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# Bounds for Laplacian-type graph energies

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# BOUNDS FOR LAPLACIAN–TYPE GRAPH ENERGIES

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*Abstract.* Let $G$ be an undirected simple and connected graph with $n$ vertices ($n \geq 3$) and $m$ edges. Denote by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$, $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ , and $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} > \rho_n = 0$ , respectively, the Laplacian, signless Laplacian, and normalized Laplacian eigenvalues of $G$. The Laplacian energy, signless Laplacian energy, and normalized Laplacian energy of $G$ are defined as $LE = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$, $SLE = \sum_{i=1}^{n} \left|\gamma_i - \frac{2m}{n}\right|$, and $NLE = \sum_{i=1}^{n} |\rho_i - 1|$, respectively. Lower bounds for $LE$, $SLE$, and $NLE$ are obtained.

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*Keywords:* energy (of graph), Laplacian energy, signless Laplacian energy, normalized Laplacian energy, Randić energy

## 1. INTRODUCTION

Let $G$ be an undirected simple and connected graph with $n$ vertices ($n \geq 2$) and $m$ edges, and let $d_1, d_2, \ldots, d_n$ be its vertex degrees.

If the $i$-th and $j$-th vertex of the graph $G$ are adjacent, we write $i \sim j$. Then the adjacency matrix $\mathbf{A} = (a_{ij})$ of $G$ is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } i \neq j \quad \text{and} \quad i \sim j \\ \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of $\mathbf{A}$ form the (ordinary) spectrum of $G$; for details on the respective spectral theory see [9].

Denote by $\mathbf{D}$ the diagonal matrix of the vertex degrees of $G$. The Laplacian matrix of $G$ is $\mathbf{L} = \mathbf{D} - \mathbf{A}$ and its eigenvalues are $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ (see [3, 16, 25]). In addition, $\mathbf{Q} = \mathbf{D} + \mathbf{A}$ is the signless Laplacian matrix of $G$ and its eigenvalues will be denoted by $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 0$ [10, 11].

Because the graph $G$ is assumed to be connected, it has no isolated vertices (i.e., $d_i > 0$ for all $1 \leq i \leq n$) and therefore the matrix $\mathbf{D}^{-1/2}$ is well–defined. Then $\mathbf{L}^* = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2}$ is called the normalized Laplacian matrix of the graph $G$. Its

eigenvalues are $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} > \rho_n = 0$. For details of the spectral theory of the normalized Laplacian matrix see [8].

It is convenient to write the normalized Laplacian matrix as $\mathbf{I} - \mathbf{R}$, where $\mathbf{R}$ is the so-called Randić matrix [4, 29, 30], whose $(i, j)$-entry is

$$r_{ij} = \begin{cases} 1/\sqrt{d_i\, d_j} & \text{if } i \neq j \quad \text{and} \quad i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

The (ordinary) energy of the graph $G$ is defined as [23]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| \,. \tag{1.1}$$

Its theory is nowadays well elaborated [23]. Energy–like spectral invariants have been introduced also for other graph matrices [18]. In this paper we are concerned with the Laplacian [21, 23], signless Laplacian [1], and normalized Laplacian (or Randić) energies [5, 20], defined as

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$$

$$SLE = SLE(G) = \sum_{i=1}^{n} \left| \gamma_i - \frac{2m}{n} \right|$$

$$NLE = NLE(G) = \sum_{i=1}^{n} |\rho_i - 1|$$

respectively. In what follows lower bounds for $LE$, $SLE$ and $NLE$ are obtained.

*Remark* 1. In analogy to (1.1), the "Randić energy" is defined as the sum of the absolute values of the eigenvalues of the Randić matrix. It has been shown in [20], that the Randić energy coincides with the normalized signless Laplacian energy.

*Remark* 2. One could also consider the normalized signless Laplacian matrix, $\mathbf{D}^{-1/2}\mathbf{Q}\mathbf{D}^{-1/2}$ and its "energy" (sum of absolute values of eigenvalues). However, the energy of this matrix is exactly the same as the normalized Laplacian energy, $NLE$ [20]. For the general definition of the energy of a matrix see [28].

The Laplacian, signless Laplacian, and normalized (or Randić) Laplacian spreads of a graph $G$ are defined as $LS(G) = \mu_1 - \mu_{n-1}$, $SLS(G) = \gamma_1 - \gamma_n$, and $NLS(G) = \rho_1 - \rho_{n-1}$, respectively (see [5, 13, 15, 24]).

## 2. Preliminaries

In this section we recall some results from spectral graph theory, and state a few analytical inequalities needed for our work.

**Lemma 1** ([3]). *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 2$, vertices and $m$ edges. Then*

$$\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m$$

*where $M_1$ is the sum of squares of the vertex degrees, usually referred to as the first Zagreb index (see [2, 7, 19]).*

**Lemma 2** ([12]). *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 2$, vertices and $m$ edges. Then*

$$\frac{M_1}{m} \geq 2\sqrt{\frac{M_1}{n}} \geq \frac{4m}{n} \,. \tag{2.1}$$

**Lemma 3** ([31]). *Let $G$ an $(n,m)$-graph, such that $n \geq 3$ and $m \geq 1$. Then*

$$LE(G) \geq \mu_1 - \mu_{n-1} + \frac{2m}{n} \tag{2.2}$$

*with equality if and only if $n = 3$ or for $n \geq 4$ if $\mu_2 = \cdots = \mu_{n-2} = \frac{2m}{n}$.*

**Lemma 4** ([26]). *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 3$, vertices and $m$ edges. Then*

$$LS(G) = \mu_1 - \mu_{n-1} \leq \sqrt{\frac{2}{n-1}} \sqrt{(n-1)(M_1 + 2m) - 4m^2} \,. \tag{2.3}$$

*Equality holds if and only if $G \cong K_n$.*

**Lemma 5** ([27]). *Let $a_1, a_2, \ldots, a_n$ be real numbers and $p_1, p_2, \ldots, p_n$ non-negative real numbers with the property $p_1 + p_2 + \cdots + p_n = 1$. Then, for each $\alpha$, $\alpha \leq 0$ and $\alpha \geq 1$,*

$$\sum_{i=1}^{n} p_i \, a_i^{\alpha} \geq \left( \sum_{i=1}^{n} p_i \, a_i \right)^{\alpha} . \tag{2.4}$$

*For the case $0 \leq \alpha \leq 1$, the opposite inequality is valid. Equality in (2.4) holds if and only if $\alpha = 0$ or $\alpha = 1$ or $a_1 = a_2 = \cdots = a_n$.*

**Lemma 6** ([6]). *Let $a_1, a_2, \ldots, a_n$ be real numbers, and assume that there are $r, R \in \mathbb{R}$ such that $-\infty < r \leq a_i \leq R < +\infty$, for each $i = 1, 2, \ldots, n$. Then for any non-negative $p_1, p_2, \ldots, p_n$ with the property $p_1 + p_2 + \cdots + p_n = 1$,*

$$0 \leq \sum_{i=1}^{n} p_i \, a_i^2 - \left( \sum_{i=1}^{n} p_i \, a_i \right)^2 \leq \frac{1}{2}(R - r) \sum_{i=1}^{n} p_i \left| a_i - \sum_{i=1}^{n} p_i \, a_i \right| . \tag{2.5}$$

*The constant $\frac{1}{2}$ is sharp.*

**Lemma 7** ([32]). *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 2$, vertices and $m$ edges. Then*

$$\sum_{i=1}^{n-1} \rho_i = n \qquad \text{and} \qquad \sum_{i=1}^{n} \rho_i^2 = n + 2R_{-1} \tag{2.6}$$

*where $R_{-1} = \sum_{i \sim j} \frac{1}{d_i d_j}$; for details on the graph invariant $R_{-1}$ see* [4, 22].

**Lemma 8** ([17]). *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 2$, vertices and $m$ edges. Then*

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n} \gamma_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m$$

*where $M_1$ is the first Zagreb index.*

**Lemma 9** ([17]). *The signless Laplacian spread has an upper bound*

$$SLS(G) \leq \sqrt{\frac{2[n(M_1 + 2m) - 4m^2]}{n}} \, .$$

**Lemma 10** ([14]). *Suppose that $G$ is a graph without isolated vertices. Then*

$$\mu_1 - \mu_{n-1} \geq \frac{2}{n-1} \sqrt{(n-1)(2m + M_1) - 4m^2} \, . \tag{2.7}$$

## 3. Main results

### 3.1. *Lower bound for Laplacian energy*

**Theorem 1.** *Let $G$ be an undirected connected graph with $n$, $n \geq 3$, vertices and $m$ edges. Then*

$$LE(G) \geq \frac{2m}{n} + \frac{2}{n-1} \sqrt{(n-1)(2m + M_1) - 4m^2} \, . \tag{3.1}$$

*Proof.* Inequality (3.1) directly follows from inequalities (2.2) and (2.7). □

**Corollary 1.** *Let $G$ be an undirected graph with $n$, $n \geq 3$, vertices and $m$ edges. Then*

$$LE(G) \geq \frac{2m}{n} + \frac{2}{n-1} \sqrt{\frac{2m(n(n-1) - 2m)}{n}} \, .$$

**Corollary 2.** *Let $G$ be an undirected simple and connected $k$-regular graph with $n$, $n \geq 3$, vertices and $m$ edges, $1 < k \leq n-1$. Then*

$$LE(G) \geq k + \frac{2}{n-1} \sqrt{nk(n-k-1)} \, .$$

**Theorem 2.** *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 3$ vertices and $m$ edges. Then*

$$LE(G) \geq \sqrt{\frac{2}{n-1}}\sqrt{(n-1)(M_1+2m)-4m^2}\,. \tag{3.2}$$

*Proof.* For $n-1$ and $p_i := \frac{1}{n-1}$, $a_i := \mu_i$, $i = 1,2,\ldots n-1$, $r := \mu_{n-1}$ and $R := \mu_1$, the inequality (2.5) transforms into

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\mu_i^2 - \frac{1}{(n-1)^2}\left(\sum_{i=1}^{n-1}\mu_i\right)^2 \leq \frac{\mu_1-\mu_{n-1}}{2(n-1)}\sum_{i=1}^{n-1}\left|\mu_i - \frac{1}{n-1}\sum_{i=1}^{n-1}\mu_i\right|$$

i.e., based on Lemma 1,

$$(n-1)(M_1+2m)-4m^2 \leq \frac{n-1}{2}(\mu_1-\mu_{n-1})\sum_{i=1}^{n-1}\left|\mu_i - \frac{2m}{n-1}\right|.$$

Since

$$\sum_{i=1}^{n-1}\left|\mu_i - \frac{2m}{n-1}\right| \leq \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| = LE(G)$$

using inequality (2.3), from the above inequality we obtain (3.2). □

Using Lemma 2, we arrive at the following $(n,m)$-type lower bound for the Laplacian energy:

**Corollary 3.** *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 3$, vertices and $m$ edges. Then*

$$LE(G) \geq \sqrt{\frac{4m(n(n-1)-2m)}{n(n-1)}}\,. \tag{3.3}$$

**Corollary 4.** *Let $G$ be an undirected simple and connected $k$-regular graph with $n$, $n \geq 3$, vertices and $m$ edges, $1 < k \leq n-1$. Then*

$$LE(G) > \sqrt{\frac{2nk(n-k-1)}{n-1}}\,.$$

*Remark* 3. Since for undirected $k$-regular graphs, $LE = E$, the inequality in Corollary 4 provides a lower bound also for the ordinary energy.

Inequalities (3.1) and (3.2) are incomparable. Thus, for example, if $G \cong K_n$, then inequality (3.1) is stronger than (3.2), but if $G \cong K_{1,n-1}$, $n \geq 8$, then the opposite is valid.

3.2. *Lower bound for signless Laplacian energy*

**Theorem 3.** *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 3$, vertices and $m$ edges. Then*

$$SLE(G) \geq \sqrt{\frac{2(n(M_1+2m)-4m^2)}{n}} \,. \tag{3.4}$$

*Proof.* For $p_i := \frac{1}{n}$, $a_i = \gamma_i$, $i = 1,2,\ldots,n$, $r = \gamma_n$ and $R = \gamma_1$, the inequality (2.5) becomes

$$\frac{1}{n}\sum_{i=1}^{n}\gamma_i^2 - \frac{1}{n^2}\left(\sum_{i=1}^{n}\gamma_i\right)^2 \leq \frac{\gamma_1-\gamma_n}{2n}\sum_{i=1}^{n}\left|\gamma_i - \frac{2m}{n}\right| .$$

Bearing in mind Lemma 8, the above inequality becomes

$$n(M_1+2m)-4m^2 \leq \frac{n}{2}\, SLS(G)\times SLE(G) \,.$$

By Lemma 9 and the above inequality, we obtain (3.4). □

Bearing in mind Lemma 2 and inequality (3.4), we arrive at a lower bound for $SLE(G)$ depending only on the parameter $m$.

**Corollary 5.** *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 3$, vertices and $m$ edges. Then*

$$SLE(G) \geq 2\sqrt{m} \,.$$

**Corollary 6.** *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 3$, vertices and $m$ edges, which is $k$-regular, $1 < k \leq n$. Then*

$$SLE(G) \geq \sqrt{2nk} \,.$$

3.3. *Lower bound for normalized Laplacian energy*

**Theorem 4.** *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 3$, vertices and $m$ edges. Let, as before, $R_{-1} = \sum_{i\sim j}\frac{1}{d_i\, d_j}$. Then*

$$NLS(G) \leq \sqrt{\frac{2}{n-1}}\,\sqrt{2(n-1)R_{-1}-n} \,. \tag{3.5}$$

*Equality holds if and only if $G \cong K_n$.*

*Proof.* According to (2.6) we have that

$$\begin{aligned}(n-1)(n+2R_{-1})-n^2 &= (n-1)\sum_{i=1}^{n-1}\rho_i^2 - \left(\sum_{i=1}^{n-1}\rho_i\right)^2 \\ &= \sum_{1\leq i<j\leq n-1}(\rho_i-\rho_j)^2 \,. \end{aligned} \tag{3.6}$$

By Lemma 5, i.e., by inequality (2.4), for $n = 2$ and $\alpha = 2$, we get

$$(\rho_1 - \rho_i)^2 + (\rho_i - \rho_{n-1})^2 \geq \frac{1}{2}(\rho_1 - \rho_{n-1})^2 \tag{3.7}$$

for each $i = 2, 3, \ldots, n-2$. Then,

$$\begin{aligned}\sum_{1 \leq i < j \leq n-1} (\rho_i - \rho_j)^2 &\geq \sum_{i=2}^{n-2} [(\rho_1 - \rho_i)^2 + (\rho_i - \rho_{n-1})^2] + (\rho_1 - \rho_{n-1})^2 \\ &\geq \frac{n-3}{2}(\rho_1 - \rho_{n-1})^2 + (\rho_1 - \rho_{n-1})^2 \\ &= \frac{n-1}{2}(\rho_1 - \rho_{n-1})^2\end{aligned}$$

which combined with (3.6) yields

$$(n-1)(n+2R_{-1}) - n^2 = 2(n-1)R_{-1} - n \geq \frac{n-1}{2}(\rho_1 - \rho_{n-1})^2$$

from which the inequality (3.5) follows.

Equality in (3.7) holds if and only if $\rho_1 = \rho_2 = \cdots = \rho_{n-1}$. Therefore, equality in (3.5) holds if and only if $G \cong K_n$. This completes the proof of Theorem 4. □

**Corollary 7.** *Let $G$ be an undirected simple nad connected $k$-regular graph, $1 < k \leq n-1$, with $n$, $n \geq 3$, vertices and $m$ edges. Then*

$$NLS(G) \leq \sqrt{\frac{2n(n-k-1)}{(n-1)k}}\,.$$

*Equality holds if and only if $k = n-1$, i.e., $G \cong K_n$.*

We now state a theorem, analogous to Theorem 2, which provides a lower bound for $NLE$ in terms of parameters $n$ and $R_{-1}$.

**Theorem 5.** *Let $G$ be an undirected simple and connected graph with $n$, $n \geq 3$, vertices and $m$ edges. Then*

$$NLE(G) \geq \sqrt{\frac{2}{n-1}}\sqrt{2(n-1)R_{-1} - n}\,. \tag{3.8}$$

*Proof.* For $n := n-1$, $p_i := \frac{1}{n-1}$, $a_i := \rho_i$, $i = 1, 2, \ldots, n-1$, $r = \rho_{n-1}$ and $R = \rho_1$, inequality (2.5) becomes

$$\frac{1}{n-1}\sum_{i=1}^{n-1}\rho_i^2 - \frac{1}{(n-1)^2}\left(\sum_{i=1}^{n-1}\rho_i\right)^2 \leq \frac{\rho_1 - \rho_{n-1}}{2(n-1)}\sum_{i=1}^{n-1}\left|\rho_i - \frac{1}{n-1}\sum_{i=1}^{n-1}\rho_i\right|.$$

Having in mind Lemma 7, the above inequality transforms into

$$(n-1)(n+2R_{-1}) - n^2 \leq \frac{n-1}{2}NLS(G)\sum_{i=1}^{n-1}\left|\rho_i - \frac{n}{n-1}\right|\,. \tag{3.9}$$

Since

$$\sum_{i=1}^{n-1}\left|\rho_i-\frac{n}{n-1}\right|\le\sum_{i=1}^{n}|\rho_i-1|$$

according to (3.9) we obtain

$$(n-1)(n+2R_{-1})-n^2\le\frac{n-1}{2}\,NLS(G)NLE(G)\;. \tag{3.10}$$

Combining (3.5) and (3.10) we arrive at (3.8). □

*Remark* 4. For a $k$-regular graph, $R_{-1}=m/k^2=n/(2k)$. Since for $k$-regular graphs, $NLE=\frac{1}{k}E=\frac{1}{k}LE$, inequality (3.8) is equivalent to the result proven in Corollary 4.

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