J. Math. Anal. Appl. 461 (2018) 219–233

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the Bishop frames of pseudo null and null Cartan curves in Minkowski 3-space

Milica Grbović, Emilija Nešović *

University of Kragujevac, Faculty of Science, Department of Mathematics and Informatics, Radoja Domanovića 12, 34000 Kragujevac, Serbia

A R T I C L E I N F O

Article history:
Received 7 February 2017
Available online 11 January 2018
Submitted by H.R. Parks

Dedicated to the memory of Acad. Prof. Dr. Mileva Prvanović

Keywords:
Bishop frame
Frenet frame
Cartan frame
Pseudo null curve
Null Cartan curve

A B S T R A C T

In this paper, we derive the Bishop frames of the pseudo null curve and show that its normal Bishop vectors N_1 and N_2 can be obtained by applying the hyperbolic rotation and the composition of three rotations about two lightlike and one spacelike axis to the Frenet vectors N and B respectively. We also derive Bishop frame of the null Cartan curve and show that among all null Cartan curves in $\mathbb{E}_1^3$, only the null Cartan cubic has two Bishop frames, one of which coincides with its Cartan frame. As an application, we obtain some solutions of the Da Rios vortex filament equation in terms of the Bishop frames of the pseudo null curves and null Cartan cubic.

© 2018 Elsevier Inc. All rights reserved.

1. Introduction

The *Bishop frame* or *relatively parallel adapted frame* $\{T, N_1, N_2\}$ of a regular curve in Euclidean 3-space is introduced by R.L. Bishop in [3]. It contains the tangential vector field T and two normal vector fields N_1 and N_2, which can be obtained by rotating the Frenet vectors N and B in the normal plane $T^\perp$ of the curve, in such a way that they become *relatively parallel*. This means that their derivatives N_1' and N_2' with respect to the arc-length parameter s of the curve are collinear with the tangential vector field T. Hence N_1' and N_2' make minimal rotation in the planes $N_1^\perp$ and $N_2^\perp$, respectively. For this reason, the Bishop frame is also known as the frame with *minimal rotation property*. Such frame is well defined even in the points where the Frenet curvature κ of the curve vanishes, which is not the case with the Frenet frame. The Frenet equations according to the Bishop frame in $\mathbb{E}^3$ have the form ([3])

* Corresponding author.
E-mail addresses: milica.grbovic@kg.ac.rs (M. Grbović), nesovickg@sbb.rs (E. Nešović).

https://doi.org/10.1016/j.jmaa.2018.01.014
0022-247X/© 2018 Elsevier Inc. All rights reserved.

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$

where $\kappa_1(s) = \kappa(s)\cos\theta(s)$ and $\kappa_2(s) = \kappa(s)\sin\theta(s)$ are the Bishop curvatures. A new version of the Bishop frame (type-2 Bishop frame) in $\mathbb{E}^3$ is introduced in [24]. For the parallel transport frame in the Euclidean 4-space, see [12].

In Minkowski spaces $\mathbb{E}^3_1$ and $\mathbb{E}^4_1$, the Bishop frames of the timelike and the spacelike curves are obtained in [20] and [9]. Some applications of the Bishop frames in Minkowski spaces can be found in [5], [6], [23].

In Minkowski 3-space, the pseudo null and null Cartan curve are known as the curves whose the Frenet and Cartan frame respectively contains two null (lightlike) vector fields. The Bishop frames of such curves are not derived jet. In this paper, we obtain the Bishop frames of the pseudo null curve and show that its normal Bishop vectors N_1 and N_2 can be obtained by applying the hyperbolic rotation and the composition of three rotations about two lightlike and one spacelike axis to its Frenet vectors N and B, respectively. We also derive Bishop frame of the null Cartan curve and show that among all null Cartan curves in $\mathbb{E}^3_1$, only the null Cartan cubic has two Bishop frames, one of which coincides with its Cartan frame.

The Bishop frames of the pseudo null curves and null Cartan curves can be used in many applications in differential geometry of curves and surfaces in Minkowski spaces, such as in characterizations of Bertrand ([1]), Mannheim ([13]) and Smarandache curves ([16]), for deriving Bäcklund transformations ([14,15]), in classifications of k-type Darboux helices ([18]), etc.

In [7] the Da Rios has shown that the velocity $v(s,t)$ of a vortex filament $x(s,t)$ regarded as a space curve in $\mathbb{E}^3$ parameterized by the arc-length parameter s for all time t is given by the *vortex filament equation* $v = x_t = x_s \times x_{ss}$. It is known that an evolving curve $x(s,t)$ in $\mathbb{E}^3$, evolving according to the vortex filament equation, generates *Hasimoto surface* ([21]). The planar (non-planar) curves in $\mathbb{E}^3$ which generate Hasimoto surfaces are the circle and elastica (helix and non-planar elastica) ([17]). In Minkowski 3-space, a spacelike curve $x(s,t)$ with a timelike (spacelike) principal normal vector field, evolving according to the vortex filament equation, generates the spacelike (timelike) Hasimoto surface ([10]). In the Lorentzian n-space, null general helices $x(s,t)$ evolving in the axis direction are the solutions of *null Betchov–Da Rios equation* (null localized induction equation) $x_t = x_{ss} \times x_{sss}$ ([11]). Explicit solutions of the Betchov–Da Rios soliton equation in terms of non-null and null curves in 3-dimensional Lorentzian space forms are given in [2]. The vortex filament equation for pseudo null curves in Minkowski 3-space is obtained in [15] and reads $v = x_t = x_s \times x_{ss}$. In this paper, we obtain some solutions of the Da Rios vortex filament equation in terms of the Bishop frames of the pseudo null curves and null Cartan cubic.

2. Preliminaries

The Minkowski 3-space $\mathbb{E}^3_1$ is the real vector space $\mathbb{E}^3$ equipped with the standard indefinite flat metric $\langle\cdot,\cdot\rangle$ defined by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3, \tag{1}$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}^3_1$. Since $\langle\cdot,\cdot\rangle$ is an indefinite metric, an arbitrary vector $x \in \mathbb{E}^3_1 \setminus \{0\}$ can have one of three causal characters: it can be *spacelike*, *timelike* or *null* (*lightlike*), if $\langle x, x\rangle$ is positive, negative or zero, respectively. In particular, the vector $x = 0$ is a spacelike. The *norm* (length) of a vector $x \in \mathbb{E}^3_1$ is given by $\|x\| = \sqrt{|\langle x, x\rangle|}$. An arbitrary curve $\alpha : I \to \mathbb{E}^3_1$ can locally be *spacelike*, *timelike* or *null (lightlike)*, if all of its velocity vectors $\alpha'(s)$ satisfy $\langle\alpha'(s), \alpha'(s)\rangle > 0$, $\langle\alpha'(s), \alpha'(s)\rangle < 0$ or $\langle\alpha'(s), \alpha'(s)\rangle = 0$ and $\alpha'(s) \neq 0$, respectively ([19]).

A spacelike curve $\alpha : I \to \mathbb{E}^3_1$ is called a *pseudo null curve*, if its principal normal vector field N and binormal vector filed B are null vector fields satisfying the condition $\langle N, B\rangle = 1$. The Frenet formulae of a non-geodesic pseudo null curve α have the form ([22])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2}$$

where the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$ is an arbitrary function in arclength parameter s of α. The Frenet's frame vectors of α satisfy the equations

$$\begin{aligned} &\langle T, T\rangle = 1, \ \langle N, N\rangle = \langle B, B\rangle = 0, \\ &\langle T, N\rangle = \langle T, B\rangle = 0, \ \langle N, B\rangle = 1, \end{aligned} \tag{3}$$

and

$$T \times N = N, \ N \times B = T, \ B \times T = B. \tag{4}$$

The frame $\{T, N, B\}$ is *positively oriented*, if $det(T, N, B) = [T, N, B] = 1$.

A curve $\beta : I \to \mathbb{E}^3_1$ is called a *null curve*, if its tangent vector $\beta' = T$ is a null vector. A null curve $\beta = \beta(s)$ is called a *null Cartan curve*, if it is parameterized by the pseudo-arc function s defined by ([4])

$$s(t) = \int_0^t \sqrt{||\beta''(u)||}\, du. \tag{5}$$

There exists a unique Cartan frame $\{T, N, B\}$ along a non-geodesic null Cartan curve β satisfying the Cartan equations ([8])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\tau & 0 & \kappa \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{6}$$

where the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$ is an arbitrary function in pseudo-arc parameter s. If $\tau(s) = 0$, the null Cartan curve is called a *null Cartan cubic*. The Cartan's frame vectors of β satisfy the relations

$$\begin{aligned} &\langle T, T\rangle = \langle B, B\rangle = 0, \ \langle N, N\rangle = 1, \\ &\langle T, N\rangle = \langle N, B\rangle = 0, \ \langle T, B\rangle = -1, \end{aligned} \tag{7}$$

and

$$T \times N = -T, \ N \times B = -B, \ B \times T = N. \tag{8}$$

Cartan frame $\{T, N, B\}$ is *positively oriented*, if $det(T, N, B) = [T, N, B] = 1$.

If β is null Cartan curve, then its normal plane $\pi = T^\perp = span\{T, N\}$ is a lightlike at each point of the curve. According to [8], the *radical space* on π is given by $Rad(T\pi) = span\{\beta'\}$. All spacelike vectors in π are orthogonal to β' and generate the *screen distribution* on π given by $S(T\pi) = \{Y \in \pi \,|\, g(Y, \beta') = 0\}$.

Note that the choice of the screen distribution $S(T\pi)$ on π is not a unique. For a given screen distribution $S(T\pi)$ on π, there exists a unique *lightlike transversal vector* $Z \in \mathbb{E}^3_1$, which satisfies the conditions ([8])

$$g(Z, Z) = 0, \quad g(Z, \beta') = -1, \quad g(Z, Y) = 0.$$

3. The Bishop frame of a pseudo null curve

In this section we prove that there are two possible Bishop frames of a non-geodesic pseudo null curve in Minkowski 3-space, which are not unique and contain one fixed vector. We also show that the normal Bishop vector N_1 (of the first Bishop frame) can be obtained by applying the hyperbolic rotation to the principal normal vector N, while the normal Bishop vector N_2 (of the first Bishop frame) can be obtained by applying the composition of three rotations about two lightlike and one spacelike axis to the binormal vector B. We first define the Bishop frame of a pseudo null curve as follows.

Definition 1. The Bishop frame $\{T_1, N_1, N_2\}$ of a pseudo null curve α in $\mathbb{E}^3_1$ is positively oriented pseudo-orthonormal frame consisting of the tangential vector field T_1 and two relatively parallel lightlike normal vector fields N_1 and N_2.

Since $\langle N_1, N_1' \rangle = 0$, the vector N_1' lies in the lightlike plane $N_1^{\perp} = span\{T_1, N_1\}$ and makes minimal rotation in that plane if it is collinear with T_1, or with N_1. If N_1' is collinear with N_1, then N_1 coincides with the principal normal vector N, so the Bishop frame would coincide with the Frenet frame, which has no minimal rotation property. Hence N_1' is collinear with T_1. Similarly, we conclude that the vector N_2' makes minimal rotation in the lightlike plane $N_2^{\perp} = span\{T_1, N_2\}$ if it is collinear with T_1.

Therefore, we define the lightlike normal vector fields N_1 and N_2 to be *relatively parallel*, if their derivatives with respect to the arc-length parameter s are tangential.

Remark 1. We can also define N_1 and N_2 to be relatively parallel, if the normal component $T_1^{\perp} = span\{N_1, N_2\}$ of their derivatives N_1' and N_2' is zero, which implies that the mentioned derivatives are collinear with T_1.

Theorem 1. *Let α be a pseudo null curve in $\mathbb{E}^3_1$ parameterized by arc-length parameter s with the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$. Then the Bishop frame $\{T_1, N_1, N_2\}$ and the Frenet frame $\{T, N, B\}$ of α are related by:*

(i)

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\kappa_2} & 0 \\ 0 & 0 & \kappa_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{9}$$

and the Frenet equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & \kappa_1 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{10}$$

where $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0 e^{\int \tau(s) ds}$, $c_0 \in \mathbb{R}_0^+$;

(ii)

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -\kappa_1 \\ 0 & -\frac{1}{\kappa_1} & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{11}$$

and the Frenet equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & \kappa_1 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{12}$$

where $\kappa_1(s) = c_0 e^{\int \tau(s) ds}$, $c_0 \in \mathbb{R}_0^-$ and $\kappa_2(s) = 0$.

Proof. Let $\{T_1, N_1, N_2\}$ be the Bishop frame of a pseudo null curve $\alpha(s)$, satisfying the conditions

$$\begin{aligned} &\langle T_1, T_1 \rangle = 1, \ \langle N_1, N_1 \rangle = \langle N_2, N_2 \rangle = 0, \\ &\langle T_1, N_1 \rangle = \langle T_1, N_2 \rangle = 0, \ \langle N_1, N_2 \rangle = 1. \end{aligned} \tag{13}$$

Since T_1 is the unit tangent vector field, assume that $T_1 = T$. We will determine the normal vector fields N_1 and N_2 such that their derivatives are tangential. In relation to that, decompose the vector N_1' with respect to the pseudo-orthonormal basis $\{T_1, N_1, N_2\}$ by

$$N_1' = aT_1 + bN_1 + cN_2, \tag{14}$$

where $a(s), b(s), c(s)$ are some differentiable functions in arc-length parameter s. By using the relations (13) and (14), we find

$$\langle N_1', T_1 \rangle = a = -\kappa_1, \quad \langle N_1', N_1 \rangle = c = 0, \quad \langle N_1', N_2 \rangle = b = 0, \tag{15}$$

where $\kappa_1(s)$ is some differentiable function. Substituting (15) in (14), we obtain

$$N_1' = -\kappa_1 T_1. \tag{16}$$

In a similar way, we get

$$N_2' = -\kappa_2 T_1, \quad T_1' = \kappa_2 N_1 + \kappa_1 N_2, \tag{17}$$

where $\kappa_2(s)$ is some differentiable function. By using the relations (16) and (17), we get

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & \kappa_1 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}. \tag{18}$$

Since $T_1 = T$, from the Frenet equations (2) it follows that T_1' is a null vector. By using the condition $\langle T_1', T_1' \rangle = 0$ and the relations (13) and (17), we obtain $\kappa_1 \kappa_2 = 0$. Hence we distinguish two cases: (A) $\kappa_1 = 0$; (B) $\kappa_2 = 0$.

(A) If $\kappa_1 = 0$, the Frenet equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & 0 \\ 0 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}. \tag{19}$$

Next we find the relationship between the function κ_2 and the torsion τ, as well as between the Frenet's frame vectors and the Bishop's frame vectors. Since N_1 and N_2 are two linearly independent null vectors lying in the timelike plane $T_1^\perp = span\{N, B\}$, they are collinear with N and B.

If

$$N_1 = \lambda N, \quad N_2 = \mu B, \tag{20}$$

for some differentiable functions $\lambda(s) \neq 0$ and $\mu(s) \neq 0$, differentiating (20) with respect to s, using the condition $\langle N_1, N_2 \rangle = 1$ and relations (2), (3) and (19), we get

$$\lambda(s) = \frac{1}{\kappa_2(s)}, \quad \mu(s) = \kappa_2(s) = c_0 e^{\int \tau(s) ds}, \tag{21}$$

where $c_0 \in \mathbb{R}_0^+$. It can be easily verified that $\{T_1, N_1, N_2\}$ is a positively oriented pseudo-orthonormal frame. Substituting (21) in (20), we find that the Frenet and the Bishop frame of α are related by (9).

If $N_1 = \lambda B$ and $N_2 = \mu N$, differentiating the equation $N_1 = \lambda B$ with respect to s and using the relations (2), (3) and (19) we get $\lambda = 0$, which is a contradiction. This proves statement (i).

(B) If $\kappa_2 = 0$, the Frenet equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \kappa_1 \\ -\kappa_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}. \tag{22}$$

Similarly as in the case (A), assume that $N_1 = \lambda N$ and $N_2 = \mu B$. Differentiating the equation $N_2 = \mu B$ with respect to s and using (2), (3) and (22), we get $\mu = 0$, which is a contradiction. It follows that

$$N_1 = \lambda B, \quad N_2 = \mu N. \tag{23}$$

Then $det(T_1, N_1, N_2) = [T_1, N_1, N_2] = -1$, which means that the frame $\{T_1, N_1, N_2\}$ is a negatively oriented. To avoid this situation, define the tangential vector field T_1 as $T_1 = -T$. Differentiating (23) with respect to s, using the condition $\langle N_1, N_2 \rangle = 1$ and the relations (2), (3) and (22), we find

$$\lambda(s) = -\kappa_1(s) = c_0 e^{\int \tau(s) ds}, \quad \mu(s) = c_0 e^{-\int \tau(s) ds}, \tag{24}$$

where $c_0 \in \mathbb{R}_0^+$. Substituting (24) in (23) we get that the Frenet and the Bishop frame of α are related by (11). This proves statement (ii) and the theorem. $\square$

We call the functions κ_1 and κ_2 the *first* and the *second Bishop curvature* of a pseudo null curve α, respectively.

If α is a regular curve in $\mathbb{E}^3$ with the Bishop curvatures $\kappa_1(s)$ and $\kappa_2(s)$, then a planar curve γ parameterized by $\gamma(s) = (\kappa_1(s), \kappa_2(s))$ is called the *normal development* of α ([3]). It is known that two regular curves in $\mathbb{E}^3$ are congruent if and only if they have the same normal development. For example, the normal development of a planar curve in $\mathbb{E}^3$ is the straight line passing through the origin, and the normal development of a helix in $\mathbb{E}^3$ is a circle centered at the origin ([3]).

If α is a pseudo null curve in $\mathbb{E}_1^3$ with the Bishop curvatures $\kappa_1(s) = 0$, $\kappa_2(s) > 0$, then its normal development is the curve $\gamma(s) = (0, \kappa_2(s))$, which represents reparametrization of (a positive part of) y-axis. Note that pseudo null straight line and pseudo null circle have the same normal development $\gamma(s) = (0, c_1)$, $c_1 \in \mathbb{R}^+$, although they are not congruent.

In the next theorem, we give geometric interpretation of the relation (9).

Theorem 2. *Let $\alpha(s)$ be a pseudo null curve in $\mathbb{E}_1^3$ parameterized by arc-length parameter s with the Frenet frame $\{T, N, B\}$ and the torsion $\tau(s)$. If the Bishop frame $\{T_1, N_1, N_2\}$ of α is given by relation (9), then:*

(i) $R_\omega(N) = N_1$, where R_ω is the hyperbolic rotation for the hyperbolic angle $\omega(s) = -\ln(\kappa_2(s))$ about spacelike axis spanned by $e_3 = (0, 0, 1)$;

(ii) $(R_{-\theta} \circ R_{-\varphi} \circ R_\theta)(B) = N_2$, where R_θ is rotation for an angle $\theta(s) = -\int \kappa_2(s)ds$ about lightlike axis spanned by $e_0 = (1, 1, 0)$ and $R_{-\varphi}$ is the hyperbolic rotation for the hyperbolic angle $-\varphi = -\int \tau(s)ds - \ln c_0$ about spacelike axis spanned by $e_3 = (0, 0, 1)$, $c_0 \in \mathbb{R}_0^+$.

Proof. From the Frenet equations (2), we have

$$N'(s) = \tau(s)N(s).$$

Let us put $N(s) = (x(s), y(s), z(s))$, where $x(s)$, $y(s)$ and $z(s)$ are some differentiable functions. Substituting this in the last equation, we obtain

$$x'(s) = \tau(s)x(s), \quad y'(s) = \tau(s)y(s), \quad z'(s) = \tau(s)z(s),$$

and thus

$$x(s) = e^{\int \tau(s)ds + c_1}, \quad y(s) = e^{\int \tau(s)ds + c_2}, \quad z(s) = e^{\int \tau(s)ds + c_3},$$

where $c_1, c_2, c_3 \in \mathbb{R}$ are constants of integration. By using relation (21), the last relation becomes

$$x(s) = \frac{1}{c_0}k_2(s)e^{c_1}, \quad y(s) = \frac{1}{c_0}k_2(s)e^{c_2}, \quad z(s) = \frac{1}{c_0}k_2(s)e^{c_3},$$

$c_0 \in \mathbb{R}_0^+$. Therefore,

$$N(s) = k_2(s)A_0, \tag{25}$$

where $A_0 = \frac{1}{c_0}(e^{c_1}, e^{c_2}, e^{c_3})$ is a constant vector. According to (3) and (25), we have $\langle N(s), N(s)\rangle = \langle k_2(s)A_0, k_2(s)A_0\rangle = 0$. Since $k_2(s) \neq 0$ we get $\langle A_0, A_0\rangle = 0$, which means that A_0 is a constant null vector. Next we show that up to isometries of $\mathbb{E}_1^3$ we may take $A_0 = (1, 1, 0)$. In relation to that, let us put $A_0 = (a, b, c)$, where

$$a = \frac{1}{c_0}e^{c_1}, \quad b = \frac{1}{c_0}e^{c_2}, \quad c = \frac{1}{c_0}e^{c_3}.$$

By applying the rotation R_α for an angle $\alpha = \dfrac{c}{b-a}$, $b \neq a$ about lightlike axis spanned by $V = (1, 1, 0)$ and using $-a^2 + b^2 + c^2 = 0$, we find

$$R_\alpha(A_0) = \begin{bmatrix} 1 + \frac{\alpha^2}{2} & -\frac{\alpha^2}{2} & \alpha \\ \frac{\alpha^2}{2} & 1 - \frac{\alpha^2}{2} & \alpha \\ \alpha & -\alpha & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{a-b}{2} \\ \frac{b-a}{2} \\ 0 \end{bmatrix}.$$

Next, by applying the symmetry S_x with respect to x-axis, we obtain

$$(S_x \circ R_\alpha)(A_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{a-b}{2} \\ \frac{b-a}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{a-b}{2} \\ \frac{a-b}{2} \\ 0 \end{bmatrix}.$$

If $a - b > 0$, by applying the hyperbolic rotation R_β for the hyperbolic angle $\beta = \ln(\frac{2}{a-b})$ about spacelike axis spanned by $e_3 = (0, 0, 1)$, we get

$$(R_\beta \circ S_x \circ R_\alpha)(A_0) = \begin{bmatrix} \cosh\beta & \sinh\beta & 0 \\ \sinh\beta & \cosh\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{a-b}{2} \\ \frac{a-b}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

If $a - b < 0$, by using the hyperbolic rotation R_γ for the hyperbolic angle $\gamma = \ln(\frac{2}{b-a})$ about spacelike axis spanned by $e_3 = (0, 0, 1)$, we obtain $(R_\gamma \circ S_x \circ R_\alpha)(A_0) = (-1, -1, 0)$. By applying the symmetry S_O with respect to the origin O, we get $(S_O \circ R_\gamma \circ S_x \circ R_\alpha)(A_0) = (1, 1, 0)$.

Consequently, up to isometries of $\mathbb{E}^3_1$ we may take $A_0 = (1, 1, 0)$. Substituting $A_0 = (1, 1, 0)$ in (25), it follows that the principal normal vector has the form

$$N(s) = (\kappa_2(s), \kappa_2(s), 0). \tag{26}$$

By applying the hyperbolic rotation R_ω for the hyperbolic angle $\omega(s) = -\ln(\kappa_2(s))$ about spacelike axis spanned by $e_3 = (0, 0, 1)$ and using relation (9), we find

$$R_\omega(N) = \begin{bmatrix} \cosh\omega & \sinh\omega & 0 \\ \sinh\omega & \cosh\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \kappa_2 \\ \kappa_2 \\ 0 \end{bmatrix} = \frac{1}{\kappa_2} N = N_1, \tag{27}$$

which proves statement (i).

By using the relations $T' = N$ and (26), we find

$$T(s) = (\int \kappa_2(s)ds, \int \kappa_2(s)ds, 1).$$

On the other hand, from (26) and (27) we get $N_1(s) = (1, 1, 0)$. By using the conditions (13), it follows that the Bishop vector N_2 has the form

$$N_2(s) = (-\frac{(\int \kappa_2(s)ds)^2}{2} - \frac{1}{2}, \frac{1}{2} - \frac{(\int \kappa_2(s)ds)^2}{2}, -\int \kappa_2(s)ds). \tag{28}$$

According to relations (9) and (28), the binormal vector is given by

$$B(s) = (-\frac{(\int \kappa_2(s)ds)^2}{2\kappa_2(s)} - \frac{1}{2\kappa_2(s)}, \frac{1}{2\kappa_2(s)} - \frac{(\int \kappa_2(s)ds)^2}{2\kappa_2(s)}, -\frac{\int \kappa_2(s)ds}{\kappa_2(s)}).$$

By applying the rotation R_θ for an angle $\theta(s) = -\int \kappa_2(s)ds$ about lightlike axis spanned by $N_1 = (1, 1, 0)$ to the binormal vector B, we find

$$R_\theta(B) = \begin{bmatrix} 1 + \frac{\theta^2}{2} & -\frac{\theta^2}{2} & \theta \\ \frac{\theta^2}{2} & 1 - \frac{\theta^2}{2} & \theta \\ \theta & -\theta & 1 \end{bmatrix} \begin{bmatrix} -\frac{(\int \kappa_2(s)ds)^2}{2\kappa_2(s)} - \frac{1}{2\kappa_2(s)} \\ \frac{1}{2\kappa_2(s)} - \frac{(\int \kappa_2(s)ds)^2}{2\kappa_2(s)} \\ -\frac{\int \kappa_2(s)ds}{\kappa_2(s)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\kappa_2(s)} \\ \frac{1}{2\kappa_2(s)} \\ 0 \end{bmatrix}.$$

Next, by applying the hyperbolic rotation $R_{-\varphi}$ for the hyperbolic angle $-\varphi(s) = -\int \tau(s)ds - \ln c_0$, $c_0 \in \mathbb{R}_0^+$ about spacelike axis spanned by $e_3 = (0, 0, 1)$ to the vector $R_\theta(B)$ and using (2), we obtain

$$(R_{-\varphi} \circ R_\theta)(B) = \begin{bmatrix} \cosh(-\varphi) & \sinh(-\varphi) & 0 \\ \sinh(-\varphi) & \cosh(-\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2\kappa_2(s)} \\ \frac{1}{2\kappa_2(s)} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}.$$

Finally, by applying the rotation $R_{-\theta}$ for an angle $-\theta(s) = \int \kappa_2(s)ds$ about lightlike axis spanned by $N_1 = (1, 1, 0)$ to the vector $(R_{-\varphi} \circ R_\theta)(B)$, we get

$$(R_{-\theta} \circ R_{-\varphi} \circ R_\theta)(B) = \begin{bmatrix} 1 + \frac{\theta^2}{2} & -\frac{\theta^2}{2} & -\theta \\ \frac{\theta^2}{2} & 1 - \frac{\theta^2}{2} & -\theta \\ -\theta & \theta & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = N_2,$$

which proves statement (ii). □

The next theorem can be proved analogously, so we omit its proof.

Theorem 3. *Let $\alpha(s)$ be a pseudo null curve in $\mathbb{E}_1^3$ parameterized by arc-length parameter s with the Frenet frame $\{T, N, B\}$ and the torsion $\tau(s)$. If the Bishop frame $\{T_1, N_1, N_2\}$ of α is given by relation (11), then:*

(i) $R_\omega(N) = N_2$, where R_ω is the hyperbolic rotation for the hyperbolic angle $\omega(s) = -\ln(-\kappa_1(s))$ about spacelike axis spanned by $e_3 = (0, 0, 1)$;

(ii) $(R_{-\theta} \circ R_{-\varphi} \circ R_\theta)(B) = N_1$, where R_θ is rotation for an angle $\theta(s) = -\int \kappa_1(s)ds$ about lightlike axis spanned by $e_0 = (1, 1, 0)$ and $R_{-\varphi}$ is the hyperbolic rotation for the hyperbolic angle $-\varphi(s) = -\int \tau(s)ds - \ln(-c_0)$ about spacelike axis spanned by $e_3 = (0, 0, 1)$, $c_0 \in \mathbb{R}_0^-$.

4. The Bishop frame of a null Cartan curve

In this section we obtain the Bishop frame of a non-geodesic null Cartan curve. We also prove that among all non-geodesic null Cartan curves in $\mathbb{E}_1^3$, only the null Cartan cubic has two Bishop frames, one of which coincides with its Cartan frame. In relation to that, we first modify the definition of the Bishop frame with respect to the Euclidean case as follows.

Definition 2. The Bishop frame $\{T_1, N_1, N_2\}$ of a non-geodesic null Cartan curve in $\mathbb{E}_1^3$ is positively oriented pseudo-orthonormal frame consisting of the tangential vector field T_1, relatively parallel spacelike normal vector field N_1 and relatively parallel lightlike transversal vector field N_2.

We define the spacelike normal vector field N_1 and the lightlike transversal vector field N_2 along a null Cartan curve to be *relatively parallel*, if the normal component $T_1^\perp = span\{T_1, N_1\}$ of their derivatives N_1' and N_2' is zero. This means that the vector fields N_1' and N_2' are collinear with N_2. Note that relatively parallel vector fields N_1 and N_2 are defined in the same way as in the case of pseudo null curve, or a curve in $\mathbb{E}^3$, but with respect to an indefinite metric. In the next theorem we obtain the Bishop frame of a null Cartan curve.

Theorem 4. *Let α be a null Cartan curve in $\mathbb{E}_1^3$ parameterized by pseudo-arc s with the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$. Then the Bishop frame $\{T_1, N_1, N_2\}$ and the Cartan frame $\{T, N, B\}$ of α are related by:*

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\kappa_2 & 1 & 0 \\ \frac{\kappa_2^2}{2} & -\kappa_2 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{29}$$

and the Cartan equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} \kappa_2 & \kappa_1 & 0 \\ 0 & 0 & \kappa_1 \\ 0 & 0 & -\kappa_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{30}$$

where the first Bishop curvature $\kappa_1(s) = 1$ and the second Bishop curvature satisfies Riccati differential equation $\kappa_2'(s) = -\dfrac{1}{2}\kappa_2^2(s) - \tau(s)$.

Proof. Assume that α has the Bishop frame $\{T_1, N_1, N_2\}$ which satisfies the conditions

$$\begin{aligned} \langle T_1, T_1 \rangle = \langle N_2, N_2 \rangle = 0,\ \langle N_1, N_1 \rangle = 1, \\ \langle T_1, N_2 \rangle = -1,\ \langle T_1, N_1 \rangle = \langle N_1, N_2 \rangle = 0, \end{aligned} \tag{31}$$

and

$$T_1 \times N_1 = -T_1,\ N_1 \times N_2 = -N_2,\ N_2 \times T_1 = N_1. \tag{32}$$

We will determine the normal vector field N_1 by using the condition that it is relatively parallel. In relation to that, decompose the vector N_1' with respect to the pseudo-orthonormal basis $\{T_1, N_1, N_2\}$ by

$$N_1' = aT_1 + bN_1 + cN_2, \tag{33}$$

where $a(s), b(s), c(s)$ are some differentiable functions in pseudo-arc parameter s. By using (31), (33) and the condition that N_1 is relatively parallel, we find

$$\langle N_1', T_1 \rangle = -c = -\kappa_1, \quad \langle N_1', N_1 \rangle = b = 0, \quad \langle N_1', N_2 \rangle = -a = 0, \tag{34}$$

where $\kappa_1(s)$ is some differentiable function. Substituting (34) in (33), we obtain

$$N_1' = \kappa_1 N_2. \tag{35}$$

In a similar way, we get

$$N_2' = -\kappa_2 N_2, \quad T_1' = \kappa_2 T_1 + \kappa_1 N_1, \tag{36}$$

where $\kappa_2(s)$ is some differentiable function. Relations (35) and (36) imply relation (30). Since T_1 is a null tangent vector field, we may assume that

$$T_1 = T. \tag{37}$$

By using (31) and (37), we get $\langle T_1, N_1 \rangle = \langle T, N_1 \rangle = 0$. Therefore, $N_1 \in T^\perp$ so relation (7) implies that the normal vector field N_1 can be written as

$$N_1 = \lambda T + \mu N,$$

where $\lambda(s)$ and $\mu(s)$ are some differentiable functions. Since $\langle N_1, N_1 \rangle = 1$, by using (7) we get $\mu(s) = \pm 1$. Let us take $\mu(s) = 1$. Thus

$$N_1 = \lambda T + N. \tag{38}$$

By using the relations (7), (31), (37) and (38), we get

$$N_2 = \frac{\lambda^2}{2} T + \lambda N + B. \tag{39}$$

In particular, differentiating (37) with respect to s and using (6), (37) and (38), we obtain

$$T_1' = T' = N = N_1 - \lambda T_1.$$

On the other hand, according to the second equation of (36), we have

$$T_1' = \kappa_2 T_1 + \kappa_1 N_1.$$

From the last two relations, we find that Bishop curvatures of α are given by

$$\kappa_1 = 1, \quad \kappa_2 = -\lambda. \tag{40}$$

Then relations (37), (38), (39) and (40) imply (29). In a similar way, by taking $\mu(s) = -1$ we get negatively oriented Bishop frame $\{T_1^* = -T_1, N_1^* = -N_1, N_2^* = -N_2\}$ with the Bishop curvatures $\kappa_1^*(s) = \kappa_1(s) = 1$, $\kappa_2^*(s) = \kappa_2(s)$ and satisfying the conditions (31). According to Definition 2, this case is not possible.

Finally, differentiating the relation (38) with respect to s and using (6), (7), (31), (35) and (40), we obtain that κ_2 satisfies Riccati differential equation

$$\kappa_2'(s) = -\frac{1}{2}\kappa_2^2(s) - \tau(s),$$

which completes the proof of the theorem. $\square$

By using Theorem 4, we easily get the next corollary, which states that among all null Cartan curves in $\mathbb{E}_1^3$ only the null Cartan cubic has two Bishop frames, one of which coincides with its Cartan frame.

Corollary 1. *Let α be a null Cartan cubic in $\mathbb{E}_1^3$ parameterized by pseudo-arc s. Then the Bishop frame $\{T_1, N_1, N_2\}$ and the Cartan frame $\{T, N, B\}$ of α are related by:*

(i)

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\kappa_2 & 1 & 0 \\ \frac{\kappa_2^2}{2} & -\kappa_2 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

and the Cartan equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} \kappa_2 & \kappa_1 & 0 \\ 0 & 0 & \kappa_1 \\ 0 & 0 & -\kappa_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{41}$$

where the Bishop curvatures have the form $\kappa_1(s) = 1$ and $\kappa_2(s) = \dfrac{2}{s}$;

(ii)

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

and the Cartan equations of α according to the Bishop frame read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} \kappa_2 & \kappa_1 & 0 \\ 0 & 0 & \kappa_1 \\ 0 & 0 & -\kappa_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix},$$

where the Bishop curvatures have the form $\kappa_1(s) = 1$ and $\kappa_2(s) = 0$.

5. Some applications

In this section, by using the Bishop frame of the pseudo null curve and null Cartan cubic, we give some solutions of the Da Rios vortex filament equation which generate lightlike Hasimoto surfaces.

Example 1. Let α be a pseudo null curve in $\mathbb{E}_1^3$ parameterized by arc-length parameter s with parameter equation

$$\alpha(s) = (\frac{s^3}{3} + \frac{s^2}{2}, \frac{s^3}{3} + \frac{s^2}{2}, s).$$

According to the statement (i) of Theorem 1, the Bishop frame $\{T_1, N_1, N_2\}$ and the Bishop curvatures of α have the form

$$\begin{aligned} T_1(s) &= (s^2 + s, s^2 + s, 1), \\ N_1(s) &= (\frac{1}{c_0}, \frac{1}{c_0}, 0), \\ N_2(s) &= c_0(-\frac{(s^2+s)^2+1}{2}, \frac{1-(s^2+s)^2}{2}, -s^2 - s), \end{aligned}$$

$\kappa_1(s) = 0$, $\kappa_2(s) = c_0(2s+1)$, where $c_0 \in \mathbb{R}_0^+$.

Denote by $x(s,t)$ an evolving curve in $\mathbb{E}_1^3$ parameterized by arc-length parameter s for all time t with evolution equation

$$x(s,t) = \alpha(s) + t(a(s)T_1(s) + b(s)N_1(s) + c(s)N_2(s)), \tag{42}$$

where $a(s)$, $b(s)$ and $c(s)$ are some differentiable functions. By taking the partial derivatives of the relation (42) with respect to s and t and using (10), we obtain

$$x_s = T_1 + t(aT_1 + bN_1 + cN_2)', \tag{43}$$

$$x_{ss} = \kappa_2 N_1 + t(aT_1 + bN_1 + cN_2)'', \tag{44}$$

$$x_t = aT_1 + bN_1 + cN_2. \tag{45}$$

Assume that the curve $x(s,t)$ evolves according to the vortex filament equation $x_t = x_s \times x_{ss}$. Substituting (43), (44) and (45) in the last equation and using relations (8) and (9), we get

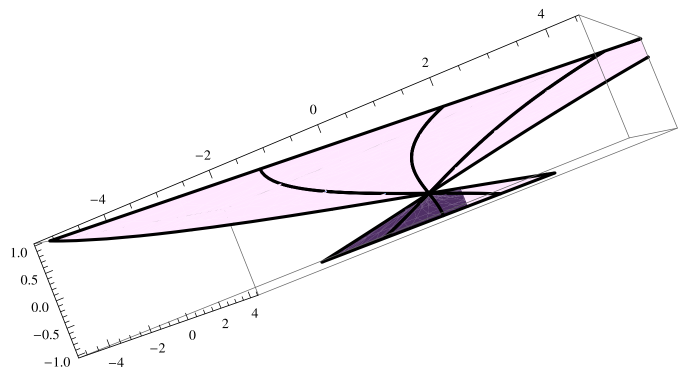

Fig. 1. Lightlike Hasimoto surface and evolving curve $x(s,t)$ for $t=-1$, $t=0$ and $t=1$.

$$aT_1+bN_1+cN_2=t^2(aT_1+bN_1+cN_2)'\times(aT_1+bN_1+cN_2)''$$
$$+t[T_1\times(aT_1+bN_1+cN_2)''+\kappa_2(aT_1+bN_1+cN_2)'\times N_1]+\kappa_2N_1.$$

The previous equation is satisfied for each t, if and only if the next relations hold

$$a=0,\quad b=\kappa_2,\quad c=0, \tag{46}$$
$$(aT_1+bN_1+cN_2)'\times(aT_1+bN_1+cN_2)''=0,$$
$$T_1\times(aT_1+bN_1+cN_2)''+\kappa_2(aT_1+bN_1+cN_2)'\times N_1=0.$$

Substituting (46) in (42), we find that evolving curve $x(s,t)$ has evolution equation of the form

$$x(s,t)=\alpha(s)+t\kappa_2(s)N_1(s). \tag{47}$$

It can be easily checked that $\langle x_s,x_s\rangle=1$ and $\langle x_{ss},x_{ss}\rangle=0$, which means that $x(s,t)$ is a pseudo null curve. In particular, since $x_s\times x_t=\kappa_2N_1$ is a null vector, evolving curve $x(s,t)$ generates lightlike cylindrical ruled surface with parameter equation (47), which represents lightlike Hasimoto surface (Fig. 1).

Example 2. Let us consider the null Cartan cubic in $\mathbb{E}^3_1$ parameterized by pseudo-arc parameter s with parameter equation

$$\alpha(s)=(\frac{s^3}{4}+\frac{s}{3},\frac{s^2}{2},\frac{s^3}{4}-\frac{s}{3}).$$

According to the statement (i) of Corollary 1, the Bishop frame $\{T_1,N_1,N_2\}$ of α is given by

$$T_1(s)=(\frac{3s^2}{4}+\frac{1}{3},s,\frac{3s^2}{4}-\frac{1}{3}),$$
$$N_1(s)=(-\frac{2}{3s},-1,\frac{2}{3s}),$$
$$N_2(s)=(\frac{2}{3s^2},0,-\frac{2}{3s^2}),$$

and the Bishop curvatures of α read $\kappa_1(s)=1$, $\kappa_2(s)=\dfrac{2}{s}$.

Denote by $x(s,t)$ an evolving curve in $\mathbb{E}^3_1$ parameterized by arc-length parameter s for all time t with evolution equation

$$x(s,t)=T_1(s)-t(a(s)T_1(s)+b(s)N_1(s)+c(s)N_2(s)), \tag{48}$$

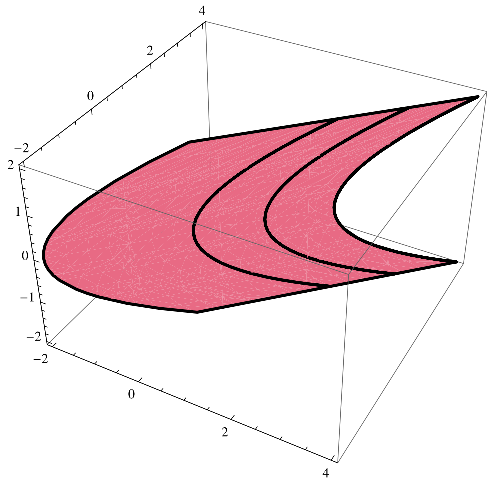

Fig. 2. Lightlike Hasimoto surface and evolving curve $x(s,t)$ for $t=-1$, $t=0$ and $t=1$.

where $a(s)$, $b(s)$ and $c(s)$ are some differentiable functions. Analogously as in Example 1, by taking the partial derivatives of the relation (48) with respect to s and t and assuming that $x(s,t)$ evolves according to the vortex filament equation $x_t = x_s \times x_{ss}$, we get

$$a(s) = \frac{2}{s^2}, \quad b(s) = \frac{2}{s}, \quad c(s) = 1. \tag{49}$$

Substituting (49) in (48), we find that evolution equation of $x(s,t)$ reads

$$x(s,t) = T_1(s) - t(\frac{2}{s^2}T_1(s) + \frac{2}{s}N_1(s) + N_2(s)). \tag{50}$$

By taking the partial derivatives of the relation (50) with respect to s and t and using relation (41), we find

$$x_s = \frac{2}{s}T_1(s) + N_1(s), \quad x_{ss} = \frac{2}{s^2}T_1(s) + \frac{2}{s}N_1(s) + N_2(s), \tag{51}$$

$$x_t = -(\frac{2}{s^2}T_1(s) + \frac{2}{s}N_1(s) + N_2(s)). \tag{52}$$

By using (31) and the first equation of (51), we find $\langle x_s, x_s \rangle = 1$. Relation (31) and the second equation of (51) yield $\langle x_{ss}, x_{ss} \rangle = 0$, which means that $x(s,t)$ is a pseudo null evolving curve. In particular, since $x_s \times x_t = (\frac{3}{2}, 0, \frac{3}{2})$ is a null constant vector and $x_{ts} = 0$, the curve $x(s,t)$ generates lightlike cylindrical ruled surface with parameter equation (50), which represents lightlike Hasimoto surface (Fig. 2).

Acknowledgments

The authors are very grateful to the referee for the useful suggestions. The authors were partially supported by the Serbian Ministry of Education, Science and Technological Development (grant number #174012).

References

[1] H. Balgetir, M. Bektas, J. Inoguchi, Null Bertrand curves in Minkovski 3-space and their characterizations, Note Mat. 23 (1) (2014) 7–13.
[2] M. Barros, A. Ferrandez, P. Lucas, M. Merono, Solutions of the Betchov–Da Rios soliton equation: a Lorentzian approach, J. Geom. Phys. 31 (1999) 217–228.

[3] L.R. Bishop, There is more than one way to frame a curve, Amer. Math. Monthly 82 (3) (1975) 246–251.
[4] W.B. Bonnor, Null curves in a Minkowski space–time, Tensor 20 (1969) 229–242.
[5] B. Bükcu, M.K. Karacan, On the slant helices according to Bishop frame of the timelike curve in Lorentzian space, Tamkang J. Math. 39 (3) (2008) 255–262.
[6] B. Bükcu, M.K. Karacan, Bishop motion and Bishop Darboux rotation axis of the timelike curve in Minkowski 3-space, Kochi J. Math. 4 (2009) 109–117.
[7] L.S. Da Rios, On the motion of an unbounded fluid with a vortex filament of an shape, Rend. Circ. Mat. Palermo 22 (1906) 117.
[8] K.L. Duggal, D.H. Jin, Null Curves and Hypersurfaces of Semi-Riemannian Manifolds, World Scientific, Singapore, 2007.
[9] M. Erdogdu, Parallel frame of non-lightlike curves in Minkowski space–time, Int. J. Geom. Methods Mod. Phys. 12 (2015), 16 pages.
[10] M. Erdogdu, M. Özdemir, Geometry of Hasimoto surfaces in Minkowski 3-space, Math. Phys. Anal. Geom. 17 (2014) 169–181.
[11] A. Ferrandez, A. Gimenez, P. Lucas, Null generalized helices and the Betchov–Da Rios equation in Lorentz–Minkowski spaces, in: Proceedings of the XI Fall Workshop on Geometry and Physics, Madrid, 2004, pp. 215–221.
[12] F. Gökcelik, Z. Bozkurt, I. Gök, F.N. Ekmekci, Y. Yayli, Parallel transport frame in 4-dimensional Euclidean space E^4, Caspian J. Math. Sci. 3 (2014) 91–103.
[13] M. Grbović, K. Ilarslan, E. Nešović, On null and pseudo null Mannheim curves in Minkowski 3-space, J. Geom. 105 (1) (2014) 177–183.
[14] M. Grbović, E. Nešović, On Bäcklund transformation and vortex filament equation for null Cartan curve in Minkowski 3-space, Math. Phys. Anal. Geom. 23 (2016) 1–15.
[15] M. Grbović, E. Nešović, On Bäcklund transformation and vortex filament equation for pseudo null curves in Minkowski 3-space, Int. J. Geom. Methods Mod. Phys. 13 (6) (2016) 1650077.
[16] N. Gürses, Ö. Bektas, S. Yüce, Special Smarandache curves in $\mathbb{R}^3_1$, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. 65 (2) (2016) 143–160.
[17] T.A. Ivey, Helices, Hasimoto surfaces and Bäcklund transformations, Canad. Math. Bull. 43 (4) (2000) 427–439.
[18] E. Nešović, U. Özturk, E.B. Koc Özturk, On k-type pseudo null Darboux helices in Minkowski 3-space, J. Math. Anal. Appl. 439 (2) (2016) 690–700.
[19] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.
[20] M. Özdemir, A.A. Ergin, Parallel frame of non-lightlike curves, Missouri J. Math. Sci. 20 (2) (2008) 127–137.
[21] C. Rogers, W.K. Schief, Bäcklund and Darboux Transformations: Geometry and Modern Application in Soliton Theory, Cambridge University Press, Cambridge, 2002.
[22] J. Walrave, Curves and Surfaces in Minkowski Space, Ph.D. thesis, Leuven University, 1995.
[23] S. Yilmaz, Position vectors of some special space-like curves according to Bishop frame in Minkowski space E^3_1, Sci. Magna 5 (1) (2009) 48–50.
[24] S. Yilmaz, M. Turgut, A new version of Bishop frame and an application to spherical images, J. Math. Anal. Appl. 371 (2010) 764–776.