

On Bishop frame of a pseudo null curve in Minkowski space-time

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Abstract: In this paper, we introduce the Bishop frame of a pseudo null curve α in Minkowski space-time. We obtain the Bishop frame's equations and the relation between the Frenet frame and the Bishop frame. We find the third order nonlinear differential equation whose particular solutions determine the form of the Bishop curvatures. By using space-time geometric algebra, we derive the Darboux bivectors D and \tilde{D} of the Frenet and the Bishop frame of α , respectively. We give geometric interpretations of the Frenet and the Bishop curvatures of α in terms of areas of the projections of the corresponding Darboux bivectors onto the planes spanned by the frame vector's fields.

Key words: Bishop frame, Darboux bivector, pseudo null curve, space-time geometric algebra, Minkowski space-time

1. Introduction

The Bishop frame $\{T, N_1, N_2\}$ (relatively parallel adapted frame, rotation-minimizing frame) of a regular curve in Euclidean space \mathbb{E}^3 can be obtained by applying rotation of the Frenet frame $\{T, N, B\}$ about the tangent vector field T for an angle $\theta(s) = \int \tau(s)ds$, where $\tau(s)$ is the second curvature (torsion) of the curve [1]. As the result of the applied rotation, the vector fields N'_1 and N'_2 are collinear with the tangent vector field T at each point of the curve. Hence, the Bishop frame is called rotation-minimizing frame with respect to T . The normal vector fields N_1 and N_2 are called relatively parallel vector fields. Such frame is well defined even in the points of the curve where the first Frenet curvature κ vanishes, which is not the case with the Frenet frame. A new versions of the Bishop frame in \mathbb{E}^3 , known as type-2 Bishop frame and N -Bishop frame, are introduced in [14, 25]. In Euclidean 4-space the Bishop frame is studied in [8] in terms of the Euler angles. In Minkowski spaces, the Bishop frame of a timelike and a spacelike curve is obtained in [5, 20]. Recently, the Bishop frame and the generalized Bishop frame of the pseudo null and null Cartan curve in Minkowski 3-space are introduced in [9, 10].

In Minkowski space-time \mathbb{E}_1^4 the Bishop frame $\{T_1, N_1, N_2, N_3\}$ of a null Cartan curve contains the tangent vector field T_1 of the curve and three vector fields whose derivatives N'_1 , N'_2 , and N'_3 with respect to pseudo-arc are collinear with N_2 [7]. Hence, they make a minimal rotations in the corresponding spaces $N_1^\perp = \text{span}\{T_1, N_2, N_3\}$, $N_2^\perp = \text{span}\{N_1, N_2, N_3\}$, and $N_3^\perp = \text{span}\{T_1, N_1, N_2\}$, respectively. Accordingly, the Bishop frame of null Cartan curve in \mathbb{E}_1^4 can be seen as rotation-minimizing frame with respect to N_2 . The Bishop frame can be used in many physical and mathematical applications related with rigid body mechanics

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[6], computer graphics [23], deformation of tubes [21], sweep surface modeling [16], and differential geometry in studying different types of curves (see for example [2, 3, 15, 24]).

The Bishop frame of a pseudo null curve α in \mathbb{E}_1^4 is not obtained yet. In this paper, we introduce the Bishop frame of a pseudo null curve α in Minkowski space-time. We derive the Bishop frame's equations and the relation between the Frenet frame and the Bishop frame. We obtain the third order nonlinear differential equation whose particular solutions determine the form of the Bishop curvatures. We emphasize that such frame cannot be introduced as the frame containing the tangent vector field T . It can be proved that if the tangent vector field T is one of the Bishop frame's vector fields of a pseudo null curve, then such curve has the third Frenet curvature $\kappa_3(s) = 0$ for each s , which is not a general case. Accordingly, we define the Bishop frame of pseudo null curve α as the frame containing the principal normal vector field $N = \alpha''$ and three vector fields whose derivatives in arc length parameter s of the curve have minimal rotation property. We derive the Bishop's frame equations and the relation between the Frenet frame and the Bishop frame. We also give some examples of the Bishop frames of pseudo null curves in \mathbb{E}_1^4 .

It is known that the Darboux vector (angular velocity vector) of the Frenet frame of a regular curve in Euclidean space \mathbb{E}^3 gives the direction of the frame's rotation and satisfies the Darboux equations ([17]). In Minkowski space-time \mathbb{E}_1^4 , the Frenet frame of a nonnull curve with a nonnull principal normal has Darboux bivector D that can be regarded as generalization of the Darboux vector ([11–13]). It is show in [11] that the Frenet frame of a timelike curve α in \mathbb{E}_1^4 has Darboux bivector which satisfies the Darboux equations in terms of the inner product of spacetime geometric algebra.

In this paper, by using space-time geometric algebra, we obtain Darboux bivector D of the Frenet frame and Darboux bivector \tilde{D} of the Bishop frame of pseudo null curve α in \mathbb{E}_1^4 . In particular, we show that D and \tilde{D} are lightlike bivectors if and only if α has the third Frenet curvature and the third Bishop curvature equal to zero respectively. We give geometric interpretations of the Frenet and the Bishop curvatures of α in terms of areas of the projections of Darboux bivectors onto the planes spanned by the frames's vector fields.

2. Preliminaries

Minkowski space-time \mathbb{E}_1^4 is the real vector space \mathbb{E}^4 equipped with the standard indefinite flat metric $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

for any two vectors $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in \mathbb{E}_1^4 . Since $\langle \cdot, \cdot \rangle$ is an indefinite metric, an arbitrary vector $x \neq 0$ in \mathbb{E}_1^4 can be *spacelike*, *timelike*, or *null (lightlike)*, if $\langle x, x \rangle$ is positive, negative, or zero, respectively. In particular, the vector $x = 0$ is a spacelike. The norm (length) of a vector $x \in \mathbb{E}_1^4$ is given by $\|x\| = \sqrt{|\langle x, x \rangle|}$. An arbitrary curve $\alpha : I \rightarrow \mathbb{E}_1^4$ can locally be spacelike, timelike, or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are spacelike, timelike or null, respectively [19].

A spacelike curve $\alpha : I \rightarrow \mathbb{E}_1^4$ is called a pseudo null curve, if its principal normal vector N and the second binormal vector B_2 are null vectors. The Frenet frame $\{T, N, B_1, B_2\}$ of pseudo null curve α satisfies the conditions

$$\begin{aligned} \langle T, T \rangle = \langle B_1, B_1 \rangle = 1, \quad \langle N, N \rangle = \langle B_2, B_2 \rangle = 0, \\ \langle T, N \rangle = \langle T, B_1 \rangle = \langle N, B_1 \rangle = \langle T, B_2 \rangle = \langle B_1, B_2 \rangle = 0, \quad \langle N, B_2 \rangle = 1. \end{aligned} \tag{2.1}$$

The Frenet equations of a pseudo null curve α in \mathbb{E}_1^4 read [22]

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 \\ 0 & \kappa_3 & 0 & -\kappa_2 \\ -\kappa_1 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.2}$$

where $\kappa_1(s) = 1$ is the first Frenet curvature and $\kappa_2(s)$ and $\kappa_3(s)$ are the second and the third Frenet curvature of α , respectively. The Frenet frame $\{T, N, B_1, B_2\}$ is positively oriented, if

$$\langle T, N \times B_1 \times B_2 \rangle = \det(T, N, B_1, B_2) = 1.$$

Note that if α has the Frenet curvatures $\kappa_2(s) \neq 0$ and $\kappa_3(s) = 0$, then it also lies in \mathbb{E}_1^4 according to its Frenet equations (2.2).

Spacetime algebra is a noncommutative associative geometric algebra $G_4(\mathbb{E}_1^4)$ of dimension 16 whose elements are called multivectors. In spacetime algebra, the geometric product of a vector a with itself is defined by [12]

$$a^2 = \epsilon_a ||a||^2,$$

where ϵ_a is the signature of a and $||a|| = \sqrt{|\langle a, a \rangle|}$ is a magnitude of a . Accordingly, the vector a is said to be timelike iff $\epsilon_a = 1$, spacelike iff $\epsilon_a = -1$, and lightlike (null) iff $\epsilon_a = 0$. The geometric product ab of two vectors a and b can be decomposed as

$$ab = a \cdot b + a \wedge b,$$

where $a \cdot b = \frac{1}{2}(ab + ba)$ is commutative inner product and $a \wedge b = \frac{1}{2}(ab - ba)$ is a noncommutative outer (wedge, exterior) product. The inner product of 1-vector (vector) a and bivector (2-vector) $b \wedge c$ is given by [12]

$$a \cdot (b \wedge c) = \langle a, b \rangle c - \langle a, c \rangle b = -(b \wedge c) \cdot a. \tag{2.3}$$

In particular, the inner product of two bivectors (2-vectors) is defined by

$$(b \wedge a) \cdot (u \wedge v) = -\langle b, u \rangle \langle a, v \rangle + \langle b, v \rangle \langle a, u \rangle. \tag{2.4}$$

The Frenet frame $\{\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2\}$ of a timelike curve in \mathbb{E}_1^4 has Darboux bivector \bar{D} of the form [11]

$$\bar{D} = \bar{\kappa}_1 \bar{N} \wedge \bar{T} + \bar{\kappa}_2 \bar{N} \wedge \bar{B}_1 + \bar{\kappa}_3 \bar{B}_1 \wedge \bar{B}_2.$$

It represents the generalization of the Darboux vector D in \mathbb{E}_1^3 . Namely, if $\bar{\kappa}_3 = 0$, then timelike curve lies in \mathbb{E}_1^3 , so its Darboux vector is given by

$$D = -I\bar{D} = -I(\bar{\kappa}_1 \bar{N} \wedge \bar{T} + \bar{\kappa}_2 \bar{N} \wedge \bar{B}_1) = \bar{\kappa}_1 \bar{B}_1 + \bar{\kappa}_2 \bar{T},$$

where I is the pseudoscalar of $G_3(\mathbb{E}_1^3)$ and $I(a \wedge b) = a \times b$. For further properties of spacetime geometric algebra, we refer to [12, 13].

3. The Bishop frame of a pseudo null curve in \mathbb{E}_1^4

In this section, we introduce the Bishop frame of a pseudo null curve α in \mathbb{E}_1^4 with the Frenet curvatures $\kappa_1(s) = 1$, $\kappa_2(s) \neq 0$, and $\kappa_3(s) = 0$ or $\kappa_3(s) \neq 0$. We derive the Bishop frame's equations and find the relation between the Frenet and the Bishop frame of α . We obtain the third order nonlinear differential equation whose particular solutions determine the form of the Bishop curvatures and provide the related examples.

Let α be a pseudo null curve in \mathbb{E}_1^4 with the Frenet frame $\{T, N, B_1, B_2\}$. Then its rectifying space $N^\perp = \text{span}\{T, N, B_1\}$ is a lightlike. The spacelike vector fields T and B_1 span a screen distribution on N^\perp ([4]). For a given T and B_1 , there exists a unique lightlike transversal vector field B_2 in \mathbb{E}_1^4 (the second binormal vector field of α), satisfying the conditions ([4])

$$\langle B_2, B_2 \rangle = 0, \quad \langle B_2, N \rangle = 1, \quad \langle B_2, B_1 \rangle = \langle B_2, T \rangle = 0.$$

Denote by $\{N_0, N_1, N_2, N_3\}$ a new frame along α , where $N_1 = N$ is the principal normal vector field, N_0 and N_2 are spacelike vector fields and N_3 is a lightlike transversal vector field satisfying the conditions

$$\begin{aligned} \langle N_1, N_1 \rangle = \langle N_3, N_3 \rangle = 0, \quad \langle N_0, N_0 \rangle = \langle N_2, N_2 \rangle = \langle N_1, N_3 \rangle = 1, \\ \langle N_0, N_1 \rangle = \langle N_0, N_2 \rangle = \langle N_0, N_3 \rangle = \langle N_1, N_2 \rangle = \langle N_2, N_3 \rangle = 0. \end{aligned} \tag{3.1}$$

Hence, the frame $\{N_0, N_1, N_2, N_3\}$ is pseudo-orthonormal. With respect to this frame, the vector fields N'_0 , N'_2 and N'_3 can be decomposed as

$$N'_0 = a_0N_0 + b_0N_1 + c_0N_2 + d_0N_3, \quad N'_2 = a_2N_0 + b_2N_1 + c_2N_2 + d_2N_3, \quad N'_3 = a_3N_0 + b_3N_1 + c_3N_2 + d_3N_3, \tag{3.2}$$

for some differentiable functions a_i, b_i, c_i, d_i , $i = 0, 2, 3$. We define the vector fields N_0 , N_2 , and N_3 to be relatively parallel as follows.

Definition 3.1 *The spacelike vector fields N_0 and N_2 and the lightlike transversal vector field N_3 of a pseudo null curve α in \mathbb{E}_1^4 are said to be relatively parallel, if the component $N_1^\perp = \text{span}\{N_0, N_1, N_2\}$ of their derivatives N'_0 , N'_2 , and N'_3 is equal to zero.*

By using the Definition (3.1) and relation (3.2), it follows that the vector fields N'_0 , N'_2 , and N'_3 are collinear with null vector field N_3 at each point of the curve. This means that N'_0 , N'_2 and N'_3 make minimal rotations in hyperplanes $N_0^\perp = \{N_1, N_2, N_3\}$, $N_2^\perp = \text{span}\{N_0, N_1, N_3\}$ and $N_3^\perp = \text{span}\{N_0, N_2, N_3\}$, respectively.

Definition 3.2 *The Bishop frame $\{N_0, N_1, N_2, N_3\}$ of a pseudo null curve α in \mathbb{E}_1^4 is positively oriented pseudo-orthonormal frame containing principal normal vector field $N_1 = N$, relatively parallel spacelike vector fields N_0 and N_2 and relatively parallel lightlike transversal vector field N_3 satisfying the conditions (3.1).*

The Bishop frame $\{N_0, N_1, N_2, N_3\}$ is positively oriented, if

$$\langle N_0, N_1 \times N_2 \times N_3 \rangle = \det(N_0, N_1, N_2, N_3) = 1.$$

Theorem 3.3 *Let α be a pseudo null curve in \mathbb{E}_1^4 parameterized by arc length s with the Frenet curvatures $\kappa_1(s) = 1$, $\kappa_2(s) \neq 0$, $\kappa_3(s)$. Then the Bishop frame $\{N_0, N_1, N_2, N_3\}$ and the Frenet frame $\{T, N, B_1, B_2\}$ of α are related by*

$$\begin{aligned} N_0 &= \frac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}T + \left(\frac{(\frac{\sigma_2}{\sigma_1})'\sigma_1^2\sigma_2}{(\sigma_1^2 + \sigma_2^2)^2} - \frac{\sigma_1\sigma_3}{\sigma_1^2 + \sigma_2^2} \right)N + \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}B_1, \\ N_1 &= N, \\ N_2 &= -\frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}T - \left(\frac{(\frac{\sigma_2}{\sigma_1})'\sigma_1^3}{(\sigma_1^2 + \sigma_2^2)^2} + \frac{\sigma_2\sigma_3}{\sigma_1^2 + \sigma_2^2} \right)N + \frac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}B_1, \\ N_3 &= -\frac{(\frac{\sigma_2}{\sigma_1})'\sigma_1^2}{(\sigma_1^2 + \sigma_2^2)^{\frac{3}{2}}}T - \frac{1}{2} \left(\frac{\sigma_3^2}{\sigma_1^2 + \sigma_2^2} + \frac{(\frac{\sigma_2}{\sigma_1})'^2\sigma_1^4}{(\sigma_1^2 + \sigma_2^2)^3} \right)N + \frac{\sigma_3}{\sqrt{\sigma_1^2 + \sigma_2^2}}B_1 + B_2. \end{aligned} \tag{3.3}$$

and the Bishop frame's equations read

$$\begin{bmatrix} N'_0 \\ N'_1 \\ N'_2 \\ N'_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\sigma_1 \\ \sigma_1 & \sigma_3 & \sigma_2 & 0 \\ 0 & 0 & 0 & -\sigma_2 \\ 0 & 0 & 0 & -\sigma_3 \end{bmatrix} \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{3.4}$$

where the Bishop curvatures σ_1 , σ_2 , and σ_3 of α have the form

$$\begin{aligned} \sigma_1(s) &= \kappa_2(s) \cos \theta(s), & \sigma_2(s) &= \kappa_2(s) \sin \theta(s), \\ \sigma_3(s) &= -\frac{\kappa_2(s)}{\theta'(s)} \left(1 + \left(\frac{\theta'(s)}{\kappa_2(s)} \right)' \right), \end{aligned} \tag{3.5}$$

and the function $\theta(s) \neq \text{constant}$ satisfies the third order nonlinear differential equation

$$\left(\frac{1}{\theta'} + \frac{1}{\theta'} \left(\frac{\theta'}{\kappa_2} \right)' \right)' - \frac{\kappa_2}{2} \left(\frac{1}{\theta'} + \frac{1}{\theta'} \left(\frac{\theta'}{\kappa_2} \right)' \right)^2 + \frac{\theta'^2}{2\kappa_2} + \kappa_3 = 0. \tag{3.6}$$

Proof Let $\{N_0, N_1, N_2, N_3\}$ be the Bishop frame of a pseudo null curve α parameterized by arc-length s . According to Definition (3.1), vector fields N_0 , N_2 , and N_3 are relatively parallel. By using this property, we will determine the Bishop frame's equations. Decompose the vector field N'_0 with respect to the pseudo-orthonormal basis $\{N_0, N_1, N_2, N_3\}$ by

$$N'_0 = a_0N_0 + b_0N_1 + c_0N_2 + d_0N_3, \tag{3.7}$$

where $a_0(s), b_0(s), c_0(s)$, and $d_0(s)$ are some differentiable functions in s . By using the relations (3.1) and (3.7), we find

$$\begin{aligned} \langle N'_0, N_0 \rangle &= a_0 = 0, & \langle N'_0, N_1 \rangle &= d_0 = -\sigma_1, \\ \langle N'_0, N_2 \rangle &= c_0 = 0, & \langle N'_0, N_3 \rangle &= b_0 = 0, \end{aligned} \tag{3.8}$$

where we have put $\sigma_1(s) = -d_0$. Substituting (3.8) in (3.7), we obtain

$$N'_0 = -\sigma_1N_3. \tag{3.9}$$

In a similar way, we get

$$N'_1 = \sigma_1N_0 + \sigma_3N_1 + \sigma_2N_2, \quad N'_2 = -\sigma_2N_3, \quad N'_3 = -\sigma_3N_3, \tag{3.10}$$

where $\sigma_2(s)$ and $\sigma_3(s)$ are some differentiable functions. By using the relations (3.9) and (3.10), we get that Bishop frame's equations have form (3.4).

Next we obtain relations between Bishop curvatures $\sigma_1(s)$, $\sigma_2(s)$, $\sigma_3(s)$ and Frenet curvatures $\kappa_1(s) = 1$, $\kappa_2(s) \neq 0$, and $\kappa_3(s)$. According to Definition (3.2), we have $N_1 = N$. Differentiating the previous equation with respect to s and using (2.2) and (3.10), we find

$$N'_1 = N' = \sigma_1 N_0 + \sigma_3 N_1 + \sigma_2 N_2 = \kappa_2 B_1.$$

The last relation gives

$$B_1 = \frac{\sigma_1}{\kappa_2} N_0 + \frac{\sigma_3}{\kappa_2} N_1 + \frac{\sigma_2}{\kappa_2} N_2. \tag{3.11}$$

By using the condition $\langle B_1, B_1 \rangle = 1$ and relations (3.1) and (3.11), we obtain

$$\sigma_1^2 + \sigma_2^2 = \kappa_2^2.$$

We may consider the next three cases: (A) $\sigma_1 = 0$, $\sigma_2 = \pm\kappa_2$; (B) $\sigma_1 = \pm\kappa_2$, $\sigma_2 = 0$; (C) $\sigma_1 = \kappa_2 \cos \theta$, $\sigma_2 = \kappa_2 \sin \theta$, where $\theta = \theta(s)$ is some differentiable function.

(A) $\sigma_1 = 0$, $\sigma_2 = \pm\kappa_2$. Substituting this in (3.11), we get

$$B_1 = \frac{\sigma_3}{\kappa_2} N_1 \pm N_2. \tag{3.12}$$

Decompose the tangent vector field T with respect to Bishop frame by

$$T = \lambda N_0 + \mu N_1 + \nu N_2 + \omega N_3, \tag{3.13}$$

where $\lambda, \mu, \nu, \omega$ are some differentiable functions. By using relations (3.1), (3.12), (3.13) and the conditions $\langle T, B_1 \rangle = \langle T, N \rangle = 0$, we find $\nu = \omega = 0$. Hence,

$$T = \lambda N_0 + \mu N_1.$$

Differentiating the last relation with respect to s and using (2.2), (3.11), and (3.12), we find

$$T' = N = N_1 = (\lambda' + \mu\sigma_1)N_0 + (\mu' + \mu\sigma_3)N_1 + \mu\sigma_2 N_2 - \lambda\sigma_1 N_3.$$

The last relation implies the system of equations

$$\lambda' + \mu\sigma_1 = 0, \quad \mu' + \mu\sigma_3 = 1, \quad \mu\sigma_2 = 0, \quad \lambda\sigma_1 = 0.$$

Since $\sigma_2 = \pm\kappa_2 \neq 0$, it follows $\mu = 0$. Then the second equation of the last system of equations gives a contradiction.

(B) $\sigma_1 = \pm\kappa_2$, $\sigma_2 = 0$. Similarly as in case (A), we get a contradiction.

(C) $\sigma_1 = \kappa_2 \cos \theta$, $\sigma_2 = \kappa_2 \sin \theta$. By using relations (3.11), (3.13) and the conditions $\langle T, N \rangle = \langle T, B_1 \rangle = 0$, we find $\lambda = \sin \theta$, $\nu = -\cos \theta$, $\omega = 0$. Substituting this in (3.13), we obtain

$$T = \sin \theta N_0 + \mu N_1 - \cos \theta N_2. \tag{3.14}$$

By taking the derivative of the last relation with respect to s and using (2.2), (3.9), and (3.10), we get

$$\begin{aligned} N_1 &= (\theta' \cos \theta + \mu \sigma_1)N_0 + (\mu' + \mu \sigma_3)N_1 + (\mu \sigma_2 + \theta' \sin \theta)N_2 \\ &+ (\sigma_1 \sin \theta + \sigma_2 \cos \theta)N_3. \end{aligned} \tag{3.15}$$

By using a linear independence of vectors $N_0, N_1, N_2,$ and $N_3,$ we get the system of differential equations

$$\begin{aligned} \theta' \cos \theta + \mu \sigma_1 &= 0, \\ \mu' + \mu \sigma_3 &= 1, \\ \mu \sigma_2 + \theta' \sin \theta &= 0, \\ -\sigma_1 \sin \theta + \sigma_2 \cos \theta &= 0. \end{aligned} \tag{3.16}$$

If $\theta' = 0,$ then $\mu \sigma_1 = \mu \sigma_2 = 0.$ If $\mu = 0,$ the second equation of (3.17) implies a contradiction. If $\mu \neq 0$ and $\sigma_1 = \sigma_2 = 0,$ then $\kappa_2 = 0,$ which is also a contradiction. Hence, $\theta' \neq 0.$ Substituting $\sigma_1 = \kappa_2 \cos \theta,$ $\sigma_2 = \kappa_2 \sin \theta$ in (3.17), we get

$$\mu = -\frac{\theta'}{\kappa_2}, \quad \sigma_3 = \frac{1 - \mu'}{\mu} = -\frac{\kappa_2}{\theta'} \left(1 + \left(\frac{\theta'}{\kappa_2} \right)' \right). \tag{3.17}$$

In particular, substituting $\sigma_1 = \kappa_2 \cos \theta$ and $\sigma_2 = \kappa_2 \sin \theta$ in (3.11), yields

$$B_1 = \cos \theta N_0 + \frac{\sigma_3}{\kappa_2} N_1 + \sin \theta N_2. \tag{3.18}$$

The Frenet frame's vector fields T, N and B_1 are expressed in relations (3.14) and (3.18) in terms of the Bishop frame's vector fields. It remains to express B_2 in terms of N_0, N_1, N_2 and $N_3.$ By using the conditions

$$\langle B_2, T \rangle = \langle B_2, B_1 \rangle = \langle B_2, B_2 \rangle = 0, \quad \langle B_2, N \rangle = 1,$$

and relations (3.14),(3.18) and $N = N_1,$ we find

$$\begin{aligned} B_2 &= \left(-\frac{\sigma_3}{\kappa_2} \cos \theta - \mu \sin \theta \right) N_0 - \frac{1}{2} \left(\left(\frac{\sigma_3}{\kappa_2} \right)^2 + \mu^2 \right) N_1 \\ &+ \left(-\frac{\sigma_3}{\kappa_2} \sin \theta + \mu \cos \theta \right) N_2 + N_3. \end{aligned} \tag{3.19}$$

Relations (3.14), (3.17), (3.18), and (3.20) imply the following relation between the Bishop and the Frenet frame

$$\begin{aligned} N_0 &= \sin \theta T + \left(\frac{\theta'}{\kappa_2} \sin \theta - \frac{\sigma_3}{\kappa_2} \cos \theta \right) N + \cos \theta B_1, \\ N_1 &= N, \\ N_2 &= -\cos \theta T - \left(\frac{\theta'}{\kappa_2} \cos \theta + \frac{\sigma_3}{\kappa_2} \sin \theta \right) N + \sin \theta B_1, \\ N_3 &= -\frac{\theta'}{\kappa_2} T - \frac{1}{2} \left(\left(\frac{\sigma_3}{\kappa_2} \right)^2 + \left(\frac{\theta'}{\kappa_2} \right)^2 \right) N + \frac{\sigma_3}{\kappa_2} B_1 + B_2. \end{aligned} \tag{3.20}$$

In particular, substituting the expressions

$$\kappa_2 = \sqrt{\sigma_1^2 + \sigma_2^2}, \quad \sin \theta = \frac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}, \quad \cos \theta = \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}, \quad \frac{\theta'}{\kappa_2} = \frac{(\frac{\sigma_2}{\sigma_1})' \sigma_1^2}{(\sigma_1^2 + \sigma_2^2)^{\frac{3}{2}}},$$

in relation (3.21), we find

$$\begin{aligned} N_0 &= \frac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} T + \left(\frac{(\frac{\sigma_2}{\sigma_1})' \sigma_1^2 \sigma_2}{(\sigma_1^2 + \sigma_2^2)^2} - \frac{\sigma_1 \sigma_3}{\sigma_1^2 + \sigma_2^2} \right) N + \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} B_1, \\ N_1 &= N, \\ N_2 &= -\frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} T - \left(\frac{(\frac{\sigma_2}{\sigma_1})' \sigma_1^3}{(\sigma_1^2 + \sigma_2^2)^2} + \frac{\sigma_2 \sigma_3}{\sigma_1^2 + \sigma_2^2} \right) N + \frac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} B_1, \\ N_3 &= -\frac{(\frac{\sigma_2}{\sigma_1})' \sigma_1^2}{(\sigma_1^2 + \sigma_2^2)^{\frac{3}{2}}} T - \frac{1}{2} \left(\frac{\sigma_3^2}{\sigma_1^2 + \sigma_2^2} + \frac{(\frac{\sigma_2}{\sigma_1})'^2 \sigma_1^4}{(\sigma_1^2 + \sigma_2^2)^3} \right) N + \frac{\sigma_3}{\sqrt{\sigma_1^2 + \sigma_2^2}} B_1 + B_2. \end{aligned}$$

This proves relation (3.3). It can be easily checked that

$$\langle N_0, N_1 \times N_2 \times N_3 \rangle = \det(N_0, N_1, N_2, N_3) = 1.$$

Hence, the Bishop frame of α is positively oriented.

In order to determine differentiable function $\theta(s)$, from the Bishop frame's equations (3.4) we have $N_2' = -\sigma_2 N_3$. Substituting the equation $\sigma_2 = \kappa_2 \sin \theta$ and the last equation of (3.21) in the previous equation, we get

$$N_2' = -\kappa_2 \sin \theta \left[-\frac{\theta'}{\kappa_2} T - \frac{1}{2} \left(\left(\frac{\sigma_3}{\kappa_2} \right)^2 + \left(\frac{\theta'}{\kappa_2} \right)^2 \right) N + \frac{\sigma_3}{\kappa_2} B_1 + B_2. \right] \quad (3.21)$$

On the other hand, differentiating the relation

$$N_2 = -\cos \theta T - \left(\frac{\theta'}{\kappa_2} \cos \theta + \frac{\sigma_3}{\kappa_2} \sin \theta \right) N + \sin \theta B_1$$

with respect to s and applying (2.2), we find

$$\begin{aligned} N_2' &= \theta' \sin \theta T - \cos \theta N - \left(\frac{\theta'}{\kappa_2} \cos \theta + \frac{\sigma_3}{\kappa_2} \sin \theta \right)' N - \left(\frac{\theta'}{\kappa_2} \cos \theta \right. \\ &\quad \left. + \frac{\sigma_3}{\kappa_2} \sin \theta \right) \kappa_2 B_1 + \theta' \cos \theta B_1 + \sin \theta (\kappa_3 N - \kappa_2 B_2). \end{aligned} \quad (3.22)$$

By using linear independence of the Frenet vector fields in relations (3.21) and (3.22), we get the differential equation

$$\left(\frac{1}{\theta'} + \frac{1}{\theta'} \left(\frac{\theta'}{\kappa_2} \right)' \right)' - \frac{\kappa_2}{2} \left(\frac{1}{\theta'} + \frac{1}{\theta'} \left(\frac{\theta'}{\kappa_2} \right)' \right)^2 + \frac{\theta'^2}{2\kappa_2} + \kappa_3 = 0. \quad (3.23)$$

In this way, relation (3.6) is proved. This completes the proof of the theorem. \square

Remark 3.4 Note that every particular solution of the differential equation (3.6) will provide the Bishop curvatures given by relation (3.5). The general solution of the third order nonlinear differential (3.6) depends on the Frenet curvatures $\kappa_2(s) \neq 0$ and $\kappa_3(s)$. If $\kappa_2(s) = \text{constant} \neq 0$ and $\kappa_3(s) = \text{constant}$, we get the third order nonlinear differential equation with constant coefficients.

Example 3.5 Let us consider pseudo null helix α in \mathbb{E}_1^4 parameterized by the arclength s given by

$$\alpha(s) = \frac{3}{\sqrt{10}}\left(\frac{1}{9} \cosh(3s), \frac{1}{9} \sinh(3s), \sin s, -\cos s\right).$$

The Frenet curvatures of α read

$$\kappa_1(s) = 1, \quad \kappa_2(s) = 3, \quad \kappa_3(s) = \frac{4}{3}. \tag{3.24}$$

A straightforward calculation shows that the Frenet frame of α has the form

$$\begin{aligned} T(s) &= \frac{3}{\sqrt{10}}\left(\frac{1}{3} \sinh(3s), \frac{1}{3} \cosh(3s), \cos(s), \sin(s)\right), \\ N(s) &= \frac{3}{\sqrt{10}}(\cosh(3s), \sinh(3s), -\sin s, \cos s), \\ B_1(s) &= \frac{1}{\sqrt{10}}(3 \sinh(3s), 3 \cosh(3s), -\cos s, -\sin s), \\ B_2(s) &= \frac{5}{3\sqrt{10}}(-\cosh(3s), -\sinh(3s), -\sin s, \cos s). \end{aligned}$$

Substituting (3.24) in (3.6), we obtain the differential equation

$$\left(\frac{1}{\theta'} + \frac{1}{\theta'}\left(\frac{\theta'}{3}\right)'\right)' - \frac{3}{2}\left(\frac{1}{\theta'} + \frac{1}{\theta'}\left(\frac{\theta'}{3}\right)'\right)^2 + \frac{\theta'^2}{6} + \frac{4}{3} = 0$$

whose one particular solution reads $\theta(s) = s$. Substituting $\theta(s) = s$ and (3.24) in (3.5), we obtain that the Bishop curvatures of α are given by

$$\sigma_1(s) = 3 \cos s, \quad \sigma_2(s) = 3 \sin s, \quad \sigma_3(s) = -3. \tag{3.25}$$

Relations (3.3) and (3.25) imply that Bishop frame of α reads

$$\begin{aligned} N_0(s) &= \sin(s)T(s) + \left(\frac{1}{3} \sin(s) + \cos(s)\right)N(s) + \cos(s)B_1(s), \\ N_1(s) &= N(s), \\ N_2(s) &= -\cos(s)T(s) + \left(-\frac{1}{3} \cos(s) + \sin(s)\right)N(s) + \sin(s)B_1(s), \\ N_3(s) &= -\frac{1}{3}T(s) - \frac{5}{9}N(s) - B_1(s) + B_2(s). \end{aligned}$$

It can be easily checked that the Bishop frame satisfies relation (3.4).

Example 3.6 Let us consider pseudo null curve α in \mathbb{E}_1^4 parameterized by the arc-length s with parameter equation

$$\alpha(s) = \left(-\frac{s^2}{2} + \frac{1}{8} \ln s, \frac{s^2}{2} + \frac{1}{8} \ln s, \frac{s\sqrt{2}}{4}(\sin(\ln s) + \cos(\ln s)), \frac{s\sqrt{2}}{4}(\sin(\ln s) - \cos(\ln s))\right).$$

A straightforward calculation shows that the Frenet frame of α has the form

$$\begin{aligned} T(s) &= \left(-s + \frac{1}{8s}, s + \frac{1}{8s}, \frac{1}{\sqrt{2}} \cos(\ln s), \frac{1}{\sqrt{2}} \sin(\ln s)\right), \\ N(s) &= \left(-1 - \frac{1}{8s^2}, 1 - \frac{1}{8s^2}, -\frac{1}{s\sqrt{2}} \sin(\ln s), \frac{1}{s\sqrt{2}} \cos(\ln s)\right), \\ B_1(s) &= \left(\frac{1}{4s}, \frac{1}{4s}, \frac{1}{\sqrt{2}}(\sin(\ln s) - \cos(\ln s)), \frac{1}{\sqrt{2}}(-\sin(\ln s) - \cos(\ln s))\right), \\ B_2(s) &= \left(\frac{s^2}{2} + \frac{5}{16}, -\frac{s^2}{2} + \frac{5}{16}, -\frac{s}{2\sqrt{2}}(\sin(\ln s) + 2\cos(\ln s)), \frac{s}{2\sqrt{2}}(\cos(\ln s) - 2\sin(\ln s))\right). \end{aligned}$$

In particular, the Frenet curvatures of α read

$$\kappa_1(s) = 1, \quad \kappa_2(s) = \frac{1}{s^2}, \quad \kappa_3(s) = -\frac{1}{2}. \tag{3.26}$$

Substituting (3.26) in (3.6), we obtain the differential equation

$$\frac{1}{\theta'} \left((s^2 \theta')' - 1 \right)' - \frac{1}{2s^2} \left(\frac{1}{\theta'} \left((s^2 \theta')' - 1 \right) \right)^2 + \frac{s^2}{2} \theta'^2 - \frac{1}{2} = 0.$$

One particular solution of the previous differential equation reads $\theta(s) = \ln s$. Substituting $\theta(s) = \ln s$ and (3.26) in (3.5), we find that the Bishop curvatures of α have the form

$$\sigma_1(s) = \frac{1}{s^2} \cos(\ln s), \quad \sigma_2(s) = \frac{1}{s^2} \sin(\ln s), \quad \sigma_3(s) = -\frac{2}{s}. \tag{3.27}$$

By using relations (3.3) and (3.27), we obtain that the Bishop frame reads

$$\begin{aligned} N_0(s) &= \sin(\ln s)T(s) + (s \sin(\ln s) + 2s \cos(\ln s))N(s) + \cos(\ln s)B_1(s), \\ N_1(s) &= N(s), \\ N_2(s) &= -\cos(\ln s)T(s) + (s \cos(\ln s) - 2s \sin(\ln s))N(s) + \sin(\ln s)B_1(s), \\ N_3(s) &= -sT(s) - \frac{5}{2}s^2N(s) - 2sB_1(s) + B_2(s), \end{aligned}$$

It can be verified that the Bishop frame satisfies relation (3.4).

4. The Frenet frame's and the Bishop frame's Darboux bivectors

In this section, we obtain Darboux bivectors of the Frenet and Bishop frame of a pseudo null curve α in \mathbb{E}_1^4 . We also give geometric interpretations of the Frenet and the Bishop curvatures in terms of areas of the projections of the Darboux bivectors onto the three 2-planes generated by the frame vector's fields.

Let us consider the Frenet frame $\{T, N, B_1, B_2\}$ of a pseudo null curve α in \mathbb{E}_1^4 with Frenet curvatures $\kappa_1(s) = 1$, $\kappa_2(s) \neq 0$, and $\kappa_3(s)$. The Frenet frame of α determines six basis bivectors $E_{TN} = T \wedge N$, $E_{TB_1} = T \wedge B_1$, $E_{TB_2} = T \wedge B_2$, $E_{NB_1} = N \wedge B_1$, $E_{NB_2} = N \wedge B_2$, $E_{B_1B_2} = B_1 \wedge B_2$ of unit or zero areas

that are parallel to six 2-planes in \mathbb{E}_1^4 spanned by $\{T, N\}$, $\{T, B_1\}$, $\{T, B_2\}$, $\{N, B_1\}$, $\{N, B_2\}$, $\{B_1, B_2\}$ respectively.

The Frenet frame's Darboux bivector D can be represented as the sum of its projections onto the six 2-planes parallel to basis bivectors [18]

$$D = aE_{TN} + bE_{TB_1} + cE_{TB_2} + dE_{NB_1} + fE_{NB_2} + hE_{B_1B_2}, \tag{4.1}$$

where a, b, c, d, f, h are areas of the projections. It is known that Darboux bivector D satisfies the Darboux equations in terms of the inner product of spacetime geometric algebra of the form [11]

$$T' = D \cdot T, \quad N' = D \cdot N, \quad B'_1 = D \cdot B_1, \quad B'_2 = D \cdot B_2. \tag{4.2}$$

By using the relations (2.1), (2.2), (2.3), (4.1), and (4.2), we find

$$\begin{aligned} D \cdot T &= (aE_{TN} + bE_{TB_1} + cE_{TB_2} + dE_{NB_1} + fE_{NB_2} + hE_{B_1B_2}) \cdot T \\ &= -aN - bB_1 - cB_2 = N. \end{aligned}$$

The last relation gives

$$a = -1, \quad b = 0, \quad c = 0. \tag{4.3}$$

By using the relations (2.1), (2.2), (2.3), (4.1), (4.2), and (4.3), we get

$$\begin{aligned} D \cdot N &= cT + fN + hB_1 = k_2B_1, \\ D \cdot B_1 &= bT + dN - hB_2 = k_3N - k_2B_2, \\ D \cdot B_2 &= aT - dB_1 - fB_2 = -T - k_3B_1. \end{aligned}$$

From the last system of equations, we get

$$f = 0, \quad h = k_2, \quad d = k_3. \tag{4.4}$$

Substituting (4.3) and (4.4) in (4.1), we obtain that the Darboux bivector of the Frenet frame reads

$$D = -\kappa_1E_{TN} + \kappa_2E_{B_1B_2} + \kappa_3E_{NB_1}. \tag{4.5}$$

Thus, the Frenet curvatures $\kappa_1 = D \cdot E_{TB_2} = 1$, $\kappa_2 = D \cdot E_{NB_1}$ and $\kappa_3 = D \cdot E_{B_1B_2}$ of α can be interpreted as areas of the projections of Darboux bivector D onto the lightlike 2-planes $span\{T, N\}$, $span\{B_1, B_2\}$, $span\{N, B_1\}$, respectively. By using the relations (2.1), (2.4), and (4.5), we find

$$D^2 = D \cdot D = 2\kappa_2\kappa_3.$$

Since $\kappa_2(s) \neq 0$, the Darboux bivector D is lightlike if and only if the third Frenet curvature $\kappa_3(s) = 0$. Otherwise, it can be spacelike or timelike.

Let us now consider the Bishop frame $\{N_0, N_1, N_2, N_3\}$ of a pseudo null curve α . Such frame induces six basis bivectors $E_{N_0N_1} = N_0 \wedge N_1$, $E_{N_0N_2} = N_0 \wedge N_2$, $E_{N_0N_3} = N_0 \wedge N_3$, $E_{N_1N_2} = N_1 \wedge N_2$, $E_{N_1N_3} = N_1 \wedge N_3$, $E_{N_2N_3} = N_2 \wedge N_3$ of unit or zero areas that are parallel to the corresponding 2-planes. Hence, the Bishop frame's Darboux bivector \tilde{D} can be written as

$$\tilde{D} = \tilde{a}E_{N_0N_1} + \tilde{b}E_{N_0N_2} + \tilde{c}E_{N_0N_3} + \tilde{d}E_{N_1N_2} + \tilde{f}E_{N_1N_3} + \tilde{h}E_{N_2N_3}, \tag{4.6}$$

where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{f}, \tilde{h}$ are areas of the projections of \tilde{D} onto corresponding 2-planes [18]. Relations (2.1), (2.2), (2.3), and (4.6) imply that Darboux bivector \tilde{D} has the form

$$\tilde{D} = \sigma_1 E_{N_0 N_1} + \sigma_2 E_{N_2 N_3} + \sigma_3 E_{N_1 N_3}. \tag{4.7}$$

Consequently, the Bishop curvatures $\sigma_1 = \tilde{D} \cdot E_{N_1 N_0}$, $\sigma_2 = \tilde{D} \cdot E_{N_1 N_2}$ and $\sigma_3 = \tilde{D} \cdot E_{N_1 N_3}$ of α can be interpreted as areas of the projections of \tilde{D} onto the planes $span\{N_0, N_1\}$, $span\{N_2, N_3\}$ and $span\{N_1, N_3\}$, respectively. By using (2.4), (3.1), and (4.7), we get

$$\tilde{D}^2 = \tilde{D} \cdot \tilde{D} = \sigma_3^2.$$

Therefore, Darboux bivector \tilde{D} is a lightlike if and only if $\sigma_3 = 0$. If $\sigma_3 \neq 0$, Darboux bivector \tilde{D} is timelike. The Darboux bivectors D and \tilde{D} cannot be both lightlike, because if $\kappa_3 = \sigma_3 = 0$ relations (3.5) and (3.6) give $\theta' = 0$, which is a contradiction.

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