold M^n of arbitrary dimension n, $n \geq 2$, and with arbitrary codimension m, $m \geq 2$ in a real space form $\tilde{M}^{n+m}(c)$ of curvature c, at every point p of M^n:*

$$\rho \leq H^2 - \rho^{\perp} + c, \tag{12}$$

and equality holds if and only if there exist orthonormal bases of the tangent space T_pM and the normal space $T_p^{\perp}M$ with respect to which the corresponding Weingarten maps are given by

$$A_1 = \begin{pmatrix} \lambda & \mu & 0 & \dots & 0 \\ \mu & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 & 0 & \dots & 0 \\ 0 & -\mu & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

for some $\lambda, \mu \in R$, and all other shape operators do vanish identically.

Submanifolds satisfying the equality in the Wintgen inequality (12) are called *Wintgen ideal submanifolds*. A justification for the terminology *"Wintgen ideal submanifolds"* M^n in $\tilde{M}^{n+m}(c)$ for those submanifolds M^n in $\tilde{M}^{n+m}(c)$ for which $\rho = H^2 - \rho^{\perp} + c$ holds at all points p of M^n, is as follows: for all possible isometric immersions of M^n in space forms $\tilde{M}^{n+m}(c)$, the value of the *intrinsic scalar curvature* ρ of M puts a lower bound to all possible values of the extrinsic curvature $H^2 - \rho^{\perp} + c$ that M in any case can not avoid to "undergo" as a submanifold in $\tilde{M}$. And, from this point of view, M is called a *Wintgen ideal submanifold*, when it actually is able to achieve a realisation in $\tilde{M}$ such that this extrinsic curvature indeed everywhere assumes its theoretically smallest possible value as given by its intrinsic normalised scalar curvature.

2. Deszcz symmetries of Wintgen ideal submanifolds

For a Riemannian manifold (M^n, g), let R also denote the $(1,1)$ *curvature operator* $R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$, besides the $(0,4)$ curvature tensor, such that, by definition

$$R(X,Y,Z,W) = g(R(X,Y)Z,W), \tag{13}$$

By the action of the curvature operator R *working as a derivation* on the curvature tensor R, the following $(0,6)$ tensor $R \cdot R$ is obtained:

$$\begin{aligned}(R \cdot R)(X_1, X_2, X_3, X_4; X, Y) &:= (R(X,Y) \cdot R)(X_1, X_2, X_3, X_4) \\ &= -R(R(X,Y)X_1, X_2, X_3, X_4) - R(X_1, R(X,Y)X_2, X_3, X_4) \\ &\quad -R(X_1, X_2, R(X,Y)X_3, X_4) - R(X_1, X_2, X_3, R(X,Y)X_4).\end{aligned} \tag{14}$$

It was recently shown by one of the authors and Haesen [12], that this tensor $R \cdot R$ can be geometrically interpreted as giving the second order measure of *the change of the sectional curvatures* $K(p, \pi)$ *for tangent* $2D$*–planes* π *at points* p *after the parallel transport of* π *all around infinitesimal co–ordinate parallelograms in* M *cornered at* p. Thus, according to [12], the *semi–symmetric* or *Szabó symmetric spaces* ([20] [21]), i.e. the spaces satisfying $R \cdot R = 0$, are the Riemannian manifolds (M^n, g) for which all *sectional curvatures remain preserved after parallel transport of their planes around all infinitesimal co–ordinate parallelograms in* M. The *locally symmetric* or *Cartan symmetric spaces*, i.e. the Riemannian manifolds (M^n, g) for which $\nabla R = 0$, constitute a *proper subclass* of the Szabó symmetric spaces. *Deszcz symmetric spaces* or *pseudo–symmetric spaces* ([7] [23]) are characterised by the fact that their $(0, 6)$ curvature tensor $R \cdot R$ is proportional to their $(0, 6)$ *Tachibana tensor* $Q(g, R) := - \wedge_g \cdot R$, whereby the metrical endomorphism $\wedge_g$ acts on the $(0, 4)$ tensor R as a derivation, i.e. by the fact that

$$R \cdot R = L\ Q(g, R), \tag{15}$$

for some function $L : M^n \to R$, (whenever $Q(g, R) \neq 0$). We recall that $Q(g, R) \equiv 0$ characterises the real space forms.

From [12] we further mention the following. Two 2-planes π and $\bar{\pi}$, spanned by vectors $\overrightarrow{u}, \overrightarrow{v}$ and $\overrightarrow{x}, \overrightarrow{y}$ respectively, at a same point p of M, are said to be *curvature dependent* if $Q(g, R)(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{v}, \overrightarrow{u}; \overrightarrow{x}, \overrightarrow{y}) \neq 0$, which condition is independent of the choices of bases for π and $\bar{\pi}$. For such planes, the *double sectional curvature* or the *sectional curvature of Deszcz* or the *Riemann curvature of Deszcz* $L(p, \pi, \bar{\pi})$ is defined as the real number given by

$$L(p, \pi, \bar{\pi}) := \frac{(R \cdot R)(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{v}, \overrightarrow{u}; \overrightarrow{x}, \overrightarrow{y})}{Q(g, R)(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{v}, \overrightarrow{u}; \overrightarrow{x}, \overrightarrow{y})}, \tag{16}$$

(which is independent of the choices of bases for π and $\bar{\pi}$); it is *a scalar valued Riemannian invariant*. The knowledge of the tensor $R \cdot R$ is *equivalent* to the knowledge of the sectional curvatures $L(p, \pi, \bar{\pi})$ of Deszcz. And just like the geometrical interpretation of the sectional curvatures $K(p, \pi)$ of Riemann in terms of the *parallelogramoïds of Levi-Civita* [15], also the sectional curvatures $L(p, \pi, \bar{\pi})$ of Deszcz can be interpreted in these terms (in this respect, we refer to [13] where in particular such interpretations are obtained for the sectional curvatures as well as for the *Ricci* and *conformal Weyl curvatures* of *Deszcz* in terms of the *squaroïds* of Levi–Civita). Finally the Deszcz symmetric spaces are characterised by the *isotropy* of the curvatures $L(p, \pi, \bar{\pi})$, i.e. by the property that at every point p of M the scalars $L(p, \pi, \bar{\pi})$ are the same for all possible pairs of curvature dependent tangent planes π and $\bar{\pi}$ at p. In the present situation however there is no lemma of Schur, which then would further force this real valued function $L : M \to R$ automatically to be constant; therefore, Kowalski and Sekizawa called the pseudo–symmetric spaces for which *the double sectional curvature* L *is* indeed a *constant*, independent of the planes π and $\bar{\pi}$ as well as of the points p of M, the *pseudo-symmetric spaces of constant type* L [14]. For instance, the standard models of the Thurston geometries [22] are the $3D$–prototypes of the Deszcz–symmetric spaces with *constant* L [1].

And similar studies concerning, in particular, the $(0,4)$ *Weyl conformal curvature tensor* C and the $(0,2)$ *Ricci tensor* S have been carried through in the mean time, (characterising the corresponding "Deszcz–symmetries" in terms of the isotropy of the corresponding scalar curvature functions which depend on two planes and on a plane and a direction, respectively). For a recent general exposition on conditions of Deszcz symmetry we refer to [8].

It was shown in [17] that *Wintgen ideal submanifolds* M^n of dimension $n > 3$ and with codimension 2 in real space forms $\tilde{M}^{n+2}(c)$ of curvature c intrinsically enjoy some *curvature symmetries in the sense of Deszcz*, i.e. *the Deszcz symmetries* of their Riemann–Christoffel curvature tensor R, of their Ricci curvature tensor S and of their conformal curvature tensor of Weyl C. Such *Wintgen ideal submanifolds* M^n of $\tilde{M}^{n+2}(c)$ are *Deszcz symmetric* if and only if M^n is *totally umbilical* in $\tilde{M}^{n+2}(c)$, in which case $L = 0$, or M^n is *minimal* in $\tilde{M}^{n+2}(c)$, in which case $L = c$. Moreover, it was proved in [9] that the *Deszcz symmetry*, or, equivalently, the property to be *quasi–Einstein*, for $3D$*–Wintgen ideal submanifolds* M^3 in $\tilde{M}^{3+m}(c)$, can be characterised in terms of the intrinsic minimal values of the *Ricci curvatures* of M and of the extrinsic notions of the *umbilicity*, the *minimality* and the *pseudo–umbilicity* of such M^3 in $\tilde{M}^{3+m}(c)$. Therefore, concerning the study of Deszcz symmetries of Wintgen ideal submanifolds only the situation of dimension $n > 3$ in case of arbitrary codimension m remains to be considered. But, in view of Theorems A and B, the proofs given in [17] obviously also hold for the general codimensions, so that accordingly we can announce the following general results.

Theorem 1. *A Wintgen ideal submanifold M^n of a real space form $\tilde{M}^{n+m}(c)$, $(n > 3, m \geq 2)$ is Deszcz symmetric, if and only if M^n is totally umbilical in $\tilde{M}^{n+m}(c)$, in which case $L = 0$, or M^n is minimal in $\tilde{M}^{n+m}(c)$, in which case $L = c$.*

Theorem 2. *A Wintgen ideal submanifold M^n of $\tilde{M}^{n+m}(c)$, $(n > 3, m \geq 2)$, is Deszcz Ricci–symmetric, i.e. satisfies $R \cdot S = L_S\, Q(g, S)$ for some function $L_S : M^n \to R$, if and only if M^n is Deszcz symmetric.*

Theorem 3. *Every Wintgen ideal submanifold M^n of $\tilde{M}^{n+m}(c)$, $(n > 3, m \geq 2)$, is a Riemannian manifold with pseudo–symmetric conformal Weyl tensor, i.e. satisfies $C \cdot C = L_C\, Q(g, C)$ for some function $L_C : M^n \to R$.*

Proposition 4. *A Wintgen ideal submanifold M^n of $\tilde{M}^{n+m}(c)$, $(n > 3, m \geq 2)$ is minimal if and only if the pseudo–symmetry function of its Weyl conformal curvature tensor is given by*

$$L_C = \frac{n-3}{(n-1)(n-2)}\,(c - K_{inf}).$$

3. Chen submanifolds

For submanifolds M^n of $\tilde{M}^{n+m}$ the notion of *allied vector field* of a given normal vector field of M^n is defined in [4] and, accordingly, for any submanifold M^n in $\tilde{M}^{n+m}$, for a local orthonormal frame $\{\xi_1 = \frac{\overrightarrow{H}}{\|\overrightarrow{H}\|}, \xi_2, \ldots, \xi_m\}$ whereby $\overrightarrow{H}$ is the *mean curvature vector field* of M^n in $\tilde{M}^{n+m}$,

$$a(\overrightarrow{H}) = \frac{1}{n}\sum_{\beta=2}^{m} tr(A_1\, A_\beta)\xi_\beta, \tag{17}$$

is the *allied vector field of* $\overrightarrow{H}$ or *allied mean curvature vector field* of M^n in $\tilde{M}^{n+m}$. A submanifold M^n is called an *$\mathcal{A}$–submanifold* or a *Chen submanifold* if the allied mean curvature vector field of M^n identically vanishes, $a(\overrightarrow{H}) \equiv \overrightarrow{0}$. By a result of B. Rouxel [19], a submanifold M^n of $\tilde{M}^{n+m}$ is a Chen submanifold if and only if the mean curvature vector at any point p of M, $\overrightarrow{H}(p)$, is an axis of symmetry of the $(m-2)$–nd polar of its *Kommerell hyperquadric curvature image* in the normal space $T_p^{\perp}M$. Minimal submanifolds, pseudo–umbilical submanifolds and hypersurfaces are *Chen submanifolds* in a trivial way.

For *Wintgen ideal submanifolds* M^n in real space forms $\tilde{M}^{n+m}(c)$, from the specific forms of the shape operators of these submanifolds given in Theorem B, we have

$$A_1\; A_2 = \begin{pmatrix} \lambda\mu & -\mu^2 & 0 & \ldots & 0 \\ \mu^2 & -\lambda\mu & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad A_1\; A_\gamma = 0,\ (\gamma \in \{3, \ldots, m\}),$$

such that their allied mean curvature vector field $a(\overrightarrow{H})$ clearly always is identically zero. This yields the following.

Theorem 5. *Every Wintgen ideal submanifold M^n of arbitrary dimension $n \geq 2$ and codimension $m \geq 2$ in a real space form $\tilde{M}^{n+m}(c)$, is a Chen submanifold of $\tilde{M}^{n+m}(c)$.*

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