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On Some New Contractive Conditions in Complete Metric Spaces

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Abstract: One of the main goals of this paper is to obtain new contractive conditions using the method of a strictly increasing mapping $\mathbf{F}\colon (0,+\infty) \to (-\infty,+\infty)$. According to the recently obtained results, this was possible (Wardowski's method) only if two more properties (F_2) and (F_3) were used instead of the aforementioned strictly increasing (F_1). Using only the fact that the function $\mathbf{F}$ is strictly increasing, we came to new families of contractive conditions that have not been found in the existing literature so far. Assuming that $\alpha(\mathfrak{u},\mathfrak{v}) = 1$ for every $\mathfrak{u}$ and $\mathfrak{v}$ from metric space Ξ, we obtain some contractive conditions that can be found in the research of Rhoades (Trans. Amer. Math. Soc. 1977, 222) and Collaco and Silva (Nonlinear Anal. TMA 1997). Results of the paper significantly improve, complement, unify, generalize and enrich several results known in the current literature. In addition, we give examples with results in line with the ones we obtained.

Keywords: α-admissible mappings; triangularly α-admissible mappings; $\mathbf{F}$-contraction; fixed point; contractive condition

MSC: 47H10; 54H25

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1. Introduction and Preliminaries

In 2012, ref. [1] Wardowski introduced a new concept of mapping in the setting of metric spaces:

Definition 1. *Let $(\Xi, \mathbf{d}_\Xi)$ be a metric space and $\mathbf{F}\colon (0,+\infty) \to (-\infty,+\infty)$ is a mapping satisfying the following conditions:*

(F_1) *$\mathbf{F}$ is increasing;*

(F_2) *for any sequence $\{r_n\}_{n=1}^{+\infty}$ of positive real numbers, $\lim_{n\to+\infty} r_n = 0$ if and only if $\lim_{n\to+\infty} \mathbf{F}(r_n) = -\infty$;*

(F_3) *there exists $k \in (0,1)$ such that $\lim_{r\to0^+} r^k \mathbf{F}(r) = 0$.*

A self-mapping $\mathcal{T}: \Xi \to \Xi$ is said to be an $\mathbf{F}$-contraction if there exists $\tau > 0$ such that

$$\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0 \text{ implies } \tau + \mathbf{F}(\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \le \mathbf{F}(\mathbf{d}_\Xi(\mathfrak{u},\mathfrak{v})), \tag{1}$$

for all $\mathfrak{u},\mathfrak{v} \in \Xi$

Let us denote by Π the collection of functions $\mathbf{F}\colon (0,+\infty) \to (-\infty,+\infty)$ that satisfy conditions $(F_1) - (F_3)$. If $\mathbf{F},\mathbf{G},\mathbf{H}\colon (0,+\infty) \to (-\infty,+\infty)$ are defined with $\mathbf{F}(\theta) = \ln(\theta)$, $\mathbf{G}(\theta) = \ln(\theta) + \theta$ and $\mathbf{H}(\theta) = -\frac{1}{\sqrt{\theta}}$ for $\theta > 0$ respectively, then it is obvious that $\mathbf{F},\mathbf{G},\mathbf{H} \in \Pi$. For other new–old types of contractive mappings, see e.g., [2–6].

In the following, we give a statement of Wardowski's theorem [1] on a fixed point which represents a generalization of the Banach Contraction Principle [7].

Theorem 1. *Let* $(\Xi, \mathbf{d}_\Xi)$ *be a complete metric space and* $\mathcal{T}\colon \Xi \to \Xi$ *be an* **F***-contraction. Then* $\mathcal{T}$ *has a unique fixed point.*

Since Wardowski gave his results, there have been various generalizations of both Theorem 1 and the notion of **F**-contraction; see e.g., [8–18]. Let Φ be the set of mappings $\phi\colon [0, +\infty) \to [0, +\infty)$ complying with conditions:

(a) ϕ is non-decreasing;

(b) $\sum_{n=1}^{+\infty} \phi^n(t) < +\infty$ for all $t > 0$.

This set is also known as a set of c-comparison functions. It is not difficult to verify that $\phi(t) < t$ for all $t > 0$ and ϕ is continuous at 0.

In [19], Samet et al. introduced two classes of mappings:

Definition 2. *Let* $\alpha\colon \Xi^2 \to [0, +\infty)$ *be a mapping where* Ξ *is nonempty set. A self-mapping* $\mathcal{T}$ *on* Ξ *is called*

(i) α*-admissible if for all* $\mathfrak{u}, \mathfrak{v} \in \Xi$,

$$\alpha(\mathfrak{u}, \mathfrak{v}) \geq 1 \text{ implies } \alpha(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \geq 1. \tag{2}$$

(ii) a triangular α*-admissible if it is* $\alpha-$*admissible and if for all* $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in \Xi$ *holds*

$$(\alpha(\mathfrak{u}, \mathfrak{v}) \geq 1 \textit{ and } \alpha(\mathfrak{v}, \mathfrak{w}) \geq 1) \text{ implies } \alpha(\mathfrak{u}, \mathfrak{w}) \geq 1. \tag{3}$$

The following lemma will be used in the sequel of this paper.

Lemma 1 ([20] [Lemma 7])**.** *Let* $\mathcal{T}$ *be a triangular* α*-admissible mapping on a nonempty set* Ξ*. Assume that there exists* $\mathfrak{u}_0 \in \Xi$ *such that* $\alpha(\mathfrak{u}_0, \mathcal{T}\mathfrak{u}_0) \geq 1$*. Define a sequence* $\{\mathfrak{u}_n\}$ *by* $\mathfrak{u}_n = \mathcal{T}^n \mathfrak{u}_0$*. Then*

$$\alpha(\mathfrak{u}_m, \mathfrak{u}_n) \geq 1 \textit{ for all } m, n \in \mathbb{N} \cup \{0\} \textit{ with } m < n.$$

In 2017, Aydi et al. [15] widened the concept of **F**-contraction as follows.

Definition 3. *Let* $(\Xi, \mathbf{d}_\Xi)$ *be a metric space. A self-mapping* $\mathcal{T}\colon \Xi \to \Xi$ *is said to be a modified* **F***-contraction via* $\alpha-$*admissible mappings if there exists* $\tau > 0$ *such that*

$$\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0 \text{ implies } \tau + \boldsymbol{F}(\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \leq \boldsymbol{F}(\phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v}))), \tag{4}$$

for all $\mathfrak{u}, \mathfrak{v} \in \Xi$*, where the mapping* $\boldsymbol{F} \in \Pi$ *and* $\phi \in \Phi$*.*

Since **F** is defined for positive real numbers only, then $\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq 0$ does not hold, which is a consequence of the condition (4) (see Example 2.1. in [15] as well as Example 1 at the end of this paper). Otherwise ([8], page 959), instead (4) there is $\tau + \alpha(\mathfrak{u}, \mathfrak{v})\mathbf{F}(d_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \leq \mathbf{F}(d_\Xi(\mathfrak{u}, \mathfrak{v}))$.

Furthermore, authors in [15] formulated and proved the following results for their modification of **F**-contraction:

Theorem 2. *Let* $(\Xi, \mathbf{d}_\Xi)$ *be a complete metric space and* $\mathcal{T}\colon \Xi \to \Xi$ *be a modified* **F***-contraction via* α*-admissible mappings. Suppose that*

(i) $\mathcal{T}$ *is* α*-admissible;*

(ii) *there exists* $\mathfrak{u}_0 \in \Xi$ *such that* $\alpha(\mathfrak{u}_0, \mathcal{T}\mathfrak{u}_0) \geq 1$*;*

(iii) $\mathcal{T}$ *is continuous.*

Then $\mathcal{T}$ *has a fixed point.*

In the next theorem they [15] replace property (iii) with the following:

(H) If $\{\mathfrak{u}_n\}$ is a sequence in Ξ such that $\alpha(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\mathfrak{u}_n \to \mathfrak{u} \in \Xi$ as $n \to +\infty$, then there exists a subsequence $\left\{\mathfrak{u}_{n(k)}\right\}$ of $\{\mathfrak{u}_n\}$ such that $\alpha\left(\mathfrak{u}_{n(k)}, \mathfrak{u}\right) \geq 1$ for all k.

Theorem 3. *Let $(\Xi, \mathbf{d}_\Xi)$ be a complete metric space and $\mathcal{T}\colon \Xi \to \Xi$ be a modified **F**-contraction via α-admissible mappings. Suppose that*

(i) *$\mathcal{T}$ is α-admissible;*
(ii) *there exists $\mathfrak{u}_0 \in \Xi$ such that $\alpha(\mathfrak{u}_0, \mathcal{T}\mathfrak{u}_0) \geq 1$;*
(iii) *(H) holds.*

Then there exists $\mathfrak{w} \in \Xi$ such that $\mathcal{T}\mathfrak{w} = \mathfrak{w}$.

Example 2.1. in [15] shows that assumptions of the previous results are not sufficient for proving that $\mathcal{T}$ has a unique fixed point. Nevertheless, adding the condition

(U) For all $\mathfrak{u}, \mathfrak{v} \in Fix(\mathcal{T})$, let $\alpha(\mathfrak{u}, \mathfrak{v}) \geq 1$ hold true, where $Fix(\mathcal{T})$ denotes the set of fixed points of $\mathcal{T}$. unicity can be obtained. After that, authors in [15] proved the following result:

Theorem 4. *Adding condition (U) to the hypotheses of Theorem 2 (resp. Theorem 3) it follows that $\mathfrak{w}$ is the unique fixed point of $\mathcal{T}$.*

Remark 1. *A general fact in mathematical analysis [21] is that if a function defined on $(0, +\infty)$ is a non-decreasing one, both its left and right limits exist at every point $h \in (0, +\infty)$. In the case of **F**-contraction, considering the condition (F_1) only, we conclude that $\mathbf{F}(h-0) \leq \mathbf{F}(h) \leq \mathbf{F}(h+0)$.*

Also, (F_1) implies one of the two following possibilities:

(1) $\mathbf{F}(0+0) = \lim_{t\to 0^+} \mathbf{F}(t) = \mathfrak{a}, \mathfrak{a} \in (-\infty, +\infty)$,
(2) $\mathbf{F}(0+0) = \lim_{t\to 0^+} \mathbf{F}(t) = -\infty$ (for more details see [16,21]).

In [14], the authors proved some Wardowski's results using only the condition (F_1), while in [16,17] some results are proved in a different way-using the condition (F_1) and the following two lemmas. Readers can find more details from the area of fixed-point theory in [22–24].

Lemma 2 (Refs. [25–27]). *Let $\{\mathfrak{u}_n\}$ be a sequence in a metric space $(\Xi, \mathbf{d}_\Xi)$ such that $\lim_{n\to+\infty} \mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = 0$. If $\{\mathfrak{u}_n\}$ is not a Cauchy sequence in $(\Xi, \mathbf{d}_\Xi)$, then there exist $\varepsilon > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers such that $n(k) > m(k) > k$, and the sequences:*

$$\left\{\mathbf{d}_\Xi\left(\mathfrak{u}_{n(k)}, \mathfrak{u}_{m(k)}\right)\right\}, \left\{\mathbf{d}_\Xi\left(\mathfrak{u}_{n(k)+1}, \mathfrak{u}_{m(k)}\right)\right\}, \left\{\mathbf{d}_\Xi\left(\mathfrak{u}_{n(k)}, \mathfrak{u}_{m(k)-1}\right)\right\},$$
$$\left\{\mathbf{d}_\Xi\left(\mathfrak{u}_{n(k)+1}, \mathfrak{u}_{m(k)-1}\right)\right\}, \left\{\mathbf{d}_\Xi\left(\mathfrak{u}_{n(k)+1}, \mathfrak{u}_{m(k)+1}\right)\right\}, \tag{5}$$

tend to ε^+, as $k \to +\infty$.

Lemma 3. *Let $\{\mathfrak{u}_{n+1}\} = \{\mathcal{T}\mathfrak{u}_n\} = \{\mathcal{T}^n\mathfrak{u}_0\}, \mathcal{T}^0\mathfrak{u}_0 = \mathfrak{u}_0$, $n \in \mathbb{N} \cup 0$ be a Picard sequence in a metric space $(\Xi, \mathbf{d}_\Xi)$ induced by a mapping $\mathcal{T}\colon \Xi \to \Xi$ and $\mathfrak{u}_0 \in \Xi$ be its initial point. If $\mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{u}_{n+1}) < \mathbf{d}_\Xi(\mathfrak{u}_{n-1}, \mathfrak{u}_n)$ for all $n \in \mathbb{N}$ then $\mathfrak{u}_n \neq \mathfrak{u}_m$ whenever $n \neq m$.*

Proof. Let us assume contrary, i.e., $\mathfrak{u}_n = \mathfrak{u}_m$ for some $n, m \in \mathbb{N}$ with $n < m$. Then $\mathfrak{u}_{n+1} = \mathcal{T}\mathfrak{u}_n = \mathcal{T}\mathfrak{u}_m = \mathfrak{u}_{m+1}$. Furthermore, we get

$$\mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = \mathbf{d}_\Xi(\mathfrak{u}_m, \mathfrak{u}_{m+1}) < \mathbf{d}_\Xi(\mathfrak{u}_{m-1}, \mathfrak{u}_m) < \ldots < \mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{u}_{n+1}), \tag{6}$$

which is a contradiction. □

2. Main Results

In this section initially we complement Definition 3 and other results given in [15]. Our approach improves, generalizes, complements and unifies several results published in recent papers as [1–13,18]. We begin with the following definition.

Definition 4. *Let $(\Xi, \mathbf{d}_\Xi)$ be a metric space. A self-mapping $\mathcal{T}: \Xi \to \Xi$ is said to be a modified $\boldsymbol{F}$-contraction via triangular α-admissible mappings if there exists $\tau > 0$ such that for all $\mathfrak{u}, \mathfrak{v} \in \Xi$ with $\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0$ yields*

$$\tau + \boldsymbol{F}(\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \leq \boldsymbol{F}(\phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v}))), \tag{7}$$

where the mapping $\boldsymbol{F} \in \Pi$ and $\phi \in \Phi$.

Now we can complete, improve and complement Theorem 3 [15] [Theorem 2.1].

Theorem 5. *Let $(\Xi, \mathbf{d}_\Xi)$ be a complete metric space and $\mathcal{T}: \Xi \to \Xi$ be a modified $\boldsymbol{F}$-contraction via triangular α-admissible mappings. Suppose that*

(i) *$\mathcal{T}$ is α-admissible;*
(ii) *there exists $\mathfrak{u}_0 \in \Xi$ such that $\alpha(\mathfrak{u}_0, \mathcal{T}\mathfrak{u}_0) \geq 1$;*
(iii) *$\mathcal{T}$ is continuous.*

Then $\mathcal{T}$ has a fixed point.

Proof. First, (F_1) and (7) yield that for all $\mathfrak{u}, \mathfrak{v} \in \Xi$ with $\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0$ we have

$$\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq \phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v})), \tag{8}$$

where $\phi \in \Phi$. The result further follows from [19] [Theorem 2.1.]. □

Here we give our version of the proof to the previous result. For that purpose, let $\mathfrak{u}_0 \in \Xi$ such that $\alpha(\mathfrak{u}_0, \mathcal{T}\mathfrak{u}_0) \geq 1$. Let us define a Picard's sequence $\{\mathfrak{u}_n\}$ in Ξ by $\mathfrak{u}_{n+1} = \mathcal{T}\mathfrak{u}_n$, for all $n \in \mathbb{N} \cup \{0\}$. If $\mathfrak{u}_m = \mathfrak{u}_{m+1}$ for some $m \in \mathbb{N} \cup \{0\}$, then $\mathfrak{w} = \mathfrak{u}_m$ is a fixed point for $\mathcal{T}$ and the proof is finished. Assume that $\mathfrak{u}_n \neq \mathfrak{u}_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathcal{T}$ is α-admissible, we can prove (for example by induction) that $\alpha(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Applying the inequality (7) with $\mathfrak{u} = \mathfrak{u}_{n-1}, \mathfrak{v} = \mathfrak{u}_n$ and using that $\alpha(\mathfrak{u}_n, \mathfrak{u}_{n+1}) \geq 1$, we get

$$\tau + \mathbf{F}(\alpha(\mathfrak{u}_{n-1}, \mathfrak{u}_n)\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}_{n-1}, \mathcal{T}\mathfrak{u}_n)) \leq \mathbf{F}(\phi(\mathbf{d}_\Xi(\mathfrak{u}_{n-1}, \mathfrak{u}_n))), \tag{9}$$

that is, since $\mathbf{F}$ satisfies (F_1),

$$\tau + \mathbf{F}(\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}_{n-1}, \mathcal{T}\mathfrak{u}_n)) \leq \mathbf{F}(\phi(\mathbf{d}_\Xi(\mathfrak{u}_{n-1}, \mathfrak{u}_n))), \tag{10}$$

i.e.,

$$\tau + \mathbf{F}(\mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{u}_{n+1})) \leq \mathbf{F}(\phi(\mathbf{d}_\Xi(\mathfrak{u}_{n-1}, \mathfrak{u}_n))) < \mathbf{F}(\mathbf{d}_\Xi(\mathfrak{u}_{n-1}, \mathfrak{u}_n)), \tag{11}$$

for all $n \in \mathbb{N}$. This, further, means that $\mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{u}_{n+1}) < \mathbf{d}_\Xi(\mathfrak{u}_{n-1}, \mathfrak{u}_n)$ for all $n \in \mathbb{N}$. Since the limit of the non-increasing sequence $\mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{u}_{n+1})$ exists and if it is for instance $\mathbf{d}^*_\Xi > 0$, then considering Remark 1 and (10) we obtain:

$$\tau + \mathbf{F}(\mathbf{d}^*_\Xi + 0) \leq \mathbf{F}(\mathbf{d}^*_\Xi + 0),$$

which is a contradiction. Hence, $\lim_{n\to+\infty} \mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{u}_{n+1}) = 0$. To prove that the sequence $\{\mathfrak{u}_n\}$ is a Cauchy one, we will use Lemma 3. Indeed, replacing $\mathfrak{u}$ with $\mathfrak{u}_{m(k)}$ and $\mathfrak{v}$ with $\mathfrak{u}_{n(k)}$ in (7) we get :

$$\tau + \mathbf{F}\Big(\alpha\Big(\mathfrak{u}_{m(k)}, \mathfrak{u}_{n(k)}\Big)\mathbf{d}_\Xi\Big(\mathcal{T}\mathfrak{u}_{m(k)}, \mathcal{T}\mathfrak{u}_{n(k)}\Big)\Big) \leq \mathbf{F}\Big(\phi\Big(\mathbf{d}_\Xi\Big(\mathfrak{u}_{m(k)}, \mathfrak{u}_{n(k)}\Big)\Big)\Big),$$

that is,

$$\tau + \mathbf{F}\Big(\alpha\Big(\mathfrak{u}_{m(k)}, \mathfrak{u}_{n(k)}\Big)\mathbf{d}_\Xi\Big(\mathfrak{u}_{m(k)+1}, \mathfrak{u}_{n(k)+1}\Big)\Big) \leq \mathbf{F}\Big(\phi\Big(\mathbf{d}_\Xi\Big(\mathfrak{u}_{m(k)}, \mathfrak{u}_{n(k)}\Big)\Big)\Big). \tag{12}$$

According to Lemma 1 , $\alpha\Big(\mathfrak{u}_{m(k)}, \mathfrak{u}_{n(k)}\Big) \geq 1$. Therefore, the last inequality becomes

$$\tau + \mathbf{F}\Big(\mathbf{d}_\Xi\Big(\mathfrak{u}_{m(k)+1}, \mathfrak{u}_{n(k)+1}\Big)\Big) \leq \mathbf{F}\Big(\phi\Big(\mathbf{d}_\Xi\Big(\mathfrak{u}_{m(k)}, \mathfrak{u}_{n(k)}\Big)\Big)\Big),$$

or due to the properties of the functions $\mathbf{F}$ and ϕ

$$\tau + \mathbf{F}\Big(\mathbf{d}_\Xi\Big(\mathfrak{u}_{m(k)+1}, \mathfrak{u}_{n(k)+1}\Big)\Big) \leq \mathbf{F}\Big(\mathbf{d}_\Xi\Big(\mathfrak{u}_{m(k)}, \mathfrak{u}_{n(k)}\Big)\Big) \tag{13}$$

for all $k \in \mathbb{N}$. Taking the limit in (13) as $k \to +\infty$, we get

$$\tau + \mathbf{F}(\varepsilon + 0) \leq \mathbf{F}(\varepsilon + 0),$$

which is a contradiction. The proof of our method (approach) is finished.

Remark 2. *It is worth noticing that in our approach we use only the property (F_1). Therefore, our method significantly improves several recent results given in current literature* [8,9,15].

Now, by using our approach we will prove Theorem 3, i.e., Theorem 2.2 from [15]. First we formulate it.

Theorem 6. *Let $(\Xi, \mathbf{d}_\Xi)$ be a complete metric space and $\mathcal{T}\colon \Xi \to \Xi$ be a modified $\mathbf{F}$-contraction via α-admissible mappings. Suppose that*
(i) $\mathcal{T}$ is α-admissible;
(ii) there exists $\mathfrak{u}_0 \in \Xi$ such that $\alpha(\mathfrak{u}_0, \mathcal{T}\mathfrak{u}_0) \geq 1$;
(iii) (H) holds.

Then there exists $\mathfrak{w} \in \Xi$ such that $\mathcal{T}\mathfrak{w} = \mathfrak{w}$.

Proof. Following the lines in the proof to the Theorem 5, we see that $\mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{u}_{n+1}) < \mathbf{d}_\Xi(\mathfrak{u}_{n-1}, \mathfrak{u}_n)$ for all $n \in \mathbb{N}$, where $\{\mathfrak{u}_n\}$ is a Picard's sequence induced by the point $\mathfrak{u}_0$. Then, according to Lemma 2 , we achieve the points $\lim_{n\to+\infty} \mathfrak{u}_n = \mathfrak{w}, \mathcal{T}\mathfrak{w} \notin \{\mathfrak{u}_n\}_{n\geq k}$ for some $k \in \mathbb{N}$. Hence, for all $n \geq k$ it follows by (7)

$$\tau + \mathbf{F}(\alpha(\mathfrak{u}_n, \mathfrak{u})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}_n, \mathcal{T}\mathfrak{w})) \leq \mathbf{F}(\phi(\mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{w}))) < \mathbf{F}(\mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{w})), \tag{14}$$

or by the property (F_1) for $\mathbf{F}$ and the property of ϕ it follows

$$\tau + \mathbf{F}(\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}_n, \mathcal{T}\mathfrak{w})) < \mathbf{F}(\mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{w})). \tag{15}$$

It is clear that (15) yields $\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}_n, \mathcal{T}\mathfrak{w}) < \mathbf{d}_\Xi(\mathfrak{u}_n, \mathfrak{w}) \to 0$, as $n \to +\infty$. This means that $\mathcal{T}\mathfrak{u}_n = \mathfrak{u}_{n+1} \to \mathfrak{w}$ as $n \to +\infty$. Hence, $\mathcal{T}\mathfrak{w} = \mathfrak{w}$, i.e., $\mathfrak{w}$ is a fixed point of the mapping $\mathcal{T}$. The proof is complete. □

In all the following corollaries we use only the property (F_1) of the mapping $\mathbf{F}\colon (0, +\infty) \to (-\infty, +\infty)$ which genuinely generalizes the ones from [8,9,15], which represents entire literature on the topic.

Corollary 1. *Let $(\Xi, \mathbf{d}_\Xi)$ be a complete metric space and $\mathcal{T}\colon \Xi \to \Xi$ be a given mapping. Suppose there exists $\tau > 0$ such that*

$$\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0 \text{ yields } \tau + \mathbf{F}(\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \leq \mathbf{F}(\phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v}))), \tag{16}$$

for all $\mathfrak{u}, \mathfrak{v} \in \Xi$ *where* $\mathbf{F}$ *satisfies* (F_1)*. Then* $\mathcal{T}$ *has a unique fixed point.*

Proof. It sufficient to take $\alpha(\mathfrak{u}, \mathfrak{v}) = 1$ in Theorem 5. □

Corollary 2. *Let* $(\Xi, \mathbf{d}_\Xi)$ *be a complete metric space and* $\mathcal{T} \colon \Xi \to \Xi$ *be a given mapping. Suppose there exists* $\tau > 0$ *such that*

$$\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0 \text{ implies } \tau + \boldsymbol{F}(\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \leq \boldsymbol{F}(\mathfrak{c}\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v})), \tag{17}$$

for all $\mathfrak{u}, \mathfrak{v} \in \Xi$ *where* $\mathbf{F}$ *satisfies* (F_1) *and* $\mathfrak{c} \in (0, 1)$*. Then* $\mathcal{T}$ *has a unique fixed point.*

Proof. It follows from Corollary 1 with $\phi(t) = \mathfrak{c}t$. □

The following are some consequences of the previously obtained results. Specifically, we get the following new contractive conditions that complement the ones from [28,29].

Corollary 3. *Let* $(\Xi, \mathbf{d}_\Xi)$ *be a complete metric space and* $\mathcal{T} \colon \Xi \to \Xi$ *be a modified* $\mathbf{F}$*-contraction via triangular* α*-admissible mappings. Suppose that there exists* $\tau_i > 0, i = \overline{1, 9}$ *and*

(i) $\mathcal{T}$ *is* α*-admissible;*
(ii) *there exists* $\mathfrak{u}_0 \in \Xi$ *such that* $\alpha(\mathfrak{u}_0, \mathcal{T}\mathfrak{u}_0) \geq 1$*;*
(iii) *either* $\mathcal{T}$ *is continuous or* (H) *holds, such that the following inequalities hold true:*

$$\tau_1 + \alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq \phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v})), \tag{18}$$

$$\tau_2 + \exp(\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \leq \exp(\phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v}))), \tag{19}$$

$$\tau_3 - \frac{1}{\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})} \leq -\frac{1}{\phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v}))}, \tag{20}$$

$$\tau_4 - \frac{1}{\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})} + \alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq -\frac{1}{\phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v}))} + \phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v})), \tag{21}$$

$$\tau_5 + \frac{1}{1 - \exp(\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}))} \leq \frac{1}{1 - \exp(\phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v})))} \tag{22}$$

$$\begin{aligned} \tau_6 &+ \frac{1}{\exp(-\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) - \exp(\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}))}, \\ &\leq \frac{1}{\exp(-\phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v}))) - \exp(\phi(\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v})))}, \end{aligned} \tag{23}$$

$$\tau_7 + \alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq \mathfrak{c}\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v}), \tag{24}$$

where $\mathfrak{c} \in (0, 1)$.

$$\tau_8 + \alpha^k(\mathfrak{u}, \mathfrak{v})\mathbf{d}^k_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq \mathbf{d}^k_\Xi(\mathfrak{u}, \mathfrak{v}), \tag{25}$$

where $k > 0$.

$$\tau_9 + \alpha^2(\mathfrak{u}, \mathfrak{v})\mathbf{d}^2_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) + \mathfrak{a}\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq \mathbf{d}^2_\Xi(\mathfrak{u}, \mathfrak{v}) + \mathfrak{a}\mathbf{d}_\Xi(\mathfrak{u}, \mathfrak{v}), \tag{26}$$

where $\mathfrak{a} > 0$.

Then there exists $\mathfrak{w} \in \Xi$ *such that* $\mathcal{T}\mathfrak{w} = \mathfrak{w}$.

Proof. If put in the previously obtained results $\mathbf{F}(\theta) = \theta, \mathbf{F}(\theta) = \exp(\theta), \mathbf{F}(\theta) = -\frac{1}{\theta}, \mathbf{F}(\theta) = \theta - \frac{1}{\theta}, \mathbf{F}(\theta) = \frac{1}{1-\exp(\theta)}, \mathbf{F}(\theta) = \frac{1}{\exp(-\theta)-\exp(\theta)}, \mathbf{F}(\theta) = \theta^k, \mathbf{F}(\theta) = \theta^2 + \mathfrak{a}\theta$, respectively, then we get the contractive conditions (18)–(26). Since every of the function $\theta \to \mathbf{F}(\theta)$ is strictly increasing on $(0, +\infty)$, the result follows by Theorems 5 and 6. □

Remark 3. *Adding condition* (U) *to the hypotheses of Corollary 3 we obtain that* $\mathfrak{w}$ *is the unique fixed point of* $\mathcal{T}$ *in all* (18)–(26) *contractive conditions.*

In [22] Ćirić has collected various contractive mappings in usual metric spaces (see also [28]). The next three contractive conditions are well known in the existing literature: The self-mapping $\mathcal{T}\colon \Xi \to \Xi$ on metric space $(\Xi, \mathbf{d}_\Xi)$ is called

- Ćirić 1: a generalized contraction of first order if there exists $a_1 \in [0,1)$ such that for all $\mathfrak{u}, \mathfrak{v} \in \Xi$ holds:

$$\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq a_1 \max\left\{\mathbf{d}_\Xi(\mathfrak{u},\mathfrak{v}), \frac{\mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{u}) + \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{v})}{2}, \frac{\mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{v}) + \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{u})}{2}\right\}. \tag{27}$$

- Ćirić 2: a generalized contraction of second order if there exists $a_2 \in [0,1)$ such that for all $\mathfrak{u}, \mathfrak{v} \in \Xi$ holds:

$$\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq a_2 \max\left\{\mathbf{d}_\Xi(\mathfrak{u},\mathfrak{v}), \mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{u}), \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{v}), \frac{\mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{v}) + \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{u})}{2}\right\}. \tag{28}$$

- Ćirić 3: a quasi-contraction if there exists $a_3 \in [0,1)$ such that for all $\mathfrak{u}, \mathfrak{v} \in \Xi$ holds:

$$\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) \leq a_3 \max\{\mathbf{d}_\Xi(\mathfrak{u},\mathfrak{v}), \mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{u}), \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{v}), \mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{v}), \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{u})\}. \tag{29}$$

Since,

$$\frac{\mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{u}) + \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{v})}{2} \leq \max\{\mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{u}), \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{v})\}$$

and

$$\frac{\mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{v}) + \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{u})}{2} \leq \max\{\mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{v}), \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{u})\}$$

it follows that (27) implies (28) and (28) implies (29).

In [22] Ćirić also proved the following result:

Theorem 7. *Each quasi-contraction $\mathcal{T}$ on a complete metric space $(\Xi, \mathbf{d}_\Xi)$ has a unique fixed point (say) $\mathfrak{w}$. Moreover, for all $\mathfrak{u} \in \Xi$ the sequence $\{\mathcal{T}^n\mathfrak{u}\}_{n=0}^{+\infty}$, $\mathcal{T}^0\mathfrak{u} = \mathfrak{u}$ converges to the fixed point $\mathfrak{w}$ as $n \to +\infty$.*

Now we can formulate the following notion and an open question:

Definition 5. *Let $(\Xi, \mathbf{d}_\Xi)$ be a metric space. A self-mapping $\mathcal{T}\colon \Xi \to \Xi$ is said to be a modified **F**-contraction via triangular α-admissible mappings if there exist $\tau > 0$ such that for all $\mathfrak{u}, \mathfrak{v} \in \Xi$ with $\alpha(\mathfrak{u},\mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0$ yields*

$$\tau + \mathbf{F}(\alpha(\mathfrak{u},\mathfrak{v})\mathbf{d}_\Xi(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \leq \mathbf{F}(\phi(\max \mathfrak{M}(\mathfrak{u},\mathfrak{v}))), \tag{30}$$

where $\mathbf{F} \in \Pi, \phi \in \Phi$ and $\mathfrak{M}(\mathfrak{u},\mathfrak{v})$ is one of the sets:

$$\left\{\mathbf{d}_\Xi(\mathfrak{u},\mathfrak{v}), \frac{\mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{u}) + \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{v})}{2}, \frac{\mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{v}) + \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{u})}{2}\right\},$$

$$\left\{\mathbf{d}_\Xi(\mathfrak{u},\mathfrak{v}), \mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{u}), \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{v}), \frac{\mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{v}) + \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{u})}{2}\right\}$$

or

$$\{\mathbf{d}_\Xi(\mathfrak{u},\mathfrak{v}), \mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{u}), \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{v}), \mathbf{d}_\Xi(\mathfrak{u},\mathcal{T}\mathfrak{v}), \mathbf{d}_\Xi(\mathfrak{v},\mathcal{T}\mathfrak{u})\}.$$

A suggestion for further research would be to find out if the following statement is true or not:

Each modified **F**-contraction $\mathcal{T}\colon \Xi \to \Xi$ via triangular α-admissible mappings defined on a complete metric space $(\Xi, \mathbf{d}_\Xi)$ has a fixed point if either $\mathcal{T}$ is continuous or the property (H) holds.

3. Some Examples

According to the consideration in previously section, we have the following three types **F**-contractions via some (triangular) α-admissible mappings:

1. (Ref. [15] Aydi et al.) There exists $\tau > 0$ such that

$$\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0 \text{ implies } \tau + \mathbf{F}(\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \leq \mathbf{F}(\phi(\mathbf{d}_{\Xi}(\mathfrak{u}, \mathfrak{v}))), \tag{31}$$

where $\mathcal{T}\colon \Xi \to \Xi, \alpha\colon \Xi^2 \to [0, +\infty)$ satisfy (1) and (2) while $\phi \in \Phi$.

2. (Ref. [8] Gopal et al.) There exists $\tau > 0$ such that

$$\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0 \text{ implies } \tau + \alpha(\mathfrak{u}, \mathfrak{v})\mathbf{F}(\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \leq \mathbf{F}(\phi(\mathbf{d}_{\Xi}(\mathfrak{u}, \mathfrak{v}))), \tag{32}$$

where $\mathcal{T}\colon \Xi \to \Xi, \alpha\colon \Xi^2 \to [0, +\infty)$ satisfy (1) and (2) while $\phi \in \Phi$.

3. (Our approach) There exists $\tau > 0$ such that

$$\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0 \text{ implies } \tau + \mathbf{F}(\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \leq \mathbf{F}(\phi(\mathbf{d}_{\Xi}(\mathfrak{u}, \mathfrak{v}))), \tag{33}$$

where $\mathcal{T}\colon \Xi \to \Xi, \alpha\colon \Xi^2 \to [0, +\infty)$ satisfy (1) and (2) while $\phi \in \Phi$.

Following are examples that support or do not support the three stated contractive conditions. These are also examples of triangular α-admissible mappings.

Example 1. *Let $\Xi = 0, 1, 2$ and $\mathcal{T}\colon \Xi \to \Xi$ defined with $\mathcal{T}0 = 0, \mathcal{T}1 = \mathcal{T}2 = 1$. Take $\alpha\colon \Xi^2 \to [0, +\infty)$ as $\alpha(1, 2) = \alpha(2, 1) = \alpha(1, 1) = 1$, and $\alpha(\mathfrak{u}, \mathfrak{v}) = 0$ for $(\mathfrak{u}, \mathfrak{v}) \notin \{(1, 2), (2, 1), (1, 1)\}$. We have that $\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0$ for $(\mathfrak{u}, \mathfrak{v}) \in \{(0, 1), (0, 2)\}$. In both cases we get $\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) = 0$. This means that for $(\mathfrak{u}, \mathfrak{v}) \in \{(0, 1), (0, 2)\}$ left hand $\tau + \mathbf{F}(\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) = \tau + \mathbf{F}(0)$ of (31) is not defined. This shows that often the condition $\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) > 0$ does not support. Hence, since $\alpha(\mathfrak{u}, \mathfrak{v})\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v}) = 0$ whenever $(\mathfrak{u}, \mathfrak{v}) \in \{(1, 2), (2, 1), (1, 1)\}$ then the given example does not support the conditions (31) and (33). Regarding to the condition (32) we get the following:*

Since for $(\mathfrak{u}, \mathfrak{v}) \in \{(0, 1), (0, 2)\}$ the condition (32) becomes

$$\tau + 0 \cdot \mathbf{F}(\mathbf{d}_{\Xi}(\mathcal{T}\mathfrak{u}, \mathcal{T}\mathfrak{v})) \leq \mathbf{F}(\phi(\mathbf{d}_{\Xi}(\mathfrak{u}, \mathfrak{v}))),$$

i.e.,

$$\tau \leq \mathbf{F}(\phi(1)).$$

The last is possible because $\mathbf{F}(\phi(1)) > 0$. Hence, the given example supports the condition (32) for each strictly increasing function $\mathbf{F}\colon (0, +\infty) \to (-\infty, +\infty)$.

Example 2. *Take $\Xi = (-\infty, +\infty)$. We define $\mathcal{T}\colon \Xi \to \Xi$ by $\mathcal{T}\mathfrak{u} = \mathfrak{u}^3$ and $\alpha\colon \Xi^2 \to [0, +\infty)$ by*

$$\alpha(\mathfrak{u}, \mathfrak{v}) = \begin{cases} \sqrt[3]{\mathfrak{u}^3 + \mathfrak{v}^3}, & (\mathfrak{u}, \mathfrak{v}) \in [1, +\infty)^2 \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ is a triangular α-admissible mapping.

Example 3. *Take $\Xi = [0, +\infty)$. We define $\mathcal{T}\colon \Xi \to \Xi$ by $\mathcal{T}\mathfrak{u} = \sqrt{\mathfrak{u}}$ and $\alpha\colon \Xi^2 \to [0, +\infty)$ by*

$$\alpha(\mathfrak{u}, \mathfrak{v}) = \begin{cases} \mathfrak{u} - \mathfrak{v} + 2, & \mathfrak{u} \geq \mathfrak{v} \\ \frac{1}{4}, & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ is a triangular α-admissible mapping.

Example 4. *Let* $\Xi = (-\infty, +\infty)$. *We define* $\mathcal{T}\colon \Xi \to \Xi$ *by* $\mathcal{T}\mathfrak{u} = r \cdot \mathfrak{u}$, $r \geq 1$ *and* $\alpha\colon \Xi^2 \to [0, +\infty)$ *by*

$$\alpha(\mathfrak{u}, \mathfrak{v}) = \begin{cases} \frac{\mathfrak{u}-\mathfrak{v}}{2} + 1, \mathfrak{u} \geq \mathfrak{v}, \textit{both } \mathfrak{u}, \mathfrak{v} \textit{ non-negative, or both } \mathfrak{u} \textit{ and } \mathfrak{v} \textit{ negative,} \\ \textit{or } \mathfrak{v} \textit{ is negative and } \mathfrak{u} \textit{ is positive} \\ 1, \mathfrak{v} \geq \mathfrak{u}, \textit{ both } \mathfrak{u} \textit{ and } \mathfrak{v} \textit{ non-negative,} \\ 0, \textit{ otherwise.} \end{cases}$$

Then $\mathcal{T}$ *is a triangular* α*-admissible mapping.*

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