Novi Sad J. Math.
Vol. XX, No. Y, 20ZZ, ??-??

Draft version September 3, 2015

ON GENERALIZED PARTIALLY NULL MANNHEIM CURVES IN MINKOWSKI SPACE-TIME

Milica Grbović[1] and Emilija Nešović[2]

Abstract. In this paper we define generalized partially null and pseudo null Mannheim curves in Minkowski space-time E_1^4. We prove that there are no non-geodesic generalized partially null Mannheim curves in E_1^4, by considering the cases when the corresponding mate curve is a spacelike, a timelike, null Cartan, partially null or pseudo null Frenet curve. We also answer the question: "Can a partially null Frenet curve be generalized mate curve of the generalized pseudo null Mannheim curve in Minkowski space-time?"

AMS Mathematics Subject Classification (2010): 53C50, 53C40

Key words and phrases: generalized Mannheim curves, partially null curves, Minkowski space-time

1. Introduction

In Euclidean 3-space there are many associated curves such as Bertrand mates ([5]), Mannheim mates ([9]), spherical images, evolutes, involutes, the principal-direction curves ([2]), etc., whose frame's vector fields satisfy some extra conditions. Mannheim curves in the Euclidean 3-space are discovered by A. Mannheim in 1887. They are defined as the curves having a property that their principal normal lines coincide with binormal lines of their mate curves at the corresponding points. It is well-known that a regular smooth curve in E^3 is a Mannheim curve if and only if its curvature functions κ and τ satisfy the relation $\kappa = a(\kappa^2 + \tau^2)$, for some positive constant a. Some characterizations of Mannheim curves in the Euclidean 3-space and Minkowski 3-space can be found in [7, 9].
Parameter equation of the Mannheim curve in E^3 is given by ([4])

$$\alpha(t) = (\int h(t)\sin(t)dt, \int h(t)\cos(t)dt, \int h(t)g(t)dt),$$

where $g : I \to R$ is any smooth function and a function $h : I \to R$ is given by

$$h = \frac{(1+g^2+g'^2)^3 + (1+g^2)^3(g+g'')^2}{(1+g^2)^{\frac{3}{2}}(1+g^2+g'^2)^{\frac{5}{2}}}.$$

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[1] Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac

[2] Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, e-mail:nesovickg@sbb.rs

In Euclidean 4-space a special Frenet curve α is called a *generalized Mannheim curve*, if there exists a special Frenet curve α^* and a bijection $\phi : \alpha \to \alpha^*$ such that the principal normal line of α at each point of α lies in the plane spanned by the first and second binormal line of α^* at the corresponding point ([10]). In particular, a curve α^* is called *generalized Mannheim mate (partner) curve* of α. Parameter equations and basic geometric properties of the generalized Mannheim curves in E^4 are given in [10]. In Minkowski space-time, generalized spacelike Mannheim curves whose Frenet frame contains only non-null vectors, are defined in [6]. Mannheim curves lying in 3-dimensional space forms E^3 and S^3 in E^4, as well as in H^3 in E^4_1, are studied in [3].

In this paper, we define generalized partially null and pseudo null Mannheim curves in Minkowski space-time. We prove that there are no non-geodesic generalized partially null Mannheim curves in E^4_1, by considering the cases when the corresponding mate curve is a spacelike, a timelike, null Cartan, partially null or pseudo null Frenet curve. We also answer the question: "Can a partially null Frenet curve be generalized mate curve of the generalized pseudo null Mannheim curve in Minkowski space-time?"

2. Preliminaries

Minkowski space-time E^4_1 is a 4-dimensional affine space endowed with an indefinite flat metric g with signature $(-,+,+,+)$. This means that there are affine coordinates (x_1, x_2, x_3, x_4) such that metric bilinear form can be written as

$$g(x, y) = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

for any two $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in E^4_1. Recall that a vector $v \in E^4_1 \backslash \{0\}$ can be *spacelike* if $g(v, v) > 0$, *timelike* if $g(v, v) < 0$ and *null (lightlike)* if $g(v, v) = 0$. In particular, the vector $v = 0$ is said to be spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v, v)|}$. Two vectors v and w are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve α in E^4_1, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors α' are respectively spacelike, timelike or null ([11]). A non-null curve α is parametrized by the arc-length parameter s (or has the unit speed), if $g(\alpha'(s), \alpha'(s)) = \pm 1$. In particular, a null curve α is said to be parameterized by a *pseudo-arc* s, if $g\left(\alpha''(s), \alpha''(s)\right) = 1$, where *pseudo-arc function* s is defined by $s(t) = \int_0^t (g(\alpha''(u), \alpha''(u)))^{\frac{1}{4}}\, du$ ([1]).

Definition 2.1. A non-geodesic null curve $\alpha : I \to E^4_1$ parameterized by the pseudo-arc s is called *Cartan curve*, if there exists a unique positively oriented Cartan frame $\{T, N, B_1, B_2\}$ along α and the three smooth functions κ_1, κ_2 and κ_3 satisfying the Cartan equations ([1])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & -\kappa_1 & 0 \\ 0 & \kappa_2 & 0 & \kappa_3 \\ -\kappa_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \tag{2.1}$$

The functions $\kappa_1(s)=1, \kappa_2(s)$ and $\kappa_3(s)$ are called the first, second and third Cartan curvature of α. The Cartan frame vector fields satisfy the conditions

$$g(T,T)=g(B_1,B_1)=0, \quad g(N,N)=g(B_2,B_2)=1,$$

$$g(T,N)=g(T,B_2)=g(N,B_1)=g(N,B_2)=g(B_1,B_2)=0, \quad g(T,B_1)=1.$$

In particular, Cartan frame is a positively oriented, if $det(T,N,B_1,B_2)=1$.

Definition 2.2. A spacelike or a timelike non-geodesic unit speed smooth curve $\alpha: I \rightarrow E_1^4$ is called *Frenet curve*, if there exists a unique positively oriented orthonormal or pseudo-orthonormal Frenet frame $\{T,N,B_1,B_2\}$ along α and the three smooth functions $\kappa_1 \neq 0$, κ_2 and κ_3 satisfying Frenet equations.

A smooth functions $\kappa_1 \neq 0$, κ_2 and κ_3 are called the first, second and third Frenet curvature of α. The Frenet frame is positively oriented, if it holds $det(T,N,B_1,B_2)=1$. Let $\{T,N,B_1,B_2\}$ be the moving Frenet frame along the unit speed Frenet curve $\alpha: I \rightarrow E_1^4$, consisting of the tangent, principal normal, first binormal and second binormal vector field respectively. Depending on the causal character of Frenet vector fields, we have three types of Frenet equations.

Type 1. Let α be a timelike or a spacelike Frenet curve whose Frenet frame $\{T,N,B_1,B_2\}$ contains only non-null vector fields. The Frenet equations are given by ([8])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2\kappa_1 & 0 & 0 \\ -\epsilon_1\kappa_1 & 0 & \epsilon_3\kappa_2 & 0 \\ 0 & -\epsilon_2\kappa_2 & 0 & -\epsilon_1\epsilon_2\epsilon_3\kappa_3 \\ 0 & 0 & -\epsilon_3\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.2}$$

where $g(T,T)=\epsilon_1$, $g(N,N)=\epsilon_2$, $g(B_1,B_1)=\epsilon_3$, $g(B_2,B_2)=\epsilon_4$, $\epsilon_1\epsilon_2\epsilon_3\epsilon_4=-1$, $\epsilon_i \in \{-1,1\}$, $i \in \{1,2,3,4\}$. In particular, the following conditions hold:

$$g(T,N)=g(T,B_1)=g(T,B_2)=g(N,B_1)=g(N,B_2)=g(B_1,B_2)=0.$$

Type 2. Let α be pseudo null Frenet curve, i.e. a spacelike Frenet curve with null principal normal and the second binormal. The Frenet formulae read ([12])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 \\ 0 & \kappa_3 & 0 & -\kappa_2 \\ -\kappa_1 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.3}$$

where the first curvature $\kappa_1(s)=1$ for each s. Then the following conditions are satisfied:

$$g(T,T)=g(B_1,B_1)=1, \quad g(N,N)=g(B_2,B_2)=0,$$

$$g(T,N)=g(T,B_1)=g(T,B_2)=g(N,B_1)=g(B_1,B_2)=0, \quad g(N,B_2)=1.$$

Type 3. Let α be partially null Frenet curve, i.e. a spacelike Frenet curve with null first and second binormal . The Frenet formulae read ([12])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 & 0 \\ 0 & -\kappa_2 & 0 & -\kappa_3 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.4}$$

where the third curvature $\kappa_3(s) = 0$ for each s. Consequently, such curve has only two curvatures $\kappa_1 \neq 0$ and κ_2 and the following conditions hold:

$$g(T,T) = g(N,N) = 1, \quad g(B_1,B_1) = g(B_2,B_2) = 0,$$

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(N,B_2) = 0, \quad g(B_1,B_2) = 1.$$

3. Generalized partially null Mannheim curves in Minkowski space-time

In this section we define generalized partially null Mannheim curves in Minkowski space-time. We first consider non-geodesic generalized partially null Mannheim curves and their non-geodesic mate curves, having the first curvatures different from zero. At the end of this section, we will consider the case of the first curvature being zero.

Definition 3.1. Partially null Frenet curve $\alpha : I \to E_1^4$ is called a *generalized partially null Mannheim curve* if there exists a null Cartan or Frenet curve $\alpha^\star : I^\star \to E_1^4$ and a bijection $\phi : \alpha \to \alpha^\star$ given by $\phi(\alpha(s)) = \alpha^\star(f(s))$ such that for each $s \in I$ the principal normal line of α contains the corresponding points of the curves α and $\alpha^\star$ and lies in the plane spanned by the first and second binormal line of $\alpha^\star$.

The curve α^* is called *generalized Mannheim mate curve* of α. By the principal normal (binormal) line, we mean a straight line in a direction of the principal normal (binormal) vector field. A function $f : I \subset R \to I^\star \subset R$ is some smooth function.

Remark 3.1. According to the Definition 3.1, the principal normal line $l = span\{N\}$ of α contains the corresponding points of the curves α and $\alpha^\star$, which implies relation $\alpha^\star - \alpha = \lambda N$ for some smooth function λ on I. In [10], a special Frenet curve C in E^4 is called a *generalized Mannheim curve*, if there exists a special Frenet curve $\hat{C}$ in E^4 such that the first normal line at each point of C is included in the plane generated by the second normal line and the third normal line of $\hat{C}$ at the corresponding point under bijection $\phi : C \to \hat{C}$. Note that this definition of a generalized Mannheim curve in E^4 in general case does not implies relation $\alpha^\star - \alpha = \lambda N$, which is used in proofs of the theorems in [10].

Let $\alpha : I \to E_1^4$ be a generalized partially null Mannheim curve in E_1^4 with the Frenet frame $\{T, N, B_1, B_2\}$ and $\alpha^\star : I^\star \to E_1^4$ a generalized Mannheim mate curve of α with Cartan or Frenet frame $\{T^\star, N^\star, B_1^\star, B_2^\star\}$. Since the principal normal vector N lies in the plane spanned by $\{B_1^\star, B_2^\star\}$, it holds $N(s) = a(s)B_1^\star(s) + b(s)B_2^\star(s)$ for some differentiable functions $a(s)$ and $b(s)$. Depending on the causal character of the plane span$\{B_1^\star, B_2^\star\}$, we distinguish the following three cases:

(A) the plane span$\{B_1^\star, B_2^\star\}$ is a spacelike;
(B) the plane span$\{B_1^\star, B_2^\star\}$ is a timelike;
(C) the plane span$\{B_1^\star, B_2^\star\}$ is a lightlike.

In what follows, we consider these three cases separately.

Case (A). The plane span$\{B_1^\star, B_2^\star\}$ is a spacelike.

Theorem 3.2. *There is no non-geodesic generalized partially null Mannheim curve α in Minkowski space-time whose non-geodesic generalized Mannheim mate curve $\alpha^\star$ is a timelike Frenet curve or a spacelike Frenet curve with a timelike principal normal.*

Proof. Assume that there exists a non-geodesic generalized partially null Mannheim curve $\alpha : I \to E_1^4$ whose non-geodesic generalized Mannheim mate curve $\alpha^\star : I^\star \to E_1^4$ is a timelike Frenet curve or a spacelike Frenet curve with a timelike principal normal. Then the principal normal N of α lies in a spacelike plane spanned by the spacelike vectors $B_1^\star$ and $B_2^\star$. Hence N is given by $N(s) = a(s)B_1^\star(s) + b(s)B_2^\star(s)$, where $a(s)$ and $b(s)$ are some differentiable functions. In particular, the curve $\alpha^\star$ can be parameterized by

$$\alpha^\star(f(s)) = \alpha(s) + \lambda(s)N(s), \tag{3.1}$$

where s is the arc-length parameter of α, $s^\star = f(s) = \int_0^s ||\alpha^{\star\,\prime}(t)||\,dt$ is the arc-length parameter of $\alpha^\star$ and $f : I \subset R \to I^\star \subset R$ and λ are some smooth functions.
Differentiating the relation (3.1) with respect to s and using Frenet equations (2.4), we find

$$T^\star f' = (1 - \lambda\kappa_1)T + \lambda' N + \lambda\kappa_2 B_1. \tag{3.2}$$

By taking the scalar product of (3.2) with $N = aB_1^\star + bB_2^\star$, we get

$$\lambda' = 0. \tag{3.3}$$

Therefore,

$$\lambda = constant \neq 0.$$

Substituting (3.3) in (3.2), we get

$$T^\star f' = (1 - \lambda\kappa_1)T + \lambda\kappa_2 B_1. \tag{3.4}$$

Differentiating the relation (3.4) with respect to s and using (2.2) and (2.4), we obtain

$$\epsilon_2^\star \kappa_1^\star N^\star f'^2 + T^\star f'' = (1 - \lambda\kappa_1)' T + (1 - \lambda\kappa_1)\kappa_1 N + \lambda\kappa_2' B_1 + \lambda\kappa_2 B_1'. \tag{3.5}$$

By taking the scalar product of relation (3.5) with $N = aB_1^\star + bB_2^\star$, it follows that

$$1 - \lambda\kappa_1 = 0. \tag{3.6}$$

Moreover, by using (3.4) we obtain

$$g(T^\star f', T^\star f') = \epsilon_1^\star f'^2 = (1 - \lambda\kappa_1)^2. \tag{3.7}$$

Substituting (3.6) in (3.7) yields

$$f' = 0, \tag{3.8}$$

which is a contradiction. □

Case (B). The plane span$\{B_1^\star, B_2^\star\}$ is a timelike. In this case, we obtain two theorems depending on the causal character of basis vectors $B_1^\star$ and $B_2^\star$. It is known that any timelike plane can be spanned by spacelike and timelike mutually orthogonal vectors, or else by the two linearly independent null vectors. The next theorem can be proved in a similar way as the Theorem 3.1, so we omit its proof.

Theorem 3.3. *There is no non-geodesic generalized partially null Mannheim curve α in Minkowski space-time whose non-geodesic generalized Mannheim mate curve $\alpha^\star$ is a spacelike Frenet curve with a spacelike (timelike) first binormal and a timelike (spacelike) second binormal.*

Theorem 3.4. *There is no non-geodesic generalized partially null Mannheim curve α in Minkowski space-time whose non-geodesic generalized Mannheim mate curve $\alpha^\star$ is partially null Frenet curve.*

Proof. Assume that there exists a non-geodesic generalized partially null Mannheim curve $\alpha : I \to E_1^4$ whose non-geodesic generalized Mannheim mate curve $\alpha^\star : I^\star \to E_1^4$ is a partially null Frenet curve. Consequently, the principal normal N of α lies in the timelike plane spanned by two linearly independent null vectors $B_1^\star$ and $B_2^\star$ and $\alpha^\star$ can be parameterized by

$$\alpha^\star(f(s)) = \alpha(s) + \lambda(s)N(s), \tag{3.9}$$

where s is the arc-length parameter of α, $s^\star = f(s) = \int_0^s ||\alpha^{\star\prime}(t)||\, dt$ is the arc-length parameter of $\alpha^\star$ and $f : I \subset R \to I^\star \subset R$ and λ are some smooth functions.
Differentiating the relation (3.9) with respect to s and using Frenet equations (2.4), we find

$$T^\star f' = (1 - \lambda\kappa_1)T + \lambda' N + \lambda\kappa_2 B_1. \tag{3.10}$$

By taking the scalar product of (3.10) with $N = aB_1^\star + bB_2^\star$, we get

$$\lambda' = 0. \tag{3.11}$$

Therefore,

$$\lambda = constant \neq 0.$$

Substituting (3.11) in (3.10), we find

$$T^\star f' = (1 - \lambda\kappa_1)T + \lambda\kappa_2 B_1. \tag{3.12}$$

Differentiating the relation (3.12) with respect to s and using the Frenet equations (2.2) and (2.4), we obtain

$$\kappa_1^\star N^\star f'^2 + T^\star f'' = (1 - \lambda\kappa_1)'T + (1 - \lambda\kappa_1)\kappa_1 N + \lambda\kappa_2' B_1 + \lambda\kappa_2 B_1'. \tag{3.13}$$

By taking the scalar product of relation (3.13) with $N = aB_1^\star + bB_2^\star$, it follows that

$$1 - \lambda\kappa_1 = 0. \tag{3.14}$$

Moreover, by using (3.12) we obtain

$$g(T^\star f', T^\star f') = f'^2 = (1 - \lambda\kappa_1)^2. \tag{3.15}$$

Substituting (3.14) in (3.15) yields

$$f' = 0, \tag{3.16}$$

which is a contradiction. □

Case (C). The plane span$\{B_1^\star, B_2^\star\}$ is a lightlike. In this case, we obtain two theorems depending on the causal character of a basis vectors of a lightlike plane, which can be spanned by a null vector $B_1^\star$ and a spacelike vector $B_2^\star$, or else by a spacelike vector $B_1^\star$ and a null vector $B_2^\star$.

Theorem 3.5. *There is no non-geodesic generalized partially null Mannheim curve α in E_1^4 whose non-geodesic generalized Mannheim mate curve is a null Cartan curve.*

Proof. Assume that there exists a non-geodesic generalized partially null Mannheim curve $\alpha : I \to E_1^4$ whose non-geodesic generalized Mannheim mate curve $\alpha^\star : I^\star \to E_1^4$ is a null Cartan curve. Hence the principal normal N of α lies in a lightlike plane spanned by a null vector $B_1^\star$ and a spacelike vector $B_2^\star$ and $\alpha^\star$ can be parameterized by

$$\alpha^\star(f(s)) = \alpha(s) + \lambda(s)N(s), \tag{3.17}$$

where s is the arc-length parameter of α, $s^\star = f(s)$ is the pseudo-arc parameter of $\alpha^\star$ and $f : I \subset R \to I^\star \subset R$ and λ are some smooth functions. Differentiating the relation (3.17) with respect to s and using the Frenet equations (2.1) and (2.4), we find

$$T^\star f' = (1 - \lambda\kappa_1)T + \lambda' N + \lambda\kappa_2 B_1. \tag{3.18}$$

By taking the scalar product of (3.18) with $N = aB_1^\star + bB_2^\star$, we get

$$af' = \lambda'. \tag{3.19}$$

Moreover, by using (3.18) we obtain

$$g(T^\star f', T^\star f') = (1 - \lambda\kappa_1)^2 + \lambda'^2 = 0. \tag{3.20}$$

It follows that

$$\lambda' = 0, \quad 1 - \lambda\kappa_1 = 0.$$

Substituting $\lambda' = 0$ in (3.19), we find

$$a = 0. \tag{3.21}$$

Therefore,

$$N = \pm B_2^\star. \tag{3.22}$$

Differentiating the last relation with respect to s and using (2.1) and (2.4), we obtain

$$-\kappa_1 T + \kappa_2 B_1 = \mp\kappa_3^\star T^\star f'.$$

The last relation implies

$$g(-\kappa_1 T + \kappa_2 B_1, -\kappa_1 T + \kappa_2 B_1) = \kappa_1^2 = 0,$$

which is a contradiction. □

If a lightlike plane span$\{B_1^\star, B_2^\star\}$ is spanned by a spacelike vector $B_1^\star$ and a null vector $B_2^\star$, the following theorem can be proved.

Theorem 3.6. *There is no non-geodesic generalized partially null Mannheim curve α in E_1^4 whose non-geodesic generalized Mannheim mate curve $\alpha^\star$ is a pseudo null Frenet curve.*

Proof. Assume that there exists a non-geodesic generalized partially null Mannheim curve $\alpha : I \to E_1^4$ whose non-geodesic generalized Mannheim mate curve $\alpha^\star : I^\star \to E_1^4$ is a pseudo null curve. Therefore, the principal normal N of α lies in a lightlike plane spanned by a spacelike vector $B_1^\star$ and a null vector $B_2^\star$, so $\alpha^\star$ can be parameterized as

$$\alpha^\star(f(s)) = \alpha(s) + \lambda(s)N(s), \tag{3.23}$$

where s is the arc-length parameter of α, $s^\star = f(s)$ is the arc-length parameter of $\alpha^\star$ and $f : I \subset R \to I^\star \subset R$ and λ are some smooth functions. Differentiating the relation (3.23) with respect to s and using the Frenet equations (2.3) and (2.4), we find

$$T^\star f' = (1 - \lambda\kappa_1)T + \lambda' N + \lambda\kappa_2 B_1. \tag{3.24}$$

By taking the scalar product of (3.24) with $N = aB_1^\star + bB_2^\star$, we obtain

$$\lambda' = 0. \tag{3.25}$$

Substituting (3.25) in (3.24), it follows that

$$T^\star f' = (1 - \lambda\kappa_1)T + \lambda\kappa_2 B_1. \tag{3.26}$$

The last relation implies

$$g(T^\star f', T^\star f') = f'^{\,2} = (1 - \lambda\kappa_1)^2.$$

Consequently,

$$|f'| = |1 - \lambda\kappa_1|. \tag{3.27}$$

Differentiating the last relation with respect to s, we find

$$|f''| = |\lambda\kappa_1'|. \tag{3.28}$$

On the other hand, differentiating the relation (3.26) with respect to s and using (2.3) and (2.4), we get

$$N^\star f'^{\,2} + T^\star f'' = -\lambda\kappa_1' T + (1 - \lambda\kappa_1)\kappa_1 N + \lambda\kappa_2' B_1 + \lambda\kappa_2 B_1'.$$

According to relation (2.4), $B_1' = 0$ so the last relation gives

$$g(N^\star f'^{\,2} + T^\star f'', N^\star f'^{\,2} + T^\star f'') = f''^{\,2} = \lambda^2\kappa_1'^{\,2} + (1 - \lambda\kappa_1)^2\kappa_1^2.$$

By using (3.28) and the last relation, we find

$$1 - \lambda\kappa_1 = 0. \tag{3.29}$$

Substituting (3.29) in (3.27), it follows that $f' = 0$, which is a contradiction. $\square$

Analogously, we define a *generalized pseudo null Mannheim curve* as follows.

Definition 3.7. Pseudo null Frenet curve $\alpha : I \to E_1^4$ is called a *generalized pseudo null Mannheim curve*, if there exists a null Cartan or Frenet curve $\alpha^\star : I^\star \to E_1^4$ and a bijection $\phi : \alpha \to \alpha^\star$ given by $\phi(\alpha(s)) = \alpha^\star(f(s))$ such that for each $s \in I$ the principal normal line of α contains the corresponding points of the curves α and $\alpha^\star$ and lies in the plane spanned by the first and second binormal line of $\alpha^\star$.

Now we can ask the following question "Can a non-geodesic partially null Frenet curve be the mate curve of a non-geodesic generalized pseudo null Mannheim curve in Minkowski space-time?" The answer is given in the following theorem.

Theorem 3.8. *There is no non-geodesic generalized pseudo null Mannheim curve α in E_1^4 whose non-geodesic generalized Mannheim mate curve $\alpha^\star$ is a partially null Frenet curve.*

Proof. Assume that there exists a non-geodesic generalized pseudo null Mannheim curve $\alpha : I \to E_1^4$ whose non-geodesic generalized Mannheim mate curve $\alpha^\star : I^\star \to E_1^4$ is a partially null Frenet curve. Consequently, the principal normal N of α lies in a timelike plane spanned by two null vectors $B_1^\star$ and $B_2^\star$. The curve $\alpha^\star$ can be parameterized by

$$\alpha^\star(f(s)) = \alpha(s) + \lambda(s)N(s), \tag{3.30}$$

where s is the arc-length parameter of α, $s^\star = f(s) = \int_0^s ||\alpha^{\star\,\prime}(t)||\,dt$ is the arc-length parameter of $\alpha^\star$ and $f : I \subset R \to I^\star \subset R$ and λ are some smooth functions.

Differentiating the relation (3.30) with respect to s and applying the Frenet formulae (2.3) and (2.4), we obtain

$$T^\star f' = T + \lambda' N + \lambda\kappa_2 B_1. \tag{3.31}$$

From the last relation we get

$$g(T^\star f', T^\star f') = f'^{\,2} = 1 + \lambda^2\kappa_2^2. \tag{3.32}$$

Since $N = aB_1^\star + bB_2^\star$, where a and b are some differentiable functions, the condition $g(N, N) = 0$ gives $2ab = 0$. Therefore, we may consider two cases: (I) $a = 0$ and (II) $b = 0$.

Case (I) $a = 0$. Then $N = bB_2^\star$. From relation (3.31) we get

$$T^\star = \frac{1}{f'}T + \Big(\frac{\lambda'}{f'}\Big)N + \Big(\frac{\lambda\kappa_2}{f'}\Big)B_1. \tag{3.33}$$

Differentiating the relation (3.33) with respect to s and using (2.3) and (2.4), we find

$$\kappa_1^\star N^\star f' = \Big(\frac{1}{f'}\Big)'T + \Big[\frac{1}{f'} + \Big(\frac{\lambda'}{f'}\Big)' + \frac{\lambda\kappa_2\kappa_3}{f'}\Big]N + \Big(\frac{\lambda'\kappa_2}{f'} + \Big(\frac{\lambda\kappa_2}{f'}\Big)'\Big)B_1 - \frac{\lambda\kappa_2^2}{f'}B_2. \tag{3.34}$$

By taking the scalar product of (3.34) with $N = bB_2^\star$, we obtain

$$\lambda\kappa_2 = 0.$$

Since $\lambda \neq 0$, it follows that

$$\kappa_2 = 0. \tag{3.35}$$

Substituting (3.35) in (3.32) we get

$$f' = \pm 1. \tag{3.36}$$

Next, by using (3.34), (3.35) and (3.36), it follows that a spacelike vector $N^\star$ is collinear with a null vector N, which is a contradiction.

Case (II) $b = 0$. Then $N = aB_1^\star$. By taking the scalar product of (3.34) with $N = aB_1^\star$, we obtain that (3.35) holds, which implies a contradiction. $\square$

Generally, a straight line in Euclidean 3-space can not define its Frenet frame. But, in the study of Bertrand and Mannheim curves, the straight line can be regarded as a Frenet curve with arbitrary Frenet frame. Assume that the straight line l in E_1^4 is the Frenet curve with a properly chosen Frenet frame $\{T, N, B_1, B_2\}$. In the next two examples, we show that some straight lines in E_1^4 can be regarded as generalized partially null Mannheim curves whose mate curves are also straight lines.

Example 3.1. Consider two parallel straight lines in E_1^4 with parameter equations $\alpha(s) = (1, 1, 1, s)$, $\alpha^\star(s) = (1, 1, 2, s)$ and with a properly chosen and positively oriented Frenet frames

$$T = T^\star = (0, 0, 0, 1), \quad N = B_1^\star = (0, 0, 1, 0), \quad B_2 = B_2^\star = \frac{1}{\sqrt{2}}(1, 1, 0, 0),$$

$$B_1 = N^\star = \frac{1}{\sqrt{2}}(-1, 1, 0, 0,).$$

Therefore, α and $\alpha^\star$ are partially null straight line and pseudo null straight line respectively. It can be easily checked that $\alpha^\star = \alpha + N$, which means that $\{\alpha, \alpha^\star\}$ is a generalized Mannheim pair of curves.

Example 3.2. Let α and $\alpha^\star$ be two parallel straight lines in E_1^4 with parameter equations $\alpha(s) = (2, 2, -4, s)$, $\alpha^\star(s) = (2, 2, -3, s)$. Assume that the Frenet frames of α and $\alpha^\star$ are properly chosen, positively oriented and given by

$$T = T^\star = (0, 0, 0, 1), \quad N = -B_1^\star = (0, 0, 1, 0), \quad N^\star = (-1, 0, 0, 0),$$

$$B_2^\star = (0, 1, 0, 0), \quad B_1 = \frac{1}{\sqrt{2}}(1, 1, 0, 0), \quad B_2 = \frac{1}{\sqrt{2}}(-1, 1, 0, 0,).$$

Hence α and $\alpha^\star$ are partially null straight line and spacelike straight line with a timelike principal normal respectively. Since $\alpha^\star = \alpha + N$, it follows that $\{\alpha, \alpha^\star\}$ is a generalized Mannheim pair of curves.

References

[1] Bonnor, W.B., Null curves in a Minkowski space-time. Tensor, Vol. 20 (1969), 229-242.

[2] Choi, J.H., Kim, Y.H., Associated curves of a Frenet curve and their applications. Applied Mathematics and Computation, Vol. 218 (2012), 9116-9124.

[3] Choi, J.H., Kang, T.H., Kim, Y.H., Mannheim curves in 3-dimensional space forms. Bull. Korean Math. Soc., Vol. 50 No. 4 (2013), 1099-1108.

[4] Eisenhart, L.P., A Treatise on the Differential Geometry of Curves and Surfaces. Dover Publication 1960.

[5] Ekmekci, N., Ilarslan, K., On Bertrand curves and their characterization. Differ. Geom. Dyn. Syst., Vol. 3 No. 2 (2001), 17-24.

[6] Ersoy, S., Tosun, M., Matsuda, H., Generalized Mannheim curves in Minkowski space-time E_1^4. Hokkaido Mathematical Journal, Vol. 41 No. 3 (2012), 441-461.

[7] Grbović, M., Nešović, E., On null and pseudo null Mannheim curves in Minkowski 3-space. J. Geom., Vol. 105 (2014), 177-183.

[8] Kuhnel, W., Differential geometry: curves-surfaces-manifolds. Braunschweig, Wiesbaden, 1999.

[9] Liu, H., Wang, F., Mannheim partner curves in 3-space. J. Geom., Vol. 88 (2008), 120-126.

[10] Matsuda, H., Yorozu, S., On generalized Mannheim curves in Euclidean 4-space. Nihonkai Math. J., Vol. 20 (2009), 33-56.

[11] O'Neill, B., Semi-Riemannian geometry with applications to relativity. New York: Academic press 1983.

[12] Walrave, J., Curves and surfaces in Minkowski space. Doctoral thesis, Faculty of Science, Leuven, 1995.