J. Math. Anal. Appl. 367 (2010) 705–709

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Note

A few remarks on divergent sequences: Rates of divergence II

D. Djurčić [a,1], Lj.D.R. Kočinac [b,*,1], M.R. Žižović [a,1]

[a] Technical Faculty, University of Kragujevac, Svetog Save 65, 32000 Čačak, Serbia
[b] Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia

ARTICLE INFO

Article history:
Received 19 October 2009
Available online 29 December 2009
Submitted by U. Stadtmueller

Keywords:
Regular variation
Quotient speed of divergence
Selection principles
Game theory

ABSTRACT

We continue (and, in a sense, close) our investigation begun in the paper Djurčić et al. (2009) [10] concerning quotient speed of divergence of sequences of positive real numbers and its relations with selection principles and games. By the way, we show a theorem of the generalized Galambos–Bojanić–Seneta type.

1. Introduction

In several recent papers, the authors found out nice relations between theory of divergent processes in asymptotic analysis and theory of selection principles and games (see [6,8–10]). This note can be viewed as an addendum to the paper [10], because here we extend and improve the following result ([10, Theorem 3.11]; for the definitions see the next section):

Theorem 1.1. *The following selection properties are equivalent and all are satisfied*:

(1) $\mathsf{S}_1(\mathsf{ARV}_\mathsf{s}, \mathsf{Tr}^{(2)}(\mathsf{R}_{\infty,\mathsf{s}}))$;
(2) $\alpha_2(\mathsf{ARV}_\mathsf{s}, \mathsf{Tr}^{(2)}(\mathsf{R}_{\infty,\mathsf{s}}))$;
(3) $\alpha_3(\mathsf{ARV}_\mathsf{s}, \mathsf{Tr}^{(2)}(\mathsf{R}_{\infty,\mathsf{s}}))$;
(4) $\alpha_4(\mathsf{ARV}_\mathsf{s}, \mathsf{Tr}^{(2)}(\mathsf{R}_{\infty,\mathsf{s}}))$.

In fact, the above theorem is valid for any fixed $k \in \mathbb{N}$. The methodology we used in the proof of this result did not allow us to show that the second coordinate here can be replaced by the smaller class $\mathsf{Tr}^{(\infty)}(\mathsf{R}_{\infty,\mathsf{s}})$ introduced also in [10]. One of main aims of this note is to show that it is possible. Even more: the first coordinate also can be replaced by a wider class PI^*_s of sequences considered in [3] and [11].

We also show a result of the generalized Galambos–Bojanić–Seneta type; such theorems describe the asymptotic behavior of a sequence by the asymptotic behavior of a corresponding real function.

---

[*] Corresponding author.
E-mail addresses: dragandj@tfc.kg.ac.rs (D. Djurčić), lkocinac@ptt.rs (Lj.D.R. Kočinac), zizo@tfc.kg.ac.rs (M.R. Žižović).
[1] Supported by MN RS.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.12.036

2. Definitions and preliminaries

We introduce the notions which will be used in this note.

2.1. Regular variation

A measurable function $\varphi : [a, \infty) \to (0, \infty)$, $a > 0$, is said to be *$\mathcal{O}$-regularly varying* (in the sense of Karamata) [1,4] if for each $\lambda > 0$ it satisfies

$$\underline{k}_{\varphi}(\lambda) := \liminf_{x \to \infty} \frac{\varphi(\lambda x)}{\varphi(x)} > 0. \tag{1}$$

The set of all these functions will be denoted by $\mathsf{ORV_f}$.

A measurable function $\varphi : [a, \infty) \to (0, \infty)$, $a > 0$, is said to be in the *class* PI_f^* if there exists $\lambda_0 \geqslant 1$ such that for each $\lambda > \lambda_0$

$$\underline{k}_{\varphi}(\lambda) := \liminf_{x \to \infty} \frac{\varphi(\lambda x)}{\varphi(x)} > 1. \tag{2}$$

A sequence $x = (x_n)_{n \in \mathbb{N}}$ of positive real numbers is said to belong to the *class* PI_s^* [3,11] if there is $\lambda_0 \geqslant 1$ such that for each $\lambda > \lambda_0$ it holds

$$\underline{k}_{x}(\lambda) := \liminf_{n \to \infty} \frac{x_{[\lambda n]}}{x_n} > 1. \tag{3}$$

Note that for $\lambda_0 = 1$ we have the well-known class $\mathsf{ARV_s}$ of sequences (see [7–10]), and it holds

$$\mathsf{R}_{\infty,\mathsf{s}} \underset{\neq}{\subseteq} \mathsf{ARV_s} \underset{\neq}{\subseteq} \mathsf{PI}_\mathsf{s}^*,$$

where $\mathsf{R}_{\infty,\mathsf{s}}$ denotes the class of all rapidly varying sequences in the sense of de Haan (see, for instance, [6]).

Let $x = (x_n)_{n \in \mathbb{N}}$ be a strictly increasing, unbounded sequence of positive real numbers. The *numerical function of x* is the function $\delta_x : [x_1, \infty) \to \mathbb{N}$ defined by

$$\delta_x(t) = \max\{n \in \mathbb{N}\colon\ x_n \leqslant t\}.$$

The numerical function of a sequence is an important characteristic of divergent sequences (see [18]).

2.2. Selection principles and games

For more details about selection principles we refer the reader to [19].

Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of the set $\mathbb{S}$ of all sequences of positive real numbers. Then $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle: For each sequence $(A_n\colon\ n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n\colon\ n \in \mathbb{N})$ such that for each n, $b_n \in A_n$ and $\{b_n\colon\ n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

The symbol $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, who play a round for each positive integer. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \ldots; A_n, b_n; \ldots)$ if $\{b_n\colon\ n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

It is evident that if TWO has (even if ONE does not have) a winning strategy in the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$, then the selection hypothesis $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is true. The converse implication is not always true.

Let us mention also that a strategy for a player TWO is a *coding strategy* if TWO remembers only the most recent move by ONE and by TWO before his next move, i.e. if the moves of TWO are: $b_1 = \sigma(A_1, \emptyset)$; $b_n = \sigma(A_n, b_{n-1})$, $n \geqslant 2$.

The symbol $\alpha_i(\mathcal{A}, \mathcal{B})$, $i = 1, 2, 3, 4$, denotes the following selection hypothesis (see [20]). For each sequence $(A_n\colon\ n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that:

$\alpha_1(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \setminus B$ is finite;
$\alpha_2(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
$\alpha_3(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is infinite;
$\alpha_4(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$ the set $A_n \cap B$ is nonempty.

2.3. Quotient speed of divergence

In this subsection we mention a consideration from [10].

Let $x=(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{S}$. For each $k\in\mathbb{N}$ define a new sequence $V^{(k)}(x)=(V_n^{(k)}(x))_{n\in\mathbb{N}}$ inductively by

$$V_n^{(1)}(x):=\frac{x_{n+1}}{x_n},\quad n\in\mathbb{N};$$

$$V_n^{(k+1)}(x):=\frac{V_{n+1}^{(k)}(x)}{V_n^{(k)}(x)},\quad n\in\mathbb{N}.$$

The sequence $V^{(k)}(x)$ we call the *quotient sequence of x of order k*. We also put $V^{(0)}(x)=x$.

Given a natural number k, the k-th quotient speed of a sequence $x=(x_n)_{n\in\mathbb{N}}$ is ∞, denoted $v_k(x)=\infty$, if

$$\lim_{n\to\infty}V_n^{(k)}(x)=\infty.$$

We also defined for each $k\in\mathbb{N}$

$$\mathsf{Tr}^{(\mathsf{k})}(\mathsf{R}_{\infty,\mathsf{s}})=\{x\in\mathbb{S}\colon\ v_k(x)=\infty\}$$

and

$$\mathsf{Tr}^{(\infty)}(\mathsf{R}_{\infty,\mathsf{s}})=\bigcap_{k=1}^{\infty}\mathsf{Tr}^{(\mathsf{k})}(\mathsf{R}_{\infty,\mathsf{s}}).$$

Then

$$\mathsf{Tr}^{(\infty)}(\mathsf{R}_{\infty,\mathsf{s}})\subsetneqq\cdots\subsetneqq\mathsf{Tr}^{(\mathsf{k})}(\mathsf{R}_{\infty,\mathsf{s}})\subsetneqq\cdots\subsetneqq\mathsf{Tr}^{(2)}(\mathsf{R}_{\infty,\mathsf{s}})\subsetneqq\mathsf{Tr}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}})\subsetneqq\mathsf{R}_{\infty,s}.$$

Notice that the class $\mathsf{Tr}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}})$ coincides with the class $\mathsf{Tr}(\mathsf{R}_{\infty,\mathsf{s}})$ of translationally rapidly varying sequences (see [9,21]).

3. Results

The following result of the Galambos–Bojanić–Seneta type ([2,17]; see also [5,6,13,14,16] for similar results) explains the importance of the class PI_s^* of sequences in the theory of qualitative analysis of divergent processes and in asymptotic analysis in general.

Theorem 3.1. *Let $x=(x_n)_{n\in\mathbb{N}}$ be a strictly increasing, unbounded sequence of positive real numbers. Then the following are equivalent*:

(a) *x belongs to the class PI_s^**;
(b) *the numerical function δ_x of x belongs to the class ORV_f*.

Proof. (a) ⇒ (b): Let $(x_n)_{n\in\mathbb{N}}\in\mathsf{PI}_\mathsf{s}^*$. There is $\lambda_0\geqslant 1$ such that $\underline{k}_x(\lambda)>1$ for each $\lambda>\lambda_0$. Also, there exists a non-trivial segment $[A,B]\subsetneqq(\lambda_0,\lambda_0^2)$ so that: there is $n_0\in\mathbb{N}$ with $\frac{x_{[\lambda n]}}{x_n}>1$ for all $n\geqslant n_0$ and each $\lambda\in[A,B]$. Then for each such $\lambda>\lambda_0^3$ and sufficiently large $t\geqslant t_0\geqslant 1$ we have

$$\frac{x_{[\lambda t]}}{x_{[t]}}=\frac{x_{[z[\eta[t]]]}}{x_{[\eta[t]]}}\cdot\frac{x_{[\eta[t]]}}{x_{[t]}},$$

where $z=z(t)\in[A,B]$ and $\eta=\frac{2\lambda}{A+B}$. Since $\eta>\lambda_0$, it follows

$$\liminf_{t\to\infty}\frac{x_{[\lambda t]}}{x_{[t]}}\geqslant\underline{k}_x(\eta)>1.$$

This just means that the function $x_{[t]}$, $t\geqslant 1$, belongs to the class PI_f^*; it is non-decreasing and unbounded. Its generalized inverse (see [1,15]) $(x_{[t]})^{\leftarrow}$, $t\geqslant x_1$, belongs to the class ORV_f (see [11]). But, since $(x_{[t]})^{\leftarrow}$ is strongly asymptotically equivalent to $\delta_x(t)$ for $t\to\infty$ [12], one has $\delta_x\in\mathsf{ORV}_\mathsf{f}$.

(b) ⇒ (a): Suppose now $\delta_x\in\mathsf{ORV}_\mathsf{f}$. But, according to [16] δ_x is strongly asymptotically equivalent to $(x_{[t]})^{\leftarrow}$ (for $t\to\infty$), so that $(x_{[t]})^{\leftarrow}\in\mathsf{ORV}_\mathsf{f}$. By a result from [11], $x_{[t]}$ belongs to the class PI_f^*. Therefore, $x\in\mathsf{PI}_\mathsf{s}^*$. □

As we mentioned in the Introduction the technique we used in the proof of Theorem 1.1 were not applicable to show that in that theorem $\mathsf{Tr}^{(2)}(\mathsf{R}_{\infty,\mathsf{s}})$ can be replaced by $\mathsf{Tr}^{(\infty)}(\mathsf{R}_{\infty,\mathsf{s}})$.

Now we define a new class of sequences which will help us to obtain an improvement of Theorem 1.1 changing both coordinates in it (see Theorem 3.4 below and its corollaries).

Definition 3.2. A sequence $x=(x_n)_{n\in\mathbb{N}}\in\mathbb{S}$ belongs to the *class* $\mathsf{Tr}_\omega^{(1)}(\mathsf{R}_{\infty,\mathsf{s}})$ if there is $b>1$ so that $\log_b x=(\log_b x_n)_{n\in\mathbb{N}}$ belongs to the class $\mathsf{Tr}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}})$.

Proposition 3.3. $\mathsf{Tr}_{\omega}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}}) \subsetneqq \mathsf{Tr}^{(\infty)}(\mathsf{R}_{\infty,\mathsf{s}})$.

Proof. Let $x = (x_n)_{n\in\mathbb{N}} \in \mathsf{Tr}_{\omega}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}})$. There is $b > 1$ such that $y := \log_b x = (\log_b x_n)_{n\in\mathbb{N}} \in \mathsf{Tr}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}})$, which means that

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \infty \quad \text{and} \quad \lim_{n\to\infty} y_n = \infty.$$

By mathematical induction one can prove that for any fixed $k \in \mathbb{N}$ we have

$$V_n^{(k)}(x) = b^{y_{n+k} - C_1^k y_{n+k-1} + C_2^k y_{n+k-2} - \cdots + (-1)^k y_n}$$

(where, as usually, $C_m^k = \binom{k}{m}$, $k, m \in \mathbb{N}$) and thus

$$v_k(x) = \lim_{n\to\infty} V_n^{(k)}(x) = \lim_{n\to\infty} b^{y_{n+k-1}\left(\frac{y_{n+k}}{y_{n+k-1}} - C_1^k + C_2^k \frac{y_{n+k-2}}{y_{n+k-1}} - \cdots + (-1)^k \frac{y_n}{y_{n+k-1}}\right)} = \infty.$$

This means that for each $k \in \mathbb{N}$ the sequence x belongs to the class $\mathsf{Tr}^{(\mathsf{k})}(\mathsf{R}_{\infty,\mathsf{s}})$ and consequently $x \in \mathsf{Tr}^{(\infty)}(\mathsf{R}_{\infty,\mathsf{s}})$. So, $\mathsf{Tr}_{\omega}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}}) \subset \mathsf{Tr}^{(\infty)}(\mathsf{R}_{\infty,\mathsf{s}})$.

It is easy to verify that the sequence x defined by $x_n = b^{b^n}$ ($b > 1$, $n \in \mathbb{N}$), belongs to $\mathsf{Tr}^{(\infty)}(\mathsf{R}_{\infty,\mathsf{s}})$ and does not belong to $\mathsf{Tr}_{\omega}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}})$. $\square$

Theorem 3.4. *The player* TWO *has a winning coding strategy in the game* $\mathsf{G}_1(\mathsf{PI}_{\mathsf{s}}^*, \mathsf{Tr}_{\omega}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}}))$.

Proof. A strategy σ for TWO will be defined in the following way. Suppose that the first move of ONE is the sequence $x_1 = (x_{1,m})_{m\in\mathbb{N}}$ from PI_{ω}^*. TWO responds by choosing an arbitrary $y_1 = x_{1,m_1} \in x_1$; let $\sigma(x_1) = y_1$. Let in the second round ONE choose $x_2 \in \mathsf{PI}_{\mathsf{s}}^*$. TWO argues as follows. There is $\lambda_0 \geqslant 1$ such that for each $\lambda > \lambda_0$ one can find $m_0 \in \mathbb{N}$ with $x_{2,[\lambda m]} \geqslant c(\lambda) \cdot x_{2,m}$ for some $c(\lambda) > 1$ and each $m \geqslant m_0$. TWO considers $i = [\lambda_0] + 1$; then $x_{2,m_0} < c(i) x_{2,m_0} \leqslant x_{2,i m_0}$ and also $x_{2,i^k m_0} \geqslant (c(i))^k \cdot x_{2,m_0}$ for each $k \in \mathbb{N}$. This means that TWO finds out a subsequence of x_2, namely $(x_{2,i^k m_0})_{k\in\mathbb{N}}$, which is unbounded. TWO responds by taking $\sigma(x_2, y_1) = y_2 = x_{2,m_2}$ such that $y_2 \geqslant (y_1)^2$. If in the third round ONE has played $x_3 = (x_{3,m})_{m\in\mathbb{N}}$ from $\mathsf{PI}_{\mathsf{s}}^*$, TWO argues as in the second round and responds by choosing $\sigma(x_3, y_2) = y_3 = x_{3,m_3}$ in x_3 such that $y_3 \geqslant (y_2)^3$. And so on.

It is easy to check that the sequence $(y_n)_{n\in\mathbb{N}}$ belongs to $\mathsf{Tr}_{\omega}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}})$. $\square$

Corollary 3.5. *The selection principle* $\mathsf{S}_1(\mathsf{PI}_{\mathsf{s}}^*, \mathsf{Tr}_{\omega}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}}))$ *is satisfied.*

Corollary 3.6. *The selection principle* $\mathsf{S}_1(\mathsf{PI}_{\mathsf{s}}^*, \mathsf{Tr}^{(\infty)}(\mathsf{R}_{\infty,\mathsf{s}}))$ *is satisfied.*

Remark 3.7. Let us emphasize that using the idea in the proof of [10, Theorem 3.11] and the argumentation similar to the argumentation by TWO in the proof of Theorem 3.4 (regarding the existence of an unbounded subsequence in a sequence from $\mathsf{PI}_{\mathsf{s}}^*$) one can prove that the selection properties

(i) $\mathsf{S}_1(\mathsf{PI}_{\mathsf{s}}^*, \mathsf{Tr}_{\omega}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}}))$,
(ii) $\alpha_2(\mathsf{PI}_{\mathsf{s}}^*, \mathsf{Tr}_{\omega}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}}))$,
(iii) $\alpha_3(\mathsf{PI}_{\mathsf{s}}^*, \mathsf{Tr}_{\omega}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}}))$,
(iv) $\alpha_4(\mathsf{PI}_{\mathsf{s}}^*, \mathsf{Tr}_{\omega}^{(1)}(\mathsf{R}_{\infty,\mathsf{s}}))$

are equivalent and all are satisfied.

References

[1] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, Cambridge University Press, Cambridge, 1987.
[2] R. Bojanić, E. Seneta, A unified theory of regularly varying sequences, Math. Z. 134 (1973) 91–106.
[3] V.V. Buldygin, O.I. Klesov, J.G. Steinebach, On some properties of asymptotically quasi-inverse functions, Theory Probab. Math. Statist. 77 (2008) 15–30.
[4] D. Djurčić, $\mathcal{O}$-regularly varying functions and strong asymptotic equivalence, J. Math. Anal. Appl. 220 (1998) 451–461.
[5] D. Djurčić, V. Božin, A proof of a S. Aljančić hypothesis on $\mathcal{O}$-regularly varying sequences, Publ. Inst. Math. (Beograd) 62 (76) (1997) 46–52.
[6] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, Some properties of rapidly varying sequences, J. Math. Anal. Appl. 327 (2007) 1297–1306.
[7] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, On selection principles and games in divergent processes, in: Lj.D.R. Kočinac (Ed.), Selection Principles and Covering Properties in Topology, in: Quad. Mat., vol. 18, Dipartimento di Matematica, Seconda Universita di Napoli, Caserta, 2006, pp. 133–155.
[8] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, Rapidly varying sequences and rapid convergence, Topology Appl. 155 (2008) 2143–2149.
[9] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, Classes of sequences of real numbers, games and selection properties, Topology Appl. 156 (2008) 46–55.
[10] D. Djurčić, Lj.D.R. Kočinac, M.R. Žižović, A few remarks on divergent sequences: rates of divergence, J. Math. Anal. Appl. 360 (2009) 588–598.
[11] D. Djurčić, R. Nikolić, A. Torgašev, The weak asymptotic equivalence and the generalized inverse, Lithuanian Math. J., in press.

[12] D. Djurčić, A. Torgašev, Strong asymptotic equivalence and inversion of functions in the class K_c, J. Math. Anal. Appl. 255 (2001) 383–390.

[13] D. Djurčić, A. Torgašev, Representation theorems for the sequences of the classes CR_c and ER_c, Siberian Math. J. 45 (2004) 834–838.

[14] D. Djurčić, A. Torgašev, On the Seneta sequences, Acta Math. Sin. (Engl. Ser.) 22 (2006) 689–692.

[15] D. Djurčić, A. Torgašev, Some asymptotic relations for the generalized inverse, J. Math. Anal. Appl. 335 (2007) 1397–1402.

[16] D. Djurčić, A. Torgašev, A theorem of Galambos–Bojanić–Seneta type, Abstr. Appl. Anal. 2009 (2009), doi:10.1155/2009/360794, Article ID 360794, 6 pp.

[17] J. Galambos, E. Seneta, Regularly varying sequences, Proc. Amer. Math. Soc. 41 (1973) 110–116.

[18] J. Karamata, Theory and Practice of the Stieltjes Integral, vol. CLIV, Serbian Academy of Sciences and Arts, Institute of Mathematics, Belgrade, 1949 (in Serbian).

[19] Lj.D.R. Kočinac, Selected results on selection principles, in: Proc. 3rd Seminar Geometry and Topology, Tabriz, Iran, July 15–17, 2004, pp. 71–104.

[20] Lj.D.R. Kočinac, Selection principles related to α_i-properties, Taiwanese J. Math. 12 (2008) 561–571.

[21] M. Tasković, Fundamental facts on translational $\mathcal{O}$-regularly varying functions, Math. Morav. 7 (2003) 107–152.