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Istratescu-Suzuki-Ćirić-type fixed points results in the framework of G-metric spaces

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Abstract

The aim of this paper is to present fixed point results of convex contraction, convex contraction of order 2, weakly Zamfirescu and Ćirić strong almost contraction mappings in the framework of G-metric spaces. Some examples are presented to support the results proved herein. As an application, we derive Suzuki type fixed point in G-metric spaces. Our results generalize and extend various results in the existing literature. We also present some examples to illustrate our new theoretical results.

Keywords: Coincidence point, convex contraction, convex contraction of order 2, Ćirić strong almost contraction.
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1. Introduction and preliminaries

Over the past two decades, the development of fixed point theory in metric spaces has attracted a considerable attention due to numerous applications in areas such as variational inequalities, optimization, and approximation theory. Mustafa and Sims [25] generalized the concept of a metric in which to every triplet of points of an abstract set, a real number is assigned. Based on the notion of generalized metric

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spaces, Mustafa et al. [22, 24, 26] obtained several fixed point theorems for mappings satisfying different contractive conditions. On the other hand, Mustafa et al. [18–24, 26] also obtained some interesting fixed point results for mappings satisfying new different contractive conditions. Chugh et al. [13] obtained some fixed point results for maps satisfying property P in G-metric spaces. Saadati et al. [30] studied fixed point of contractive mappings in partially ordered G-metric spaces. Shatanawi [33] obtained fixed points of Φ-maps in G-metric spaces. For more details, we refer to [1–7, 9, 11, 12, 27–29, 34, 35].

Jleli and Samet [16] (see also, [32]) observed that some fixed point results in the context of a G-metric space can be deduced by some existing results in the setting of a (quasi-) metric space. In fact, if the contraction condition of the fixed point theorem on a G-metric space can be reduced to two variables instead of three variables, then one can construct an equivalent fixed point theorem in the setting of a usual metric space. More precisely, they noticed that $d(x, y) = G(x, y, y)$ forms a quasi-metric. Therefore, if one can transform the contraction condition of existence results in a G-metric space in terms such as $G(x, y, y)$, then the related fixed point results become the known fixed point results in the context of a quasi-metric space.

On the other hands, Istratescu [15] introduced the notion of a convex contraction mapping. Recently, Miandaragh et al. [17] proved some fixed point results for generalized convex contractions on complete metric space.

The aim of this paper is to study the notion of convex contraction, convex contraction of order 2, weakly Zamfirescu mappings and Ćirić strong almost contraction in the setup of G-metric spaces. We obtain several fixed point results for such mappings in the setting of generalized metric spaces. As an application, Suzuki type fixed point result is also derived. Some examples are provided to support the results proved herein. Our results extend and generalize various existing results in the literature.

Consistent with Mustafa and Sims [25], the following definitions and results will be needed in the sequel.

Definition 1.1. Let X be a nonempty set. A mapping $G : X \times X \times X \to \mathbb{R}^+$ is said to be a G-metric on X, if for any $x, y, z \in X$, the following conditions hold:

(a) $G(x, y, z) = 0$, if $x = y = z$;

(b) $0 < G(x, y, z)$, for all $x, y \in X$ with $x \neq y$;

(c) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $y \neq z$;

(d) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry);

(e) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$.

The pair (X, G) is called a G-metric space [25].

Definition 1.2. A sequence $\{x_n\}$ in a G-metric space X is called:

(i) a G-Cauchy sequence if for any $\varepsilon > 0$, there is an $n_0 \in \mathbb{N}$ (the set of natural numbers) such that for all $n, m, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;

(ii) a G-convergent sequence if for any $\varepsilon > 0$, there is an $x \in X$ and an $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$, $G(x, x_n, x_m) < \varepsilon$.

A G-metric space is said to be G-complete, if every G-Cauchy sequence in X is G-convergent in X. It is known that a sequence $\{x_n\}$ converges to $x \in X$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$ [25].

Proposition 1.3 ([25])**.** *Let X be a G-metric space. Then following are equivalent:*

1. *$\{x_n\}$ is G-convergent to x.*
2. *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*

3. $G(x_n, x, x) \to 0$ *as* $n \to \infty$.
4. $G(x_n, x_m, x) \to 0$ *as* $n, m \to \infty$.

Definition 1.4. A G-metric on X is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 1.5. *Every G-metric on X will define a metric d_G on X given by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \text{for all } x, y \in X.$$

For a symmetric G-metric, we have

$$d_G(x, y) = 2G(x, y, y), \quad \text{for all } x, y \in X.$$

However, if G is not symmetric, then the following inequality holds:

$$\frac{3}{2}G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y), \quad \text{for all } x, y \in X.$$

Definition 1.6 ([31])**.** Let φ be the collection of all mappings $\psi : [0, \infty) \to [0, \infty)$ that satisfy the following conditions:

- $\sum\limits_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ;
- ψ is nondecreasing.

Definition 1.7 ([8])**.** Let X be a nonempty set and α: $X \times X \times X \to [0, \infty)$. A self mapping T on X is said to be α-admissible, if for any $x, y, z \in X$

$$\alpha(x, y, z) \geq 1 \quad \text{implies that} \quad \alpha(Tx, Ty, Tz) \geq 1.$$

Example 1.8 ([8])**.** Let $X = [0, \infty)$ and $T : X \to X$ by

$$T(x) = \begin{cases} 2 \ln x, & if\ x \neq 0; \\ e, & \text{otherwise.} \end{cases}$$

Define $\alpha : X \times X \times X \to [0, \infty)$ by

$$\alpha(x, y, z) = \begin{cases} e, & x \geq y \geq z; \\ 0, & x < y < z. \end{cases}$$

Then the mapping T is α-admissible.

2. Fixed point results for convex contractions

Definition 2.1. Let X be a G-metric space, T a self-map on X and $\epsilon > 0$ a given number. A point x in X is called

(a) an ϵ-fixed point of T, if $G(x, Tx, T^2x) < \epsilon$;

(b) approximate fixed point of T, if T has an ϵ-fixed point for all $\epsilon > 0$.

Definition 2.2. Let X be a G-metric space. A self-mapping T on X is called asymptotic regular if for any x in X, we have $G(T^n x, T^{n+1} x, T^{n+2} x) \to 0$ as $n \to \infty$.

Now, we have the following simple lemma.

Lemma 2.3. *Let X be a G-metric space and T an asymptotic regular map on X. Then T has an approximate fixed point.*

Proof. Let x_0 be a given point in X and $\epsilon > 0$. Since T is an asymptotic regular map on X, we can choose $n_0(\epsilon) \in \mathbb{N}$ such that

$$G(T^n x_0, T^{n+1} x_0, T^{n+2} x_0) < \epsilon$$

for all $n \geq n_0(\epsilon)$. That is, $G(T^n x_0, T(T^n x_0), T^2(T^n x_0)) < \epsilon$ for all $n \geq n_0(\epsilon)$. If we put $T^n(x_0) = y_0$ then $G(y_0, T(y_0), T^2(y_0)) < \epsilon$ implies that $y_0 = T^n(x_0)$ is an ϵ-fixed point of T in X. As $\epsilon > 0$ is an arbitrary number, so T has an approximate fixed point. □

Definition 2.4. Let X be a nonempty set and $\alpha, \eta : X \times X \times X \to [0, \infty)$ two mappings. A self-mapping T on X is said to be α-admissible with respect to η if for any $x, y, z \in X$

$$\alpha(x, y, z) \geq \eta(x, y, z) \quad \text{implies that} \quad \alpha(Tx, Ty, Tz) \geq \eta(Tx, Ty, Tz).$$

Example 2.5. Let $X = [0, \infty)$ and $T : X \to X$ by

$$T(x) = \begin{cases} 2 \ln x, & \text{if } \ x \neq 0; \\ e, & \text{otherwise.} \end{cases}$$

Define $\alpha, \eta : X \times X \times X \to [0, \infty)$ by

$$\alpha(x, y, z) = \begin{cases} e, & x \geq y \geq z; \\ 2, & x < y < z, \end{cases}$$

and $\eta(x, y, z) = 1$. Then the mapping T is α-admissible with respect to η.

Definition 2.6. Let X be a nonempty set and $\alpha, \eta : X \times X \times X \to [0, \infty)$ two mappings. A self-mapping T on X is said to be convex contraction, if for any x, y, z in X

$$\eta(x, Tx, Ty) \leq \alpha(x, y, z) \quad \text{implies that} \quad G(T^2x, T^2y, T^2z,) \leq aG(Tx, Ty, Tz) + bG(x, y, z), \tag{2.1}$$

where $a, b \geq 0$ with $a + b < 1$.

Definition 2.7. Let X be a nonempty set and $\alpha, \eta : X \times X \times X \to [0, \infty)$ two mappings. A self-mapping T on X is said to be convex contraction if for any x, y, z in X

$$\eta(x, Tx, Ty) \leq \alpha(x, y, z),$$

implies that

$$\begin{aligned} G(T^2x, T^2y, T^2z,) \leq a_1 G(x, Tx, Tx) + a_2 G(Tx, T^2x, T^2x) + b_1 G(y, Ty, Ty) \\ + b_2 G(Ty, T^2y, T^2y) + c_1 G(z, Tz, T^2z) + c_2 G(Tz, T^2z, T^2z), \end{aligned} \tag{2.2}$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \geq 0$ with $a_1 + a_2 + b_1 + b_2 + c_1 + c_2 < 1$.

Theorem 2.8. *Let X be a complete G-metric space and T an α-admissible convex contraction with respect to η. If $\alpha(x, Tx, Tx) \geq \eta(x, Tx, Tx)$ for any $x \in X$, then T has an approximate fixed point.*

Proof. Let x_0 be a given point in X. Since $\alpha(x_0, Tx_0, Tx_0) \geq \eta(x_0, Tx_0, Tx_0)$ and T is an α-admissible mapping with respect to η, we have $\alpha(Tx_0, T^2x_0, T^3x_0) \geq \eta(Tx_0, T^2x_0, T^3x_0)$. By continuing this way, we obtain that $\alpha(T^n x_0, T^{n+1} x_0, T^{n+2} x_0) \geq \eta(T^n x_0, T^{n+1} x_0, T^{n+2} x_0)$, for all $n \in \mathbb{N} \cup \{0\}$. Put $\vartheta = G(T^3x_0, T^2x_0, Tx_0) + G(T^2x_0, Tx_0, x_0)$ and $r = a + b$. Obviously, $G(T^3x_0, T^2x_0, Tx_0)$, $G(T^2x_0, Tx_0, x_0) \leq \vartheta$. By using $x = x_0$, $y = Tx_0$ and $z = T^2x_0$ in (2.1), we have

$$G(T^2x_0, T^3x_0, T^4x_0,) \leq aG(Tx_0, T^2x_0, T^3x_0) + bG(x_0, Tx_0, T^2x_0) \leq r\vartheta.$$

Similarly, we have

$$G(T^3x_0, T^4x_0, T^5x_0,) \leq aG(T^2x_0, T^3x_0, T^4x_0) + bG(Tx_0, T^2x_0, T^3x_0) \leq r^2\vartheta.$$

By continuing this process, we arrive at

$$G(T^m x_0, T^{m+1}x_0, T^{m+2}x_0) \leq r^l\vartheta,$$

where $m = 2l$ or $m = 2l + 1$. On taking the limit as $m \to \infty$ on both sides of above inequality, we have $G(T^m x_0, T^{m+1}x_0, T^{m+2}x_0) \to 0$, for any $x_0 \in X$. By Lemma 2.3, T has an approximate fixed point. □

Let T be a self-mapping on a nonempty set X and $\alpha,\ \eta : X \times X \times X \to [0, \infty)$. We say that the set X has $H^\star-$property if for any $x, y \in Fix(T)$ with $\alpha(x, y, y) < \eta(x, Tx, Tx)$, there exists $z \in X$ such that $\alpha(x, z, z) \geq \eta(x, z, z)$ and $\alpha(y, z, z) \geq \eta(y, z, z)$. Also for any $x, y \in X$, we have $\eta(x, Tx, Tx) \leq \eta(x, y, z)$.

Theorem 2.9. *Let X be a complete G-metric space and T a continuous convex contraction and α-admissible mapping with respect to η. Suppose that there exists a point x_0 in X such that*

$$\alpha(x_0, Tx_0, Tx_0) \geq \eta(x_0, Tx_0, Tx_0).$$

Then T has a fixed point. Moreover, T has a unique fixed point provided that X has $H^\star-$property.

Proof. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0$, for all $n \in \mathbb{N}$. Since T is an α-admissible mapping with respect to η and $\alpha(x_0, x_1, x_1) = \alpha(x_0, Tx_0, , Tx_0) \geq \eta(x_0, Tx_0, Tx_0)$, we have $\alpha(Tx_0, T^2x_0, T^2x_0) = \alpha(x_1, x_2, x_2) \geq \eta(Tx_0, T^2x_0, T^2x_0) = \eta(x_1, x_2, x_2)$. By continuing this way, we obtain that $\alpha(x_n, x_{n+1}, z_{n+1}) \geq \eta(x_n, x_{n+1}, x_{n+1}) = \eta(x_n, Tx_n, Tx_n)$, for all $n \in \mathbb{N} \cup \{0\}$. Also, from (2.1), we have

$$\begin{aligned} G(x_{n+2}, x_{n+3}, x_{n+4}) &= G(T^{n+2}x_0, T^{n+3}x_0, T^{n+4}x_0) = G(T^2(T^n x_0), T^2(T^{n+1}x_0), T^2(T^{n+2}x_0)) \\ &\leq aG(T(T^n x_0), T(T^{n+1}x_0), T(T^{n+2}x_0)) + bG(T^n x_0, T^{n+1}x_0, T^{n+2}x_0) \\ &= aG(x_{n+1}, x_{n+2}, x_{n+3}) + bG(x_n, x_{n+1}, x_{n+2}). \end{aligned}$$

We set $\vartheta = G(x_3, x_2, x_1) + G(x_2, x_1, x_0)$ and $r = a + b$. Then

$$G(x_m, x_{m+1}, x_{m+2}) \leq r^l\vartheta,$$

where $m = 2l$ or $m = 2l + 1$. Suppose that $m = 2l$. Then for $n, k = 2p$ with $p > 2$, $l \geq 1$ and $m < n, k$ we have

$$\begin{aligned} G(x_m, x_n, x_k) &\leq G(x_m, x_{m+1}, x_{m+1}) + G(x_{m+1}, x_{m+2}, x_{m+2}) + G(x_{m+2}, x_{m+3}, x_{m+3}) \\ &\quad + \cdots + G(x_{n-2}, x_{n-1}, x_{n-l}) + G(x_{n-1}, x_n, x_k) \\ &= G(x_{2l}, x_{2l+1}, x_{2l+1}) + G(x_{2l+1}, x_{2l+2}, x_{2l+2}) \\ &\quad + G(x_{2l+2}, x_{2l+3}, x_{2l+3}) + \cdots + G(x_{2p-2}, x_{2p-1}, x_{2p-l}) + G(x_{2p-1}, x_{2p}, x_{2p}) \\ &\leq r^l\vartheta + r^l\vartheta + r^{l+1}\vartheta + \cdots + r^{p-1}\vartheta \\ &= 2r^l\vartheta + 2r^{l+1}\vartheta + 2r^{l+2}\vartheta + \cdots + 2r^{p-1}\vartheta \\ &\leq \frac{2r^l}{1-r}\vartheta. \end{aligned}$$

Similarly, for $n, k = 2p + 1$ with $p \geq 1$, $l \geq 1$ and $m < n, k$ we have

$$G(x_m, x_n, x_k) \leq \frac{2r^l}{1-r}\vartheta.$$

Now, assume that $m = 2l + 1$. Then for $n = 2p$ with $p \geq 2$, $l \geq 1$ and $m < n$, we have

$$\begin{aligned} G(x_m, x_n, x_k) &\leq G(x_m, x_{m+1}, x_{m+1}) + G(x_{m+1}, x_{m+2}, x_{m+2}) + G(x_{m+2}, x_{m+3}, x_{m+3}) \\ &\quad + \cdots + G(x_{n-2}, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_n, x_k) \\ &= G(x_{2l+1}, x_{2l+2}, x_{2l+2}) + G(x_{2l+2}, x_{2l+3}, x_{2l+3}) \\ &\quad + G(x_{2l+3}, x_{2l+4}, x_{2l+4}) + \cdots + G(x_{2p-2}, x_{2p-1}, x_{2p-l}) + G(x_{2p-1}, x_{2p}, x_{2p}) \\ &\leq r^l\vartheta + r^{l+1}\vartheta + r^{l+1}\vartheta + \cdots + r^p\vartheta \\ &\leq 2r^l\vartheta + 2r^{l+1}\vartheta + 2r^{l+2}\vartheta + \cdots + 2r^p\vartheta \\ &\leq \frac{2r^l}{1-r}\vartheta. \end{aligned}$$

Similarly, for $n, k = 2p + 1$ with $p \geq 1$, $l \geq 1$ and $m < n, k$ we obtain that

$$G(x_m, x_n, x_k) \leq \frac{2r^l}{1-r}\vartheta.$$

Hence, for all $m, n, k \in \mathbb{N}$ with $m < n, k$, we have

$$G(x_m, x_n, x_k) \leq \frac{2r^l}{1-r}\vartheta.$$

On taking the limit as $l \to \infty$ on both sides of above inequality, we have $G(x_m, x_n, x_k) \to 0$ which implies that $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $z \in X$ such that $x_n \to z$ and $n \to \infty$ and hence $Tz = z$ as T is a continuous mapping.

Let $x, y \in Fix(T)$ where $x \neq y$. To prove the uniqueness, we consider the following cases.

(i) If $\alpha(x, y, y) \geq \eta(x, Tx, Tx)$.

As T is a convex contraction, so we have

$$\begin{aligned} G(x, y, y) &= G(T^2x, T^2y, T^2y) \leq aG(Tx, Ty, Ty) + bG(x, y, y) \\ &= aG(x, y, y) + bG(x, y, y) \\ &= (a + b)G(x, y, y) \\ &< G(x, y, y), \end{aligned}$$

a contraction.

(ii) If $\alpha(x, y, y) < \eta(x, Tx, Tx)$.

Since X has $H^\star-$property, there exists $z \in X$ such that $\alpha(x, z) \geq \eta(x, z)$ and $\alpha(y, z, z) \geq \eta(y, z, z)$. Also, T is an α-admissible mapping with respect to η, we have $\alpha(x, T^nz, T^nz) \geq \eta(x, T^nz, T^nz) \geq \eta(x, Tx, Tx)$ and $\alpha(y, T^nz, T^nz) \geq \eta(y, T^nz, T^nz) \geq \eta(y, Ty, Ty)$. If $\alpha(x, T^nz, T^nz) \geq \eta(x, Tx, Tx)$, then we have

$$G(x, T^{n+2}z, T^{n+2}z) \leq aG(x, T^{n+1}z, T^{n+1}z) + bG(x, T^nz, T^nz).$$

By taking $\vartheta = G(x, Tz, Tz) + G(x, z, z)$ and $r = a + b < 1$, we have

$$G(x, T^mz, T^mz) \leq r^l\vartheta,$$

where $m = 2l$ or $m = 2l+1$ which on taking the limit as $m \to \infty$ implies that $T^mz \to x$. Similarly, $T^mz \to y$ as $m \to \infty$. Hence $x = y$, a contradiction. Thus the result follows. □

Example 2.10. Let $X = \{0, 1, 2\}$ be a set. Let $G : X \times X \times X \to [0, \infty)$ be defined by

(x, y, z)	$G(x, y, z)$
(0,0,0), (1,1,1), (2,2,2)	0
(0,1,1), (1,0,1), (1,1,0)	1
(0,0,1), (0,1,0), (1,0,0)	2
(1,2,2), (2,1,2), (2,2,1)	2
(0,0,2), (0,2,0), (2,0,0)	3
(0,2,2), (2,0,2), (2,2,0)	3
(1,1,2), (1,2,1), (2,1,1)	4
(0,1,2), (0,2,1), (1,0,2)	4
(1,2,0), (2,0,1), (2,1,0)	4

It is clear that G is a non-symmetric G-metric as $G(0, 0, 1) \neq G(0, 1, 1)$. Let $T : X \to X$ be defined by

x	0	1	2
$T(x)$	1	0	2

Now,

(Tx, Ty, Tz)	$G(Tx, Ty, Tz)$	(T^2x, T^2y, T^2z)	$G(T^2x, T^2y, T^2z)$
(1,1,1), (0,0,0), (2,2,2)	0	(0,0,0), (1,1,1), (2,2,2)	0
(1,0,0), (0,1,0), (0,0,1)	2	(0,1,1), (1,0,1), (1,1,0)	1
(1,1,0), (1,0,1), (0,1,1)	1	(0,0,1), (0,1,0), (1,0,0)	2
(0,2,2), (2,0,2), (2,2,0)	3	(1,2,2), (2,1,2), (2,2,1)	2
(1,1,2), (1,2,1), (2,1,1)	4	(0,0,2), (0,2,0), (2,0,0)	3
(1,2,2), (2,1,2), (2,2,1)	2	(0,2,2), (2,0,2), (2,2,0)	3
(0,0,2), (0,2,0), (2,0,0)	3	(1,1,2), (1,2,1), (2,1,1)	4
(1,0,2), (1,2,0), (0,1,2)	4	(0,1,2), (0,2,1), (1,0,2)	4
(0,2,1), (2,1,0), (2,0,1)	4	(1,2,0), (2,0,1), (2,1,0)	4

Define $\alpha, \eta : X \times X \times X \to [0, \infty)$ by

$$\alpha(x, y, z) = 4 + xyz \quad \text{and} \quad \eta(x, y, z) = xyz.$$

For $x \neq y \neq z$ we consider the following cases to check that T is convex contraction.

Case-I: For $x = 0$, $y = 1$ and $z = 2$.

$$\begin{aligned} G(T^2x, T^2y, T^2z) = 4 &\leq a(4) + b(4) \\ &= aG(0, 1, 2) + bG(0, 1, 2) \\ &= aG(Tx, Ty, Tz) + bG(x, y, z). \end{aligned}$$

Case-II: For $x = 0$, $y = 2$ and $z = 1$.

$$\begin{aligned} G(T^2x, T^2y, T^2z) = 4 &\leq a(4) + b(4) \\ &= aG(0, 2, 1) + bG(0, 2, 1) \\ &= aG(Tx, Ty, Tz) + bG(x, y, z). \end{aligned}$$

Case-III: For $x = 1$, $y = 0$ and $z = 2$.

$$\begin{aligned} G(T^2x, T^2y, T^2z) = 4 &\leq a(4) + b(4) \\ &= aG(1, 0, 2) + bG(1, 0, 2) \\ &= aG(Tx, Ty, Tz) + bG(x, y, z). \end{aligned}$$

Case-IV: For $x = 1$, $y = 2$ and $z = 0$.

$$\begin{aligned} G(T^2x, T^2y, T^2z) = 4 &\leq a(4) + b(4) \\ &= aG(1, 2, 0) + bG(1, 2, 0) \\ &= aG(Tx, Ty, Tz) + bG(x, y, z). \end{aligned}$$

Case-V: For $x = 2$, $y = 0$ and $z = 1$.

$$\begin{aligned} G(T^2x, T^2y, T^2z) = 4 &\leq a(4) + b(4) \\ &= aG(2, 0, 1) + bG(2, 0, 1) \\ &= aG(Tx, Ty, Tz) + bG(x, y, z). \end{aligned}$$

Case-VI: For $x = 2$, $y = 1$ and $z = 0$.

$$\begin{aligned} G(T^2x, T^2y, T^2z) = 4 &\leq a(4) + b(4) \\ &= aG(2, 1, 0) + bG(2, 1, 0) \\ &= aG(Tx, Ty, Tz) + bG(x, y, z). \end{aligned}$$

Thus, in all cases T is convex contraction with $a, b \leq \frac{1}{2}$ with $a \neq b$. Hence all the conditions of Theorem 2.8 are satisfied and 2 is a fixed point of T.

Remark 2.11. A G-metric naturally induces a metric d_G given by $d_G(x, y) = G(x, y, y) + G(x, x, y)$. If the G-metric is not symmetric, the inequalities (2.1) do not reduce to any metric inequality with the metric d_G. Hence our results do not reduce to fixed point problems in the corresponding metric space (X, d_G). For instance, if we take $x = 0$ and $y = 2$ in above example, we obtain $d_G(T^2x, T^2y) = 2$, $d_G(Tx, Ty) = 1$, $d_G(x, y) = 2$, so there does not exist any $a, b \geq 0$ with $a + b < 1$ such that $d_G(T^2x, T^2y) \leq ad_G(Tx, Ty) + bd_G(x, y)$ holds. So, we can not apply the result of [14] to obtain fixed point of T.

Theorem 2.12. *Let X be a complete G-metric space, T a convex contraction of order 2 α-admissible with respect to η and $\alpha(x, Tx, Tx) \geq \eta(x, Tx, Tx)$ for all $x \in X$. Then T has an approximate fixed point.*

Proof. Let x_0 be a given point in X. The following arguments similar to those in the proof of Theorem 2.8 we obtain that $\alpha(T^n x_0, T^{n+1}x_0, T^{n+2}x_0) \geq \eta(T^n x_0, T^{n+1}x_0, T^{n+2}x_0)$, for all $n \in \mathbb{N}$. We set $r = a_1 + a_2 + b_1 + c_1$, $\beta = 1 - b_2 - c_2$ and $\vartheta = G(T^2x_0, T^2x_0, Tx_0) + G(Tx_0, Tx_0, x_0)$. From (2.2) with $x = x_0$ and $y = z = Tx_0$ we have

$$\begin{aligned} G(T^2x_0, T^3x_0, T^3x_0) \leq\ & a_1G(x_0, Tx_0, Tx_0) + a_2G(Tx_0, T^2x_0, T^2x_0) + b_1G(Tx_0, T^2x_0, T^2x_0) \\ & + b_2G(T^2x_0, T^3x_0, T^3x_0) + c_1G(Tx_0, T^2x_0, T^2x_0) + c_2G(T^2x_0, T^3x_0, T^3x_0), \end{aligned}$$

which implies that

$$\begin{aligned} \beta G(T^2x_0, T^3x_0, T^3x_0) &= (1 - b_2 - c_2)G(T^2x_0, T^3x_0, T^3x_0) \\ &\leq a_1G(x_0, Tx_0, Tx_0) + (a_2 + b_1 + c_1)G(Tx_0, T^2x_0, T^2x_0) \\ &\leq r\vartheta. \end{aligned}$$

Thus $G(T^2x_0, T^3x_0, T^3x_0) \leq \left(\frac{r}{\beta}\right)\vartheta$. Again from (2.2) with $x = Tx_0$ and $y = z = T^2x_0$, we have

$$\begin{aligned} G(T^3x_0, T^4x_0, T^4x_0) \leq\ & a_1G(Tx_0, T^2x_0, T^2x_0) + a_2G(T^2x_0, T^3x_0, T^3x_0) + b_1G(T^2x_0, T^3x_0, T^3x_0) \\ & + b_2G(T^3x_0, T^4x_0, T^4x_0) + c_1G(T^2x_0, T^3x_0, T^3x_0) + c_2G(T^3x_0, T^4x_0, T^4x_0), \end{aligned}$$

$$\begin{aligned}\beta G(T^3x_0, T^4x_0, T^4x_0) &\leq r\vartheta,\\ G(T^3x_0, T^4x_0, T^4x_0) &\leq \left(\frac{r}{\beta}\right)\vartheta,\end{aligned}$$

which implies that

$$\begin{aligned}\beta G(T^3x_0, T^4x_0, T^4x_0) &= (1-b_2-c_2)G(T^3x_0, T^4x_0, T^4x_0)\\ &\leq a_1G(Tx_0, T^2x_0, T^2x_0) + (a_2+b_1+c_1)G(T^2x_0, T^3x_0, T^3x_0)\\ &\leq r\vartheta.\end{aligned}$$

Hence $G(T^3x_0, T^4x_0, T^4x_0) \leq \left(\frac{r}{\beta}\right)\vartheta$. Similarly, we have

$$\begin{aligned}G(T^4x_0, T^5x_0, T^5x_0) &\leq \left(\frac{r}{\beta}\right)^2\vartheta,\\ G(T^5x_0, T^6x_0, T^6x_0) &\leq \left(\frac{r}{\beta}\right)^2\vartheta.\end{aligned}$$

By continuing this way, we can obtain that $G(T^mx_0, T^{m+1}x_0, T^{m+1}x_0) \leq \left(\frac{r}{\beta}\right)^l\vartheta$, where $m = 2l$ or $m = 2l+1$. Hence $G(T^mx_0, T^{m+1}x_0, T^{m+1}x_0) \to 0$ as $m \to \infty$, which by Lemma 2.3 implies that T has an approximate fixed point. □

Theorem 2.13. *Let X be a complete G-metric space, T a continuous convex contraction of order 2 and α-admissible mapping with respect to η. Suppose that there exists a point x_0 in X such that $\alpha(x_0, Tx_0, Tx_0) \geq \eta(x_0, Tx_0, Tx_0)$. Then T has a fixed point. Moreover, T has a unique fixed point provided that X has $H^\star-$property.*

Proof. Define a sequence $\{x_n\}$ in X by $x_n = T^nx_0$, for all $n \in \mathbb{N}$. We set $r = a_1+a_2+b_1+c_1$, $\beta = 1-b_2-c_2$ and $\vartheta = G(T^2x_0, T^2x_0, Tx_0) + G(Tx_0, Tx_0, x_0)$. From (2.2) with $x = x_0$ and $y = z = Tx_0$, we have

$$\begin{aligned}G(T^2x_0, T^3x_0, T^3x_0) &\leq a_1G(x_0, Tx_0, Tx_0) + a_2G(Tx_0, T^2x_0, T^2x_0) + b_1G(Tx_0, T^2x_0, T^2x_0)\\ &\quad + b_2G(T^2x_0, T^3x_0, T^3x_0) + c_1G(Tx_0, T^2x_0, T^2x_0) + c_2G(T^2x_0, T^3x_0, T^3x_0)\\ &\leq r\vartheta,\end{aligned}$$

which implies that

$$\begin{aligned}\beta G(T^2x_0, T^3x_0, T^3x_0) &= (1-b_2-c_2)G(T^2x_0, T^3x_0, T^3x_0)\\ &\leq a_1G(x_0, Tx_0, Tx_0) + (a_2+b_1+c_1)G(Tx_0, T^2x_0, T^2x_0)\\ &\leq r\vartheta.\end{aligned}$$

Thus, $G(T^2x_0, T^3x_0, T^3x_0) \leq \left(\frac{r}{\beta}\right)\vartheta$.

Again from (2.2) with $x = Tx_0$ and $y = z = T^2x_0$, we have

$$\begin{aligned}G(T^3x_0, T^4x_0, T^4x_0) &\leq a_1G(Tx_0, T^2x_0, T^2x_0) + a_2G(T^2x_0, T^3x_0, T^3x_0) + b_1G(T^2x_0, T^3x_0, T^3x_0)\\ &\quad + b_2G(T^3x_0, T^4x_0, T^4x_0) + c_1G(T^2x_0, T^3x_0, T^3x_0) + c_2G(T^3x_0, T^4x_0, T^4x_0),\end{aligned}$$

which implies that

$$\begin{aligned}\beta G(T^3x_0, T^4x_0, T^4x_0) &= (1-b_2-c_2)G(T^3x_0, T^4x_0, T^4x_0)\\ &\leq a_1G(Tx_0, T^2x_0, T^2x_0) + (a_2+b_1+c_1)G(T^2x_0, T^3x_0, T^3x_0)\\ &\leq r\vartheta.\end{aligned}$$

Hence $G(T^3x_0, T^4x_0, T^4x_0) \leq \left(\frac{r}{\beta}\right)\vartheta$.

Similarly, we have

$$G(T^4x_0, T^5x_0, T^5x_0) \leq \left(\frac{r}{\beta}\right)^2 \vartheta,$$
$$G(T^5x_0, T^6x_0, T^6x_0) \leq \left(\frac{r}{\beta}\right)^2 \vartheta.$$

By continuing this way, we obtain that $G(T^m x_0, T^{m+1}x_0, T^{m+1}x_0) \leq \left(\frac{r}{\beta}\right)^l \vartheta$, where $m = 2l$ or $m = 2l+1$. Now for $m = 2l$, $n, k = 2p$ with $p > 2, l \geq 1$ and $m < n, k$, we have

$$\begin{aligned}
G(x_m, x_n, x_k) &\leq G(x_m, x_{m+1}, x_{m+1}) + G(x_{m+1}, x_{m+2}, x_{m+2}) + G(x_{m+2}, x_{m+3}, x_{m+3}) \\
&\quad + \cdots + G(x_{n-2}, x_{n-1}, x_{n-l}) + G(x_{n-1}, x_n, x_k) \\
&= G(x_{2l}, x_{2l+1}, x_{2l+1}) + G(x_{2l+1}, x_{2l+2}, x_{2l+2}) \\
&\quad + G(x_{2l+2}, x_{2l+3}, x_{2l+3}) + \cdots + G(x_{2p-2}, x_{2p-1}, x_{2p-l}) + G(x_{2p-1}, x_{2p}, x_{2p}) \\
&\leq \left(\frac{r}{\beta}\right)^l \vartheta + \left(\frac{r}{\beta}\right)^l \vartheta + \left(\frac{r}{\beta}\right)^{l+1} \vartheta + \cdots + \left(\frac{r}{\beta}\right)^{p-1} \vartheta \\
&= 2\left(\frac{r}{\beta}\right)^l \vartheta + 2\left(\frac{r}{\beta}\right)^{l+1} \vartheta + 2\left(\frac{r}{\beta}\right)^{l+2} \vartheta + \cdots + 2\left(\frac{r}{\beta}\right)^{p-1} \vartheta \\
&\leq \frac{2\left(\frac{r}{\beta}\right)^l}{1 - \left(\frac{r}{\beta}\right)} \vartheta.
\end{aligned}$$

Similarly, for $m = 2l$ and $n, k = 2p + 1$ with $p \geq 1$, $l \geq 1$ and $m < n, k$ we get

$$G(x_m, x_n, x_k) \leq \frac{2\left(\frac{r}{\beta}\right)^l}{1 - \left(\frac{r}{\beta}\right)} \vartheta.$$

If $m = 2l + 1$, then for $n = 2p$ with $p \geq 2$, $l \geq 1$ and $m < n$ we have

$$\begin{aligned}
G(x_m, x_n, x_k) &\leq G(x_m, x_{m+1}, x_{m+1}) + G(x_{m+1}, x_{m+2}, x_{m+2}) + G(x_{m+2}, x_{m+3}, x_{m+3}) \\
&\quad + \cdots + G(x_{n-2}, x_{n-1}, x_{n-l}) + G(x_{n-1}, x_n, x_k) \\
&= G(x_{2l+1}, x_{2l+2}, x_{2l+2}) + G(x_{2l+2}, x_{2l+3}, x_{2l+3}) \\
&\quad + G(x_{2l+3}, x_{2l+4}, x_{2l+4}) + \cdots + G(x_{2p-2}, x_{2p-1}, x_{2p-l}) + G(x_{2p-1}, x_{2p}, x_{2p}) \\
&\leq \left(\frac{r}{\beta}\right)^l \vartheta + \left(\frac{r}{\beta}\right)^{l+1} \vartheta + \left(\frac{r}{\beta}\right)^{l+1} \vartheta + \cdots + \left(\frac{r}{\beta}\right)^{p} \vartheta \\
&\leq 2\left(\frac{r}{\beta}\right)^l \vartheta + 2\left(\frac{r}{\beta}\right)^{l+1} \vartheta + 2\left(\frac{r}{\beta}\right)^{l+2} \vartheta + \cdots + 2\left(\frac{r}{\beta}\right)^{p} \vartheta \\
&\leq \frac{2\left(\frac{r}{\beta}\right)^l}{1 - \left(\frac{r}{\beta}\right)} \vartheta.
\end{aligned}$$

Similarly, for $m = 2l + 1$ and $n, k = 2p + 1$ with $p \geq 1$, $l \geq 1$ and $m < n, k$, we have

$$G(x_m, x_n, x_k) \leq \frac{2\left(\frac{r}{\beta}\right)^l}{1 - \left(\frac{r}{\beta}\right)} \vartheta.$$

Hence, for all $m, n, k \in \mathbb{N}$ with $m < n, k$ we obtain that

$$G(x_m, x_n, x_k) \leq \frac{2\left(\frac{r}{\beta}\right)^l}{1 - \left(\frac{r}{\beta}\right)} \vartheta.$$

By taking the limit as $l \to \infty$ in the above inequality we get $\{G(x_m, x_n, x_k)\}$ converges to 0. Since (X, G) is a complete G-metric space, we have $x_n \to z$ and $n \to \infty$ for some $z \in X$. By continuity of T, $Tz = z$. By following arguments similar to those in proof of Theorem 2.9, we obtain the uniqueness of fixed point of T provided that X has $H^\star$-property. □

3. α-η-weakly Zamfirescu mappings

In this section we obtain fixed point results of α-η-weakly Zamfirescu mapping in the framework of G-metric spaces.

Definition 3.1. Let T be a self-mapping on a G-metric space X and $a, b \in \mathbb{R}^+$ with $0 < a \leq b$. If there exists a mapping $\gamma : X \times X \times X \to [0, 1]$ with $\theta(a, b) := \sup\{\gamma(x, y, z) : a \leq G(x, y, z) \leq b\} < 1$ such that for any $x, y, z \in X$

$$\eta(x, Tx, Tx) \leq \alpha(x, y, z),$$

implies that

$$G(Tx, Ty, Tz) \leq \gamma(x, y, z) \max\left\{G(x, y, z), \frac{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)}{2}, \frac{G(x, Ty, Tz) + G(y, Tz, Tx) + G(z, Tx, Ty)}{2}\right\},$$

then T is α-η-weakly Zamfirescu mapping.

Theorem 3.2. *Let X be a G-metric space and T a self-mapping on X. If T is an α-η-weakly Zamfirescu mapping and α-admissible with respect to η with $\alpha(x, Tx, Tx) \geq \eta(x, Tx, Tx)$ for any $x \in X$, then T has an approximate fixed point.*

Proof. If G is symmetric, then we have

$$d_G(x, y) = 2G(x, y, y), \tag{3.1}$$

and (3.1) becomes

$$d_G(Tx, Ty) \leq \gamma(x, y) \max\left\{d_G(x, y), \frac{d_G(x, Tx) + d_G(y, Ty)}{2}, \frac{d_G(x, Ty) + d_G(y, Tx)}{2}\right\},$$

by taking $\gamma : X \times X \to [0, 1]$ instead of $\gamma : X \times X \times X \to [0, 1]$. The result then follows from Theorem 20 in [14]. Suppose that G is non-symmetric. We proceed as follows. Let x_0 be a given point in X. We define a sequence $\{x_n\}$ by $x_n = T^n x_0$. By following arguments similar to those in the proof of Theorem 2.8 we obtain that $\alpha(T^n x_0, T^{n+1} x_0, T^{n+1} x_0) \geq \eta(T^n x_0, T^{n+1} x_0, T^{n+1} x_0)$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(TT^{n-1} x_0, TT^n x_0, TT^n x_0) \\ &\leq \gamma(T^{n-1} x_0, T^n x_0, T^n x_0) \max\Big\{G(T^{n-1} x_0, T^n x_0, T^n x_0), \\ &\quad \frac{G(T^{n-1} x_0, TT^{n-1} x_0, TT^{n-1} x_0) + 2G(T^n x_0, TT^n x_0, TT^n x_0)}{2}, \end{aligned}$$

$$\left.\frac{G(T^{n-1}x_0, TT^nx_0, TT^nx_0) + 2G(T^nx_0, TT^{n-1}x_0, TT^{n-1}x_0)}{2}\right\}$$
$$= \gamma(x_{n-1}, x_n, x_n) \max\left\{G(x_{n-1}, x_n, x_n), \frac{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})}{2},\right.$$
$$\left.\frac{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_n, x_n)}{2}\right\}$$
$$\leq \gamma(x_{n-1}, x_n, x_n) \max\left\{G(x_{n-1}, x_n, x_n), \frac{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})}{2}\right\}.$$

If
$$\max\left\{G(x_{n-1}, x_n, x_n), \frac{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})}{2}\right\} = G(x_{n-1}, x_n, x_n),$$
then we have
$$G(x_n, x_{n+1}, x_{n+1}) \leq \gamma(x_{n-1}, x_n, x_n)G(x_{n-1}, x_n, x_n).$$

If
$$\max\left\{G(x_{n-1}, x_{n+1}, x_{n+1}), \frac{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})}{2}\right\}$$
$$= \frac{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})}{2},$$
then
$$G(x_n, x_{n+1}, x_{n+1}) \leq \gamma(x_{n-1}, x_n, x_n) \left[\frac{G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})}{2}\right],$$
gives
$$(2 - 2\gamma)(x_{n-1}, x_n, x_n))G(x_n, x_{n+1}, x_{n+1}) \leq \gamma(x_{n-1}, x_n, x_n)G(x_{n-1}, x_n, x_n),$$
$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{\gamma(x_{n-1}, x_n, x_n)}{2 - 2\gamma(x_{n-1}, x_n, x_n)}G(x_{n-1}, x_n, x_n)$$
$$\leq \gamma(x_{n-1}, x_n, x_n)G(x_{n-1}, x_n, x_n),$$
which implies that
$$G(x_n, x_{n+1}, x_{n+1}) \leq \gamma(x_{n-1}, x_n, x_n)G(x_{n-1}, x_n, x_n).$$

Hence $\{G(x_{n-1}, x_n, x_n)\}$ is a non-increasing sequence which converges to a real number
$$s = \inf_{n\geq 1} G(x_{n-1}, x_n, x_n).$$

Assume that $s > 0$. Since $0 < s \leq G(x_n, x_{n+1}, x_{n+1}) \leq \cdots \leq G(x_0, x_1, x_1)$ and $\gamma(x_{n-1}, x_n, x_n) \leq \theta$ for all $n \in \mathbb{N} \cup\{0\}$, where $\theta = \theta(s, G(x_0, x_1, x_1))$, we obtain that
$$G(x_n, x_{n+1}, x_{n+1}) \leq \theta G(x_{n-1}, x_n, x_n),$$
and
$$s \leq G(x_{n-1}, x_n, x_n) \leq \theta^n G(x_0, x_1, x_1).$$

This implies $s = 0$ (on taking the limit as $n \to \infty$), a contradiction. Therefore,
$$\lim_{n\to\infty} G(x_{n-1}, x_n, x_n) = \lim_{n\to\infty} G(T^{n-1}x_0, T^nx_0, T^nx_0) = 0.$$

Now,
$$G(T^nx_0, T^{n+1}x_0, T^{n+2}x_0) = G(x_n, x_{n+1}, x_{n+2})$$

$$\begin{aligned}&\leq G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+1},x_{n+2})\\&\leq G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+2},x_{n+2})\\&\quad+G(x_{n+2},x_{n+1},x_{n+2})\\&=G(x_n,x_{n+1},x_{n+1})+2G(x_{n+1},x_{n+2},x_{n+2}),\end{aligned}$$

gives $G(T^nx_0,T^{n+1}x_0,T^{n+2}x_0)\to 0$, as $n\to\infty$. Hence, by Lemma 2.3, T has an approximate fixed point. □

Theorem 3.3. *Let X be a complete G-metric space and T a continuous, α-η-weakly Zamfirescu and α-admissible mapping with respect to η. If there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq\eta(x_0,Tx_0)$, then T has a fixed point.*

Proof. Let $x_0\in X$ be such that $\alpha(x_0,Tx_0,Tx_0)\geq\eta(x_0,Tx_0,Tx_0)$. Define a sequence $\{x_n\}$ as in Theorem 2.8. By following arguments similar to those in proof of Theorem 3.2, we obtain that

$$G(x_n,x_{n+1},x_{n+1})\leq\gamma(x_{n-1},x_n,x_n)G(x_{n-1},x_n,x_n)$$

for all $n\in\mathbb{N}\cup\{0\}$. Also, we deduce that $\{x_n\}$ is a Cauchy sequence. Since X is a complete G-metric space, there exists $z\in X$ such that $x_n\to z$. The result follows by the continuity of T. □

Example 3.4. Let $X=[0,\infty)$ and $G(x,y,z)=\max\{|x-y|,|y-z|,|z-x|\}$ be a G-metric on X. Define $T:X\to X$ and $\alpha,\eta:X\times X\times X\to[0,\infty)$ by

$$Tx=\begin{cases}\dfrac{x}{5}, & \text{if } x\in[0,2],\\ \dfrac{(3x^2+x^{x+1})(4-x)}{60}+\dfrac{x-1}{x^{2+1}}, & \text{if } x\in(2,4],\\ \dfrac{3(20-x)}{16(x^2+1)}+\dfrac{100}{16}(x-4), & \text{if } x\in(4,20),\\ 5x, & \text{if } x\in[20,\infty),\end{cases}$$

and

$$\alpha(x,y,z)=\begin{cases}15, & \text{if } x,y,z\in[0,1],\\ \dfrac{1}{2}, & \text{otherwise,}\end{cases}\qquad \eta(x,y,z)=1.$$

Let $\gamma:X\times X\times X\to[0,1]$ be a given function. If $\alpha(x,y,z)\geq 1$ then $x,y,z\in[0,1]$. Therefore,

$$\begin{aligned}G(Tx,Ty,Ty)&=\frac{1}{5}G(x,y,y)\\&\leq\frac{1}{3}\max\left\{G(x,y,y),\frac{G(x,Tx,Tx)+2G(y,Ty,Ty)}{2},\frac{G(x,Ty,Ty)+2G(y,Tx,Tx)}{2}\right\}.\end{aligned}$$

Take $\gamma(x,y,z)=\frac{1}{3}$ and so,

$$\begin{aligned}G(Tx,Ty,Ty)&=\frac{1}{5}G(x,y,y)\\&\leq\gamma(x,y,z)\max\left\{G(x,y,z),\frac{G(x,Tx,Tx)+2G(y,Ty,Ty)}{2},\frac{G(x,Ty,Ty)+2G(y,Tx,Tx)}{2}\right\}.\end{aligned}$$

That is, there exists $\gamma:X\times X\times X\to[0,1]$ with $\theta(a,b):=\sup\{\gamma(x,y,z):a\leq G(x,y,z)\leq b\}<1$ for all $0<a\leq b$, such that $\eta(x,Tx,Tx)\leq\alpha(x,y,y)$,

$$G(Tx,Ty,Ty)\leq\gamma(x,y,y)\max\left\{G(x,y,y),\frac{G(x,Tx,Tx)+2G(y,Ty,Ty)}{2},\frac{G(x,Ty,Ty)+2G(y,Tx,Tx)}{2}\right\}$$

holds for all $x,y\in X$. Then T is α-η-weakly Zamfirescu mapping. Thus, T has a fixed point by Theorem 3.3.

4. From α-η-Ćirić strong almost contraction to Suzuki type contraction

Definition 4.1. Let X be a G-metric space and $\alpha, \eta : X \times X \times X \to [0, \infty)$. A mapping $T : X \to X$ is called an α-η-Ćirić strong almost contraction, if there exists a constant $r \in [0, 1)$ such that for any $x, y, z \in X$,

$$\eta(x, Tx, Tx) \leq \alpha(x, y, z) \quad \text{implies that} \quad G(Tx, Ty, Tz) \leq rM(x, y, z) + LG(y, Tz, Tx), \tag{4.1}$$

where $L \geq 0$ and

$$M(x, y, z) = \max\Big\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz) \\ \frac{G(x, Ty, Tz) + G(y, Tz, Tx) + G(z, Tx, Ty)}{2}\Big\}.$$

Theorem 4.2. *Let X be a G-metric space and T be a continuous α-η-Ćirić strong almost contraction on X. Also suppose that, T is an α-admissible mapping with respect to η. If there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$, then T has a fixed point.*

Proof. If G is symmetric, then we have

$$d_G(x, y) = 2G(x, y, y),$$

and (4.1) becomes

$$d_G(Tx, Ty) \leq rM(x, y) + Ld_G(y, Tx).$$

The result then follows from Theorem 25 in [10]. Suppose that G is non-symmetric. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0, Tx_0) \geq \eta(x_0, Tx_0, Tx_0)$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$. As T is an α-admissible mapping with respect to η, so we have $\alpha(x_0, x_1, x_1) = \alpha(x_0, Tx_0, Tx_0) \geq \eta(x_0, Tx_0, Tx_0) = \eta(x_0, x_1, x_1)$. By continuing this process, we have

$$\eta(x_{n-1}, Tx_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n, x_n) \leq \alpha(x_{n-1}, x_n, x_n)$$

for all $n \in \mathbb{N}$. From given assumption we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq rM(x_{n-1}, x_n, x_n) + LG(x_n, Tx_{n-1}, Tx_{n-1}) \\ &= rM(x_{n-1}, x_n, x_n), \end{aligned}$$

where

$$\begin{aligned} M(x_{n-1}, x_n, x_n) &= \max\Big\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, Tx_{n-1}, Tx_{n-1}), G(x_n, Tx_n, Tx_n), \\ &\qquad \frac{G(x_{n-1}, Tx_n, Tx_n) + G(x_n, Tx_n, Tx_{n-1}) + G(x_n, Tx_{n-1}, Tx_n)}{2}\Big\} \\ &= \max\Big\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ &\qquad \frac{G(Tx_n, Tx_n, Tx_n) + G(x_n, x_{n+1}, x_n) + G(x_n, x_n, x_{n+1})}{2}\Big\} \\ &= \max\Big\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ &\qquad \frac{G(x_n, x_{n+1}, x_n) + G(x_n, x_n, x_{n+1})}{2}\Big\} \\ &= \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}, G(x_n, x_n, x_{n+1}))\}. \end{aligned}$$

Thus,

$$G(x_n, x_{n+1}, x_{n+1}) \leq \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_{n+1})\}.$$

If

$$\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_{n+1})\} = G(x_n, x_{n+1}, x_{n+1}),$$

for some n, then

$$G(x_n, x_{n+1}, x_{n+1}) \leq rG(x_n, x_{n+1}, x_{n+1}) < G(x_n, x_{n+1}, x_{n+1}),$$

gives a contradiction. Similarly,

$$\max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_{n+1})\} = G(x_n, x_n, x_{n+1}),$$

leads to a contradiction. Hence,

$$G(x_n, x_{n+1}, x_{n+1}) \leq rG(x_{n-1}, x_n, x_n),$$

which further implies that

$$G(x_n, x_{n+1}, x_{n+1}) \leq rG(x_{n-1}, x_n, x_n) \leq \cdots \leq r^n G(x_0, x_1, x_1),$$

for all $n \in \mathbb{N}$. Now for $m < n$

$$\begin{aligned} G(x_m, x_n, x_n) &\leq G(x_m, x_{m+1}, x_{m+1}) + G(x_{m+1}, x_{m+2}, x_{m+2}) + \cdots + G(x_{n-1}, x_n, x_n) \\ &\leq r^m G(x_0, x_1, x_1) + r^{m+1} G(x_0, x_1, x_1) + \cdots + r^{n-1} G(x_0, x_1, x_1) \\ &= (r^m + r^{m+1} + \cdots + r^{n-1}) G(x_0, x_1, x_1) \\ &\leq \frac{r^m}{1-r} G(x_0, x_1, x_1). \end{aligned}$$

By taking the limit as $m, n \to \infty$, we get that $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $z \in X$ such that $x_n \to z$, as $n \to \infty$. The result follows by the continuity of T. □

Theorem 4.3. *Let X be a G-metric space, $\alpha, \eta : X \times X \times X \to [0, \infty)$, T an α-admissible with respect to η and α-η-Ćirić strong almost contraction on X. If there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq \eta(x_0, Tx_0, Tx_0)$ and for any sequence $\{x_n\}$ in X such that*

$$\alpha(x_n, x_{n+1}, x_{n+1}) \geq \eta(x_n, x_{n+1}, x_{n+1}),$$

with $x_n \to x$ as $n \to \infty$, then either

$$\eta(Tx_n, T^2x_n, T^2x_n) \leq \alpha(Tx_n, x, x), \quad \text{or} \quad \eta(T^2x_n, T^3x_n, T^3x_n) \leq \alpha(T^2x_n, x, x),$$

holds for all $n \in \mathbb{N}$. Then T has a fixed point.

Proof. Let x_0 be a given point in X such that $\alpha(x_0, Tx_0, Tx_0) \geq \eta(x_0, Tx_0, Tx_0)$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$. As in proof of Theorem 3.3, we obtain that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq \eta(x_n, x_{n+1}, x_{n+1})$ for all $n \in \mathbb{N}$. Also, there exists $z \in X$ such that, $x_n \to z$ as $n \to \infty$. If $G(z, Tz, Tz) \neq 0$, then

$$\eta(Tx_{n-1}, T^2x_{n-1}, T^2x_{n-1}) \leq \alpha(Tx_{n-1}, z, z), \quad \text{or} \quad \eta(T^2x_{n-1}, T^3x_{n-1}, T^3x_{n-1}) \leq \alpha(T^2x_{n-1}, z, z),$$

holds for all $n \in \mathbb{N}$. Thus

$$\eta(x_n, Tx_n, Tx_n) \leq \alpha(x_n, z, z), \quad \text{or} \quad \eta(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \alpha(x_{n+1}, z, z),$$

holds for all $n \in \mathbb{N}$. Suppose that $\eta(x_n, Tx_n, Tx_n) \leq \alpha(x_n, z, z)$ holds for all $n \in \mathbb{N}$. By given assumption we have

$$\begin{aligned} G(x_{n+1}, Tz, Tz) &= G(Tx_n, Tz, Tz) \\ &\leq rM(x_n, z, z) + LG(z, Tx_n, Tx_n) \\ &= rM(x_n, z, z) + LG(z, x_{n+1}, x_{n+1}), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} M(x_n, z, z) &= \max\Big\{G(x_n, z, z), G(x_n, Tx_n, Tx_n), G(z, Tz, Tz), \\ &\qquad \frac{G(x_n, Tz, Tz) + G(z, Tz, Tx_n) + G(z, Tx_n, Tz)}{2}\Big\} \\ &= \max\Big\{G(x_n, z, z), G(x_n, x_{n+1}, x_{n+1}), G(z, Tz, Tz), \\ &\qquad \frac{G(x_n, Tz, Tz) + G(z, Tz, x_{n+1}) + G(z, x_{n+1}, Tz)}{2}\Big\}. \end{aligned} \tag{4.3}$$

By using (4.3) in (4.2) and taking the limit as $n \to \infty$, we have

$$G(z, Tz, Tz) \leq rG(z, Tz, Tz) < G(z, Tz, Tz),$$

a contradiction. Hence $G(z, Tz, Tz) = 0$. By following arguments similar to those given above, we obtain that $Tz = z$, if $\eta(x_{n+1}, Tx_{n+1}, Tx_{n+1}) \leq \alpha(x_{n+1}, z, z)$ holds for all $n \in \mathbb{N}$. □

Example 4.4. Let $X = [0, +\infty)$ and $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$. Define $T : X \to X$ and $\alpha, \eta : X \times X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \dfrac{x^2}{4}, & \text{if } x \in [0, 1], \\ \dfrac{(x^3 + 2x + 1)}{\sqrt{x^2 + 1}}, & \text{if } x \in (1, 2], \\ 3x, & \text{if } x \in [2, +\infty), \end{cases}$$

and

$$\alpha(x, y, z) = \begin{cases} \dfrac{1}{2}, & \text{if } x, y, z \in [0, 1], \\ \dfrac{1}{8}, & \text{otherwise}, \end{cases} \qquad \eta(x, y, z) = \frac{1}{4}.$$

Let $\alpha(x, y, z) \geq \eta(x, y, z)$, then $x, y, z \in [0, 1]$. Also, $Tw \in [0, 1]$ for all $w \in [0, 1]$. Then $\alpha(Tx, Ty, Tz) \geq \eta(Tx, Ty, Tz)$. This shows T is α-admissible mapping with respect to η. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ as $n \to \infty$ and that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq \eta(x_n, x_{n+1}, x_{n+1})$. Then $Tx_n, T^2x_n, T^3x_n \in [0, 1]$ for all $n \in \mathbb{N}$. That is,

$$\eta(Tx_n, T^2x_n, T^2x_n) \leq \alpha(Tx_n, x, x),$$

and

$$\eta(T^2x_n, T^3x_n, T^3x_n) \leq \alpha(T^2x_n, x, x)$$

hold for all $n \in \mathbb{N}$. Clearly, $\alpha(0, T0, T0) \geq \eta(0, T0, T0)$. Let $\alpha(x, y, z) \geq \eta(x, Tx, Tx)$. Now, if $x \notin [0, 1]$, then, $\frac{1}{8} \geq \frac{1}{4}$ which is not possible. So, $x, y, z \in [0, 1]$. Therefore,

$$\begin{aligned} G(Tx, Ty, Tz) &= \frac{1}{4} \max\{|x^2 - y^2|, |y^2 - z^2|, |z^2 - x^2|\} \\ &= \frac{1}{4} \max\{|x - y||x + y|, |y - z||y + z|, |z - x||z + x|\} \\ &\leq \frac{1}{2} \max\{|x - y|, |y - z|, |z - x|\} \\ &\leq \frac{1}{2} M(x, y, z) + LG(y, Tz, Tx). \end{aligned}$$

Therefore, T is an α-η-Ćirić strong almost contraction. Hence all the conditions for Theorem 4.3 are satisfied. Hence T has a fixed point.

As an application of the above result, we obtain the following Suzuki type fixed point theorem in the setup of G-metric spaces.

Theorem 4.5. *Let X be a complete G-metric space and T a self-mapping on X. If there exists $r \in [0,1)$ such that for any x, y in X*

$$\frac{1}{1+r}G(x,Tx,Tx) \le G(x,y,y) \quad \text{implies that} \quad G(Tx,Ty,Ty) \le rM(x,y,y) + LG(y,Tx,Tx), \tag{4.4}$$

where

$$M(x,y,y) = \max\left\{G(x,y,y), G(x,Tx,Tx), G(y,Ty,Ty), \frac{G(x,Ty,Ty)+G(y,Ty,Tx)+G(y,Tx,Ty)}{2}\right\},$$

then T has a fixed point.

Proof. Define $\alpha, \eta : X \times X \times X \to [0,\infty)$ by

$$\alpha(x,y,y) = G(x,y,y) \quad \text{and} \quad \eta(x,y,y) = \lambda(r)G(x,y,y),$$

where $0 \le r < 1$ and $\lambda(r) = \frac{1}{1+r}$. As for any $x, y \in X$, $\lambda(r)G(x,y,y) \le G(x,y,y)$ so we have $\eta(x,y,y) \le \alpha(x,y,y)$. Thus, conditions (i) and (ii) of Theorem 4.3 hold. Let $\{x_n\}$ be a sequence in X with $x_n \to x$ as $n \to \infty$. If $G(Tx_n, T^2x_n, T^2x_n) = 0$ for some n. Then $T^2x_n = Tx_n$ implies that Tx_n is a fixed point of T and the result follows. Suppose that $T^2x_n \ne Tx_n$ for all $n \in \mathbb{N}$. As $\lambda(r)G(Tx_n, T^2x_n, T^2x_n) \le G(Tx_n, T^2x_n, T^2x_n)$ for all $n \in \mathbb{N}$, so from (4.4) we get,

$$\begin{aligned} G(T^2x_n, T^3x_n, T^3x_n) &\le rM(Tx_n, T^2x_n, T^2x_n) + LG(T^2x_n, T^2x_n, T^2x_n) \\ &= rM(Tx_n, T^2x_n, T^2x_n), \end{aligned} \tag{4.5}$$

where,

$$\begin{aligned} M(Tx_n, T^2x_n, T^2x_n) &= \max\Big\{G(Tx_n, T^2x_n, T^2x_n), G(Tx_n, T^2x_n, T^2x_n), G(T^2x_n, T^3x_n, T^3x_n), \\ &\qquad \frac{G(Tx_n, T^3x_n, T^3x_n) + 2G(T^2x_n, T^2x_n, T^2x_n)}{2}\Big\} \\ &\le \max\Big\{G(Tx_n, T^2x_n, T^2x_n), G(T^2x_n, T^3x_n, T^3x_n), \\ &\qquad \frac{G(Tx_n, T^2x_n, T^2x_n) + G(T^2x_n, T^3x_n, T^3x_n)}{2}\Big\} \\ &= \max\{G(Tx_n, T^2x_n, T^2x_n), G(T^2x_n, T^3x_n, T^3x_n)\}. \end{aligned}$$

If

$$\max\{G(Tx_n, T^2x_n, T^2x_n), G(T^2x_n, T^3x_n, T^3x_n)\} = G(T^2x_n, T^3x_n, T^3x_n),$$

for some n, then (4.5) becomes

$$G(T^2x_n, T^3x_n, T^3x_n) \le rG(T^2x_n, T^3x_n, T^3x_n),$$

a contradiction.

Hence

$$G(T^2x_n, T^3x_n, T^3x_n) \le rG(Tx_n, T^2x_n, T^2x_n). \tag{4.6}$$

If there exists $n_0 \in \mathbb{N}$ such that

$$\eta(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}) > \alpha(Tx_{n_0}, x, x), \quad \text{and} \quad \eta(T^2x_{n_0}, T^3x_{n_0}, T^3x_{n_0}) > \alpha(T^2x_{n_0}, x, x),$$

then we have

$$\lambda(r)G(Tx_{n_0}, T^2x_{n_0}, Tx_{n_0}^2) > G(Tx_{n_0}, x, x),$$

and

$$\lambda(r)G(T^2x_{n_0}, T^3x_{n_0}, T^3x_{n_0}) > G(T^2x_{n_0}, T^2x_{n_0}, x_0).$$

Thus from (4.6) we obtain that

$$\begin{aligned} G(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}) &\le G(Tx_{n_0}, x, x) + G(x, T^2x_{n_0}, T^2x_{n_0}) \\ &< \lambda(r)G(T^2x_{n_0}, T^2x_{n_0}, Tx_{n_0}) + \lambda(r)G(T^2x_{n_0}, T^3x_{n_0}, T^3x_{n_0}) \\ &\le \lambda(r)G(Tx_{n_0}, T^2x_{n_0}, Tx_{n_0}) + r\lambda(r)G(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}) \\ &= \lambda(r)(1+r)G(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}) \\ &= G(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}), \end{aligned}$$

a contradiction. Hence, either

$$\eta(Tx_n, T^2x_n, T^2x_n) \le \alpha(Tx_n, x, x), \quad \text{or} \quad \eta(T^2x_n, T^3x_n, T^3x_n) \le \alpha(T^2x_n, x, x)$$

holds for all $n \in \mathbb{N}$ and the condition (iv) of Theorem 4.3 holds. Now $\eta(x, Tx, Tx) \le \alpha(x, y, y)$ gives that $\lambda(r)G(x, Tx, Tx) \le G(x, y, y)$. Thus from (4.4) we obtain $G(Tx, Ty, Ty) \le rM(x, y, y) + LG(y, Tx, Tx)$. Hence all the conditions of Theorem 4.3 are satisfied and the result follows. □

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