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Dynamic modelling of elastic plates reinforced by strong fibres

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Abstract. Here are considered composite materials reinforced which strong fibres, which have an important property that they are anisotropic, and in many cases this anisotropy may be very strong, in the sense that mechanical properties are strongly dependent on direction. Such materials are highly resistant to deformation by extension in the fibre direction compared to other deformation modes. Thus material has been modelled as a transversely isotropic for which the extensional modulus in the fibre direction is much greater than that in a direction perpendicular to the fibres. Material is modelled as coordinate free and such model is employed for examination of waves propagating in an infinite plate reinforced by strong fibres. Wave propagation in layer reinforced by one family of fibres with wave direction parallel with stress free boundaries, but otherwise with angle arbitrary to fibre direction, are examined here. The dispersion relations, in specific carbon fibres - epoxy resin composites, are examined. Expressions are also obtained for the variation of displacements and stresses through the thickness of the plate.

1. Introduction

Fibre reinforced material is considered using continuum mechanics approach to describe macroscopic properties, in which there no distinction between fibres and matrices material.

A composite is modeled as linear elastic continuum with one preferred direction. Problems in which fibre reinforced materials are used may be simplified by assumption that fibres may be considered as inextensible. This constraint is good mathematical simplification for strongly anisotropic material, for which extensional moduli in the fibre direction may be hundred time higher than those in transversal direction. Naturally there are not composites which have inextensible fibres, but fibres are almost inextensible. In spite of that constraint, predictions of such idealized theory is good approximation of real behavior. To have impression on level of approximation it is recommended to consider models of continuum as reinforced with inextensible as well as extensible fibres and make conclusions by comparison of results.

Constitutive relations for such materials are developed by Spencer [1, 2], wave propagation along the fibres in plate reinforced by one family of fibres was examined by Green [3], for bending waves, and by Green and Milosavljevic [4], for extensional waves. Some aspects of bulk waves are examined by Milosavljevic et al in [5]. Dynamical behavior of materials reinforced by two families of fibres are examined by Milosavljevic [6–8]. Reference [9] is focused on assessment of different options of creating long fibre composite models for 3D printing. These models are reinforced using long aramid, carbon and glass fibres. Various aspects of dynamical consideration of fibre reinforced materials are examined in references [10–13].

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Wave propagation in layer reinforced by one family of fibres with wave direction parallel with stress free boundaries, but otherwise with angle arbitrary to fibre direction, are examined here.

2. Governing equations

Here is examined linear elastic material reinforced by one family of fibres with preferred direction defined with unit vector field with components a_i, which are continual function of displacement vector with Cartesian coordinates x_i.

When developing constitutive equations for elastic materials, it is usually the best to find an equation for the *strain energy density* of the material as a function of the strain, rather than trying to write down stress-strain laws directly. This has several advantages, among them is that one can work with a scalar function and because the existence of a strain energy density guarantees that deformations of the material are perfectly reversible. Since the fibre direction depends on the position, here is considered a coordinate free formulation of the constitutive equations developed by Spencer [1, 2]. That is particularly useful when fibres are strong and one may separate the stiffness that depends on the fibre strength, which gives the possibility to consider cases when fibres gradually become stronger and stronger.

A material reinforced by one family of fibres may be described with five independent invariants, two of which depend on the fibre direction, say a_i, which is in agreement with the fact that material is transversely isotropic. In the case of linear elasticity, following Spencer [1, 2], it may be shown that the strain energy function may be expressed in terms of five irreducible invariants:

$$tr\mathbf{e},\quad tr\mathbf{e}^2,\quad tr\mathbf{e}^3,\quad \mathbf{a}\cdot\mathbf{e}\cdot\mathbf{a},\quad \mathbf{a}\cdot\mathbf{e}^2\mathbf{a} \tag{1}$$

where $\mathbf{e}$ is infinitesimal strain tensor whose components are $e_{ij}=(u_{i,j}+u_{j,i})/2$, u_i components of displacement vector, and $\mathbf{a}$ vector of the fibre direction whose components are a_i, all in Cartesian coordinate system x_i, with i = 1, 2, 3.

The most general quadratic function in $\mathbf{e}$ may be written as:

$$W=\tfrac{1}{2}\lambda(tr\mathbf{e})^2+\mu_T tr\mathbf{e}^2+\alpha(\mathbf{a}\cdot\mathbf{e}\cdot\mathbf{a})tr\mathbf{e}+2(\mu_L-\mu_T)\mathbf{a}\cdot\mathbf{e}^2\cdot\mathbf{a}+\tfrac{1}{2}\beta(\mathbf{a}\cdot\mathbf{e}\cdot\mathbf{a})^2 \tag{2}$$

which is the *strain energy density* of the linear elastic material. Coefficients $\lambda,\mu_T,\mu_L,\alpha,\beta$ are elastic constants with stress dimensions, and δ_{ij} is Kronecker's δ symbol.

Stress strain relation for transversely isotropic material is:

$$\begin{aligned}\sigma_{ij}=\frac{\partial W}{\partial e_{ij}}&=\lambda e_{kk}\delta_{ij}+2\mu_T e_{ij}+\alpha(a_\ell a_m e_{\ell m}\delta_{ij}+e_{kk}a_i a_j)\\&+2(\mu_L-\mu_T)(a_i a_k e_{kj}+a_j a_k e_{ki})+\beta a_\ell a_m e_{\ell m}a_i a_j\end{aligned} \tag{3}$$

where σ_{ij} are components of Cauchy stress tensor. Here, and throughout the paper, summation convention over repeated indices will be employed, and coma represents partial derivative.

The stiffness tensor in index notation may be expressed in the form:

$$\begin{aligned}C_{ijkl}=\frac{\partial^2 W}{\partial e_{ij}\partial e_{kl}}&=\lambda\delta_{ij}\delta_{kl}+\mu_T(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})+\alpha(a_k a_l\delta_{ij}+a_i a_j\delta_{kl})\\&+(\mu_L-\mu_T)(a_i a_k\delta_{jl}+a_i a_l\delta_{jk}+a_j a_k\delta_{il}+a_j a_l\delta_{ik})+\beta a_i a_j a_k a_l\end{aligned} \tag{4}$$

When fibres become inextensible, the material behaves as a material with constrains, which gives rise to the introduction of Lagrange's multipliers that are proportional to the generated reaction stresses. Such a material is generally referred to as an "ideal fibre reinforced material". That is particularly useful when fibres are strong and one may separate the stiffness that depends on the fibre strength, which gives the possibility to consider cases when fibres gradually become stronger and stronger.

Kinematical constraint of inextensibility in the fibre direction may be expressed as follows:

$$e_{ij}a_i a_j = 0 \tag{5}$$

leading to constitutive equation for constrained material:

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + 2(\mu_L - \mu_T)(a_i e_{jn} a_n + a_m e_{mi} a_j) + \hat{T}\, a_i a_j \tag{6}$$

where $\hat{T}$ is Lagrange's multiplier and $\hat{T} a_i a_j$ represent components of reaction stresses to constraint of inextensibility.

Equations of motion with no body forces are in the form:

$$\sigma_{ij,j} = \rho \ddot{u}_i, \qquad i = 1,2,3. \tag{7}$$

where ρ density, and superimposed dot is time derivative. The boundary conditions of infinite plate, which has thickness h, with stress free boundaries, which are parallel to $x_2 x_3$ plane, may be expressed as:

$$\sigma_{1i} = 0, \quad (i = 1,2,3) \qquad \text{at} \qquad x_1 = \pm h \tag{8}$$

Equations (1) to (8) are the governing equations of system.

3. Dispersion relations for extensional waves in plate reinforced by one family of inextensible fibres

Let us suppose that in above described plate, with fibres parallel to plate boundaries, plane wave propagates with wave direction parallel to stress free boundaries. To examine plane wave motion, displacements will be assumed in a form:

$$u_1 = U_{(x_1)}\cos\varphi, \qquad u_2 = V_{(x_1)}\sin\varphi, \qquad u_3 = W_{(x_1)}\sin\varphi, \qquad \hat{T} = T_{(x_1)}\cos\varphi \tag{9}$$

where:

$$\varphi = k(x_2 s + x_3 c - vt), \qquad s \equiv \sin\alpha, \qquad c \equiv \cos\alpha \tag{10}$$

and α angle between wave normal and x_3 axe, along which is fibre direction.

By employment of constitutive equation (3) and assumptions (9) and (10), equations of motion (7) lead to set of equations:

$$\begin{aligned} &c_1^2 U'' + k^2(v^2 - c_2^2 s^2 - c_3^2 c^2)U + (c_1^2 - c_2^2)ksV' = 0, \\ &-(c_1^2 - c_2^2)ksU' + c_2^2 V'' + k^2(v^2 - c_1^2 s^2 - c_3^2 c^2)V = 0, \\ &c_3^2 W'' + k^2(v^2 - c_3^2 s^2)W - c_3^2(U' + ksV)kc - Tck/\rho = 0 \end{aligned} \tag{11}$$

where "prime" denote derivation with respact to x_1 and new material constants are in relation to those introduced in equation (2) as follows:

$$c_1^2 = \frac{\lambda + 2\mu_T}{\rho}, \qquad c_2^2 = \frac{\mu_T}{\rho}, \qquad c_3^2 = \frac{\mu_L}{\rho} \tag{12}$$

Inextensible constraint (5), for displacements given in (9), has the following form:

$$kcW = 0 \tag{13}$$

Here attention will be restricted to symmetric deformations that excite extensional waves and functions of x_1 in (11) may be assumed as follows:

$$U = A\sin kpx_1,\qquad V = B\cos kpx_1,\qquad W = D\cos kpx_1,\qquad T = \rho c_3^2 kE\cos kpx_1 \tag{14}$$

where A, B, D, and E are arbitrary constants. Assumed general solutions (14), substituted in equations of motion (11), leads to system of algebraic equations:

$$\begin{aligned} &c_1^2 p^2 A - (v^2 - c_2^2 s^2 - c_3^2 c^2)A + (c_1^2 - c_2^2)spB = 0, \\ &(c_1^2 - c_2^2)spA + c_2^2 p^2 B - (v^2 - c_1^2 s^2 - c_3^2 c^2)B = 0, \\ &c_3^2(pA + sB)c + c_3^2 p^2 D - (v^2 - c_3^2 s^2)D + c_3^2 cE = 0 \end{aligned} \tag{15}$$

which, with constraint condition (13),

$$cD=0 \tag{16}$$

lead to system of homogeneous algebraic equations for determination the constants A, B, D and E. Constant E appears in equation $(2)_3$ only, and it should be employed to determine E. For $c \neq 0$ equation (16) implies $D = 0$, which means that $W = 0$ leading, from $(15)_{1,2}$, to set of two algebraic equations for determination the constants A and B. Thus, non-trivial solutions for A and B may be obtained provided:

$$(c_1^2 p^2 - v^2 + c_2^2 s^2 + c_3^2 c^2)(c_2^2 p^2 - v^2 + c_1^2 s^2 + c_3^2 c^2) - (c_1^2 - c_2^2)^2 s^2 p^2 = 0 \qquad 17)$$

is satisfied, leading to solutions for p^2 as:

$$p_1^2 = \frac{v^2 - c_3^2 c^2}{c_1^2} - s^2,\qquad p_2^2 = \frac{v^2 - c_3^2 c^2}{c_2^2} - s^2 \tag{18}$$

and to general solutions:

$$\begin{aligned} &U = p_1 A_1 \sin kp_1 x_1 + sA_2 \sin kp_2 x_1,\qquad V = sA_1 \cos kp_1 x_1 - p_2 A_2 \cos kp_2 x_1, \\ &W = 0,\qquad T = -\rho k c_3^2 A_1 (p_1^2 + s^2)\cos kp_1 x_1 \end{aligned} \tag{19}$$

Obtained general solutions (19) should satisfy stress free boundary conditions (8) and it may be concluded that following relations should be satisfied:

$$\begin{aligned} &A_1[c_1^2(p_1^2 + s^2) - 2c_2^2 s^2]\cos kp_1 h + 2sp_2 c_2^2 A_2 \cos kp_2 h = 0, \\ &-2sp_1 A_1 \sin kp_1 h + (p_2^2 - s^2)A_2 \sin kp_2 h = 0, \\ &p_1 A_1 \sin kp_1 h + sA_2 \sin kp_2 h = 0 \end{aligned} \tag{20}$$

That is system of three homogeneous equations for two unknown which can be satisfied for trivial solutions $A_1 = A_2 = 0$ only, that is when $U = V = 0$.

Therefore, to obtain non trivial solutions, it is necessary to postulate existence of singular layers on upper and lower boundary surface. In the singular layer reaction stress, due to constraint of inextensibility and to finite stress on both upper and lower boundary surface, become infinite.

Through singular layers shear stress σ_{13} is discontinuous having zero values on stress free boundaries $x_1 = \pm h$, but finite value when $x_1 \to \pm h$ from inside of the plate. Existence of such singular surfaces may be proved if one integrate equation (7), for i = 3, from $x_1 = h(1-\varepsilon)$ to $x_1 = h$ on the upper

boundary and from $x_1 = -h(1-\varepsilon)$ to $x_1 = -h$ on the lower boundary, for any ε that satisfies condition $0 < \varepsilon < 1$.
On the upper boundary is easy to obtain:

$$\left[\sigma_{13}\right]_{h(1-\varepsilon)}^{h} = kc\sin\phi \int_{h(1-\varepsilon)}^{h} (\rho ksc_3^2 V + T)\, dx_1 \tag{21}$$

which, taking into account that has zero value on the upper boundary, for $x_1 = h$, and finite value in the points near to the boundary, leads to:

$$L^+ = \lim_{\varepsilon \to 0} \int_{h(1-\varepsilon)}^{h} T(x_1)\, dx_1 = \rho c_3^2\, U(h) \tag{22}$$

with a corresponding load:

$$\underline{L}^+ = L^+ \cos\varphi \tag{23}$$

Here L^+ represents load that, on the upper boundary, carry singular fibres.

If one repeats same procedure on the lower boundary it is easy to conclude $L^- = L^+$, that is load that singular fibres carry on the lower boundary is same as that on the upper boundary.

In order that first two boundary conditions are satisfied, following equation:

$$(p_2^2 - s^2)^2 \cos kp_1 h \sin kp_2 h + 4s^2 p_1 p_2 \sin kp_1 h \cos kp_2 h = 0 \tag{24}$$

should be satisfied, which is frequency equation of considered system. From the equation (24), for fundamental mode, may be expressed phase speed v as a function of non-dimensional wave number kh representing dispersion relation. For long waves, when $kh \to 0$, from the equation (24) is easy to obtain solution for phase speed:

$$v_I^2 = c_3^2 c^2\,, \qquad v_{II}^2 = c_3^2 c^2 + 4s^2 c_2^2 \frac{c_1^2 - c_2^2}{c_1^2} \tag{25}$$

Equation (24) may be solved numerically by semi-inverse method in which, for assumed value of phase speed, one calculate p_i, for $i = 1, 2$, and then value kh that satisfies this equation. In such way it is possible to calculate pairs of values (v, kh) which, substituted in (19), lead to displacements distribution throughout plate thickness in form:

$$\begin{aligned} U &= p_1 A \left[\frac{\sin kp_1 x_1}{\sin kp_1 h} + \frac{2s^2}{p_2^2 - s^2} \frac{\sin kp_2 x_1}{\sin kp_2 h} \right], \\ W &= sA \left[\frac{\cos kp_1 x_1}{\sin kp_1 h} - \frac{2p_1 p_2}{p_2^2 - s^2} \frac{\cos kp_2 x_1}{\sin kp_2 h} \right] \end{aligned} \tag{26}$$

and, in similar way, from equation (9), distribution of stresses throughout plate thickness as a function of one constant only.

4. Dispersion relations for extensional waves in plate reinforced by one family of extensible fibres

If one assumes same geometry of plate, plan wave propagation in case when plate is made of one family of strong, but extensible, fibres displacements may be assumed in form:

$$u_1 = U_{(x_1)} \cos\varphi, \qquad u_2 = V_{(x_1)} \sin\varphi, \qquad u_3 = W_{(x_1)} \sin\varphi \tag{27}$$

where relation (10) are valid.

Substitution of prescribed displacements (27) into equations of motion (7), and using (4), leads to:

$$\begin{aligned} &c_1^2 U'' + k^2(v^2 - c_2^2 s^2 - c_3^2 c^2)U + (c_1^2 - c_2^2)ksV' + kc(c_3^2 + c_4^2)W' = 0, \\ &c_1^2 U'' + k^2(v^2 - c_2^2 s^2 - c_3^2 c^2)U + (c_1^2 - c_2^2)ksV' + kc(c_3^2 + c_4^2)W' = 0, \\ &-(c_3^2 + c_4^2)kcU' - (c_3^2 + c_4^2)k^2 scV + c_3^2 W'' + k^2(v^2 - c_3^2 s^2 - c_5^2 c^2)W = 0 \end{aligned} \tag{28}$$

where new material constants, on top of those defined in (12), are expressed as:

$$c_4^2 = \frac{\lambda + \alpha}{\rho}, \qquad c_5^2 = \frac{\lambda + 4\mu_L - 2\mu_T + 2\alpha + \beta}{\rho} \tag{29}$$

Extensional waves in materials with extensible fibres, which represent symmetric deformations, may be represented with $U_{(x_1)}$ as odd function, whereas $V_{(x_1)}$ and $W_{(x_1)}$ as even function of x_1. Therefore:

$$U = A\sin kpx_1, \qquad V = B\cos kpx_1, \qquad W = D\cos kpx_1, \tag{30}$$

where A, B and D are arbitrary constants. Substitution of (30) into (28) leads to homogeneous system of algebraic equations which has nontrivial solutions provided that equation:

$$\begin{vmatrix} d_{11} & -sp(c_1^2 - c_2^2) & -cp(c_3^2 + c_4^2) \\ -sp(c_1^2 - c_2^2) & d_{22} & -sc(c_3^2 + c_4^2) \\ -cp(c_3^2 + c_4^2) & -sc(c_3^2 + c_4^2) & d_{33} \end{vmatrix} = 0 \tag{31}$$

where:

$$\begin{aligned} d_{11} &= -c_1^2 p^2 + v^2 - c_2^2 s^2 - c_3^2 c^2, \\ d_{22} &= -c_2^2 p^2 + v^2 - c_1^2 s^2 - c_3^2 c^2, \\ d_{33} &= -c_3^2 p^2 + v^2 - c_3^2 s^2 - c_5^2 c^2 \end{aligned} \tag{32}$$

is satisfied. Solutions of cubic equation (31), with respect to p^2, may be expressed as:

$$\begin{aligned} p_1^2 &= \frac{v^2 - c_3^2 c^2}{c_1^2} - s^2 + \frac{c^2(c_3^2 + c_4^2)^2(p_1^2 + s^2)}{c_1^2[c_3^2(p_1^2 + s^2) - v^2 + c_5^2 c^2]}, \\ p_2^2 &= \frac{v^2 - c_3^2 c^2}{c_2^2} - s^2, \\ p_3^2 &= \frac{v^2 - c_5^2 c^2}{c_3^2} - s^2 + \frac{c^2(c_3^2 + c_4^2)^2(p_3^2 + s^2)}{c_3^2[c_1^2(p_3^2 + s^2) - v^2 + c_3^2 c^2]} \end{aligned} \tag{33}$$

which may be used to construct general solutions for variations of displacements with respect to thickness coordinate x_1 as follows:

$$\begin{aligned}
&U = p_1 A_1 \sin kp_1 x_1 + sA_2 \sin kp_2 x_1 + p_3 A_3 \sin kp_3 x_1 \ ,\\
&V = sA_1 \cos kp_1 x_1 - p_2 A_2 \cos kp_2 x_1 + sA_3 \cos kp_3 x_1 \ ,\\
&W = \frac{-c_1^2(p_1^2+s^2)+v^2-c_3^2c^2}{c(c_3^2+c_4^2)} A_1 \cos kp_1 x_1 + \frac{-c_1^2(p_3^2+s^2)+v^2-c_3^2c^2}{c(c_3^2+c_4^2)} A_3 \cos kp_3 x_1
\end{aligned} \tag{34}$$

The boundary conditions (8) lead to dispersion equation:

$$\begin{aligned}
&[c_1^2c_3^2(p_1^2+s^2)+c_4^2(v^2-c_3^2c^2)-2c_2^2s^2(c_3^2+c_4^2)][c_1^2(p_3^2+s^2)(p_2^2-s^2)\\
&-(p_2^2-s^2)(v^2+c_4^2c^2)-2s^2c^2(c_3^2+c_4^2)]\, p_3 \cot\, kp_1h\\
&+4c_1^2c_2^2(c_3^2+c_4^2)s^2p_1p_2p_3(p_3^2-p_1^2)\cot\, kp_2h\\
&-[c_1^2c_3^2(p_3^2+s^2)+c_4^2(v^2-c_3^2c^2)-2c_2^2s^2(c_3^2+c_4^2)]\times[c_1^2(p_1^2+s^2)(p_2^2-s^2)\\
&-(p_2^2-s^2)(v^2+c_4^2c^2)-2s^2c^2(c_3^2+c_4^2)]\, p_1 \cot\, kp_3h = 0
\end{aligned} \tag{35}$$

It is easy to show that dispersion equation (35), for fundamental mode, give phase velocity as a function of dimensionless wave number kh. When one examines long waves, when $kh \to 0$, it may be shown that:

$$\cot khp_\alpha \to \frac{1}{khp_\alpha}, \qquad \alpha = 1,2,3 \tag{36}$$

and from equation (32) it is easy to obtain:

$$\begin{aligned}
&(p_3^2-p_1^2)(c_3^2+c_4^2)(v^2-c_3^2c^2)\{[c_1^2(v^2-c_5^2c^2-c_3^2s^2)+c_4^4c^2]\\
&\times(v^2-c_3^2c^2-4c_2^2s^2)-s^2c^2c_3^2(c_1^2c_3^2+4c_2^2c_4^2)\\
&+4c_2^4s^2(v^2-c_3^2c^2-c_5^2s^2)\}=0.
\end{aligned} \tag{37}$$

For carbon fibre epoxy resin composite, for which is $c_1^2 > c_2^2$, and material constant c_5 become dominant, and in limiting process when $c_5 \to \infty$, may be obtained two distinct solutions for phase speed, as it has been obtained in (25) for the case of constrained inextensible materials, and one of order of c_5, which tends to infinity, corresponding to quasi-longitudinal waves. That solution is, in the case of inextensible material, lost in limiting process and inextensible theory cannot predict it.

Equations (33) and (20) may be solved numerically by semi-inverse method in which, for assumed value of phase speed, one calculate p_i, for $i = 1,2,3$, and then value kh that satisfies this equation. In such way it is possible to calculate pairs of values (v, kh) which, substituted in (34), lead to displacements distribution throughout plate thickness and, by using constitutive equations (6) stress distribution through plate thickness using one constant only.

5. Conclusions

Analysis in present paper shaws that mathematical effect, of idealized assumption that fibres are inextensible along the fibres, is reduction of order of diferential equations, which leads to reduction of number of parameters necessary to handle. However, a consequence of reduction of the diferential equations orders is that boundary conditions may not be satisfied unless additional assumption are introduced.

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The boundary conditions may not be satisfied unless assumption of existence of layers with singular fibres which carry finite loads through which shear stress components may be discontinuous and normal stress infinite.

Inextensibility is mathematical idealization, which may be used to consider propagation of quasi-longitudinal waves in restricted range of wave lengths of order one. To examine either very long waves, when $kh \to 0$, or very short waves, when $kh \to \infty$, it is necessary to employ models which include extensibility of the fibres.

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