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Note on constructing a family of solvable sine-type difference equations

Ahmed El-Sayed Ahmed[1], Bratislav Iričanin[2,3], Witold Kosmala[4], Stevo Stević[5,6*] and Zdeněk Šmarda[7]

*Correspondence: sstevic@ptt.rs
[5]Mathematical Institute of the Serbian Academy of Sciences and Arts, Knez Mihailova 36/III, 11000, Beograd, Serbia
[6]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, 40402, Taiwan, Republic of China
Full list of author information is available at the end of the article

Abstract
We obtain a family of first order sine-type difference equations solvable in closed form in a constructive way, and we present a general solution to each of the equations.

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1 Introduction

Throughout the paper we use the standard notations $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ for the sets of all natural numbers, nonnegative integers, integers, real numbers, and complex numbers, respectively. If $l, m \in \mathbb{Z}$, then instead of using the notation $l \le s \le m$, $s \in \mathbb{Z}$, we simply write $s = \overline{l,m}$ and understand that the variable s belongs to $\mathbb{Z}$.

If f is a self-mapping of a set X, then by $f^{[0]}(x)$ we denote the identity map $f(x) = x$, $x \in X$, whereas by $f^{[k]}(x)$, where $k \in \mathbb{N}$, we denote the iterated composition

$$f^{[k]}(x) := \underbrace{f\bigl(f\bigl(\cdots f}_{k \text{ times}}(x)\cdots\bigr)\bigr)$$

for $x \in X$, consisting of the composition of k functions each of which is equal to the given function f.

One of the first studied problems connected to recursive relations/difference equations was finding closed-form formulas for their solutions. First results on the problem was obtained at the beginning of the eighteenth century by de Moivre [7, 8] and Bernoulli [4] for the case of homogeneous linear difference equations with constant coefficients. The investigation was continued by Euler [9] and to a larger extent by Lagrange (see, e.g., [12, 13]). Some information on classical solvable difference equations can be found in many classical books on calculus of finite differences or difference equations, for example, in [5, 10, 11, 15–17, 19], whereas some information on recently studied solvable difference equations and systems of difference equations and their invariants can be found, for example, in [3, 20–23, 25–35], as well as in the related references quoted therein.

Another important figure in the investigation of solvability of difference equations and systems of difference equations of one or several independent variables was Laplace.

Among other things, he also noticed that some one-dimensional difference equations which have forms of some well-known trigonometric formulas are solvable in closed form.

One of the difference equations appearing in [14] is the following nonlinear equation of first order:

$$x_{n+1} = x_n^2 - 2, \quad n \in \mathbb{N}_0. \tag{1}$$

Laplace wrote the initial value x_0 in the form $a + \frac{1}{a}$ (such an a can be found for any $x_0 \in \mathbb{C}$), and based on calculations of the first few members of the sequence concluded that

$$x_n = a^{2^n} + \frac{1}{a^{2^n}}, \quad n \in \mathbb{N}_0. \tag{2}$$

The closed-form formula in (2) can be regarded as a form of general solution to Eq. (1).

By using the linear change of variables

$$x_n = 2y_n, \quad n \in \mathbb{N}_0, \tag{3}$$

Equation (1) is transformed to the following one:

$$y_{n+1} = 2y_n^2 - 1, \quad n \in \mathbb{N}_0. \tag{4}$$

Equation (4) can be found in many books. Bearing in mind the simple relation (3) between solutions to Eqs. (1) and (4), it is highly expected that Eq. (4) was also known to Laplace and some other mathematicians of that period of time. Note that Eq. (4) is similar to the well-known double-angle identity for the cosine,

$$\cos 2x = 2\cos^2 x - 1.$$

This observation naturally suggests that the substitution

$$y_n = \cos z_n, \quad n \in \mathbb{N}_0, \tag{5}$$

is used in dealing with Eq. (4). Therefore, the equation is of cosine-type, as well as its relative (1).

It seems that difference equations (1) and (4) were among the first nonlinear equations for which some closed-form formulas of their general solutions were found.

The difference equation

$$x_{n+1} = x_n^3 - 3x_n, \quad n \in \mathbb{N}_0, \tag{6}$$

can be also found in some problem books and those dealing with sequences or difference equations. Convergence of solutions to the equation can be studied in several ways. An interesting fact is that Eq. (6) can be also solved in closed form in the same way as Laplace did for the case of Eq. (1), that is, by taking $x_0 = a + \frac{1}{a}$, calculating the first few members of the sequence, and guessing a formula for general solution to the equation, which can be verified, say, by the method of induction. But, unlike the case of Eq. (1) the use of the method is less obvious and requires more experience.

The above-mentioned facts together with our recent studies of some difference equations and systems of difference equations related to the hyperbolic cotangent function (see, e.g., [29–31] and the related references therein), which are related to some product-type ones (see, e.g., [33–35]), motivated us to construct a sequence of solvable difference equations in closed form extending Eqs. (1) and (6) in a natural way, and to present their solutions in all possible cases.

Therefore, in [32] we have recently conduced a detailed analysis of connections between the quantity $a+\frac{1}{a}$ and polynomials. By using the analysis the following result was proved therein (it should be folklore, but it seems not to have been published in this form in the literature).

Theorem 1 *Consider the difference equation*

$$x_{n+1}=P_k(x_n),\quad n\in\mathbb{N}_0, \tag{7}$$

where $k\in\mathbb{N}\setminus\{1\}$ and P_k is the polynomial given by

$$P_k(t)=\left(\frac{t+\sqrt{t^2-4}}{2}\right)^k+\left(\frac{t-\sqrt{t^2-4}}{2}\right)^k. \tag{8}$$

Then the following statements hold:

(a) *If $x_0\in\mathbb{C}$ is given by*

$$x_0=a+\frac{1}{a}, \tag{9}$$

then the solution to Eq. (7) *with the initial value x_0 is given by*

$$x_n=a^{k^n}+\frac{1}{a^{k^n}},$$

for $n\in\mathbb{N}_0$.

(b) *If $x_0\in\mathbb{R}$ is such that $x_0\geq 2$, then the solution to Eq.* (7) *with the initial value x_0 is given by*

$$x_n=\left(\frac{x_0+\sqrt{x_0^2-4}}{2}\right)^{k^n}+\left(\frac{x_0+\sqrt{x_0^2-4}}{2}\right)^{-k^n},$$

for $n\in\mathbb{N}_0$.

(c) *If $x_0\in\mathbb{R}$ is such that $x_0\leq -2$, then the solution to Eq.* (7) *with the initial value x_0 is given by*

$$x_n=\left(\frac{P_k(x_0)-\sqrt{(P_k(x_0))^2-4}}{2}\right)^{k^{n-1}}+\left(\frac{P_k(x_0)-\sqrt{(P_k(x_0))^2-4}}{2}\right)^{-k^{n-1}}, \tag{10}$$

for $n\in\mathbb{N}$.

(d) *If* $|x_0| \le 2$ *and* $x_0 = 2\cos\theta$ *for some* $\theta \in [0, 2\pi)$, *then the solution to Eq.* (7) *with the initial value* x_0 *is given by*

$$x_n = 2\cos\bigl(k^n\theta\bigr), \quad n \in \mathbb{N}_0.$$

Remark 1 We want to say that in [32] was made a minor oversight, and use the opportunity to explain how it should be corrected. Namely, instead of Eq. (10), in [32, Theorem 1] is written the following one:

$$x_n = \left(\frac{x_0^2 - 2 - x_0\sqrt{x_0^2 - 4}}{2}\right)^{k^{n-1}} + \left(\frac{x_0^2 - 2 - x_0\sqrt{x_0^2 - 4}}{2}\right)^{-k^{n-1}}, \quad n \in \mathbb{N},$$

which is only true for $k = 2$, that is, instead of $P_k(x_0)$ is written $P_2(x_0) = x_0^2 - 2$. The same oversight was made in Propositions 1–4 therein, where instead of $P_2(x_0)$ should have been used $P_j(x_0)$, $j = \overline{3,6}$, respectively, in the corresponding special cases of Eq. (10).

Remark 2 Note also that since the following relation holds:

$$\frac{P_k(x_0) - \sqrt{(P_k(x_0))^2 - 4}}{2} = \left(\frac{P_k(x_0) + \sqrt{(P_k(x_0))^2 - 4}}{2}\right)^{-1}.$$

Equation (10) can also be written it the following form:

$$x_n = \left(\frac{P_k(x_0) + \sqrt{(P_k(x_0))^2 - 4}}{2}\right)^{k^{n-1}} + \left(\frac{P_k(x_0) + \sqrt{(P_k(x_0))^2 - 4}}{2}\right)^{-k^{n-1}},$$

for $n \in \mathbb{N}$.

One of the main points in [32] is the fact that the polynomial P_k satisfies the following relation:

$$P_k\left(a + \frac{1}{a}\right) = a^k + \frac{1}{a^k},$$

for every $a \in \mathbb{C} \setminus \{0\}$, and for each $k \in \mathbb{N}$.

This property of the polynomial P_k considerably helps in finding closed-form formulas for the solutions to the equations in (7). The forms of the solutions in Theorem 1 show that they are related to the cosine and hyperbolic cosine functions. One can construct other types of difference equations whose solutions are cosine of some sequences by using other trigonometric relations containing the function (see, e.g., [5, p.169]).

Another important feature of the sequence of polynomials P_k, $k \in \mathbb{N}$, is that there is a recursive relation of second order which is satisfied by them. Moreover, the recursive relation is solvable in closed form. By using the recursive relation it was obtained the representation of the polynomials given in (8) (at first sight the representation does not look like a polynomial, but by using the binomial formula and some simple calculations it is not difficult to see that it really produces some polynomials).

Motivated by all the facts above mentioned, and by the importance of the quantity $a + \frac{1}{a}$ in the investigation in [32], it is natural to see what can be obtained if we use the following related quantity: $a - \frac{1}{a}$, instead of $a + \frac{1}{a}$. This note is devoted to investigating of the problem.

Our main aim here is to obtain a family/sequence of first order sine-type difference equations solvable in closed form in a constructive way, and present general solution to each of the equations.

2 Main results

This section presents a detailed analysis which leads to a construction of a sequence of polynomials which will be effectively employed in the main result of this note. It is shown that the sequence of polynomials satisfies a recursive relation of second order which, similar to the case of the polynomials defined in Theorem 1, can be also solved in closed form. After the construction of the sequence of polynomials we present and prove the main result in this note, on the existence of a sequence of sine-type difference equations of first order which are also solvable in closed form.

2.1 Construction of a sequence of polynomials

Our considerations in [32] essentially started from the following simple relation:

$$z^2+\frac{1}{z^2}=\left(z+\frac{1}{z}\right)^2-2,\quad z\in\mathbb{C}\setminus\{0\},\tag{11}$$

which was used by Laplace in [14]. Based on the relation and few other related ones, in [32] we shown that important things in the study therein were some relations between the functions

$$f(z)=z+\frac{1}{z}\tag{12}$$

and

$$f_k(z)=z^k+\frac{1}{z^k},\quad k\in\mathbb{N}.$$

The function in (12), by which the initial value x_0 in Theorem 1 is represented, is frequently applied in various fields of mathematics. For example, in solving polynomial equations (see, e.g., [2, p.27]), and in conformal mappings (see, e.g., [1]). Recall that $\cos z$ and $\cosh z$, which are some of the basic analytic functions are defined by using the function.

In a majority of cases in the scientific literature, along with cosine and hyperbolic cosine functions, one considers simultaneously their counterparts, that is, the sine and hyperbolic sine functions. Unlike $\cos z$ and $\cosh z$, these two functions are defined by using the function

$$g(z)=z-\frac{1}{z}.\tag{13}$$

However, unlike the case of functions $f(z)$ and $f(z^2)$, where according to (11), the following very useful relation exists:

$$f\left(z^2\right)=f^2(z)-2,$$

there is no such a useful relation between the functions $g(z)$ and $g(z^2)$. Note that

$$g^2(z)=f\left(z^2\right)-2.\tag{14}$$

On the other hand, there is a useful relation between $g(z)$ and $g(z^3)$. Indeed, since

$$\left(z-\frac{1}{z}\right)^3 = z^3 - 3z + \frac{3}{z} - \frac{1}{z^3},$$

we have

$$g\left(z^3\right) = z^3 - \frac{1}{z^3} = \left(z-\frac{1}{z}\right)^3 + 3\left(z-\frac{1}{z}\right) = g^3(z) + 3g(z). \tag{15}$$

Further, note that

$$\left(z-\frac{1}{z}\right)^4 = z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4},$$

from which along with (14) it follows that

$$f\left(z^4\right) = z^4 + \frac{1}{z^4} = \left(z-\frac{1}{z}\right)^4 + 4\left(z-\frac{1}{z}\right)^2 + 2 = g^4(z) + 4g^2(z) + 2, \tag{16}$$

which is a relation between $f(z^4)$ and $g(z)$. However, similar to Eq. (14), the relation in (16) is also not very useful for our present investigation.

Further, we also have

$$\left(z-\frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5},$$

from which along with (15) it follows that

$$g\left(z^5\right) = z^5 - \frac{1}{z^5} = \left(z-\frac{1}{z}\right)^5 + 5\left(z-\frac{1}{z}\right) = g^5(z) + 5g^3(z) + 5g(z), \tag{17}$$

which is a relation between $g(z)$ and $g(z^5)$.

From (15) and (17) we have

$$\begin{aligned} g(z) &= Q_1\big(g(z)\big), \\ g\left(z^3\right) &= Q_2\big(g(z)\big), \\ g\left(z^5\right) &= Q_3\big(g(z)\big), \end{aligned} \tag{18}$$

where

$$\begin{aligned} Q_1(t) &= t, \\ Q_2(t) &= t^3 + 3t, \\ Q_3(t) &= t^5 + 5t^3 + 5t. \end{aligned} \tag{19}$$

By using the method of induction it can be proved that

$$g\left(z^{2k-1}\right) = Q_k\big(g(z)\big), \quad k \in \mathbb{N}, \tag{20}$$

for some polynomials $Q_k(t)$ of the following form:

$$Q_k(t) = t^{2k-1} + (2k-1)t^{2k-3} + \sum_{j=1}^{k-2} a_j t^{2j-1}, \tag{21}$$

where it seems not quite easy to calculate the coefficients a_j, $j = \overline{1, k-2}$, in this way.

To present the polynomials $Q_k(t)$ in a better form for applications in this note, we use here our idea in [32] on finding a recursive relation which these polynomials satisfy.

Now note that

$$\left(z^{2k-1} - \frac{1}{z^{2k-1}}\right)\left(z^2 + \frac{1}{z^2}\right) = z^{2k+1} - \frac{1}{z^{2k+1}} + z^{2k-3} - \frac{1}{z^{2k-3}},$$

from which along with (14) and (20) it follows that

$$Q_{k+1}\bigl(g(z)\bigr) = \bigl(Q_1^2\bigl(g(z)\bigr) + 2\bigr)Q_k\bigl(g(z)\bigr) - Q_{k-1}\bigl(g(z)\bigr),$$

for $k \geq 2$.

Hence, the sequence of polynomials $Q_k(t)$, satisfies the following second order recursive relation:

$$Q_{k+1}(t) = \bigl(t^2 + 2\bigr)Q_k(t) - Q_{k-1}(t), \tag{22}$$

for $k \geq 2$.

By using relation (22) and the first two (initial) conditions in (19), we can find any member of the sequence $Q_k(t)$. Moreover, Eq. (22) is solvable in closed form as a homogeneous second order linear difference equation (for a fixed t it is a difference equation with constant coefficients).

The polynomial

$$\widehat{P}_2(\lambda) = \lambda^2 - \bigl(t^2 + 2\bigr)\lambda + 1$$

is the characteristic one associated to Eq. (22), and the (characteristic) zeros of the polynomial are

$$\lambda_1 = \frac{t^2 + 2 + t\sqrt{t^2+4}}{2} \quad \text{and} \quad \lambda_2 = \frac{t^2 + 2 - t\sqrt{t^2+4}}{2}. \tag{23}$$

Hence, by a well-known theorem general solution to the difference equation (22) has the following form:

$$Q_k(t) = c_1\left(\frac{t^2 + 2 + t\sqrt{t^2+4}}{2}\right)^k + c_2\left(\frac{t^2 + 2 - t\sqrt{t^2+4}}{2}\right)^k, \tag{24}$$

for $k \in \mathbb{N}$, where c_1 and c_2 are arbitrary constants.

From (19) and (24) with $k = 1, 2$, we obtain the following two-dimensional linear system of algebraic equations:

$$\begin{aligned} &c_1\left(\frac{t^2+2+t\sqrt{t^2+4}}{2}\right)+c_2\left(\frac{t^2+2-t\sqrt{t^2+4}}{2}\right)=t, \\ &c_1\left(\frac{t^2+2+t\sqrt{t^2+4}}{2}\right)^2+c_2\left(\frac{t^2+2-t\sqrt{t^2+4}}{2}\right)^2=t^3+3t. \end{aligned} \tag{25}$$

The determinant of system (25) is

$$\Delta=\begin{vmatrix} \frac{t^2+2+t\sqrt{t^2+4}}{2} & \frac{t^2+2-t\sqrt{t^2+4}}{2} \\ \left(\frac{t^2+2+t\sqrt{t^2+4}}{2}\right)^2 & \left(\frac{t^2+2-t\sqrt{t^2+4}}{2}\right)^2 \end{vmatrix}=-t\sqrt{t^2+4}.$$

Hence, after some calculations, we have

$$c_1=\frac{1}{\Delta}\begin{vmatrix} t & \frac{t^2+2-t\sqrt{t^2+4}}{2} \\ t^3+3t & \left(\frac{t^2+2-t\sqrt{t^2+4}}{2}\right)^2 \end{vmatrix}=\lambda_2\frac{\sqrt{t^2+4}+t}{2} \tag{26}$$

and

$$c_2=\frac{1}{\Delta}\begin{vmatrix} \frac{t^2+2+t\sqrt{t^2+4}}{2} & t \\ \left(\frac{t^2+2+t\sqrt{t^2+4}}{2}\right)^2 & t^3+3t \end{vmatrix}=-\lambda_1\frac{\sqrt{t^2+4}-t}{2}. \tag{27}$$

By using (26) and (27) in Eq. (24), as well as Vieta's formulas, it follows that

$$\begin{aligned} Q_k(t)={}&\frac{\sqrt{t^2+4}+t}{2}\left(\frac{t^2+2+t\sqrt{t^2+4}}{2}\right)^{k-1} \\ &-\frac{\sqrt{t^2+4}-t}{2}\left(\frac{t^2+2-t\sqrt{t^2+4}}{2}\right)^{k-1}, \end{aligned} \tag{28}$$

for $k\in\mathbb{N}$.

2.2 A sequence of solvable sine-type difference equations

Now after the construction of polynomials $Q_k(t)$, $k\in\mathbb{N}$, which has been done in the previous subsection, we are in a position to state and prove our main result on the existence of a sequence of sine-type difference equations of first order which are solvable in closed form.

Theorem 2 *Consider the difference equation*

$$x_{n+1}=Q_k(x_n),\quad n\in\mathbb{N}_0, \tag{29}$$

where $k\in\mathbb{N}\setminus\{1\}$ *and the polynomial* Q_k *is given by* (28). *Then the following statements hold*:

(a) *If $x_0 \in \mathbb{C}$, then the solution to Eq.* (29) *with the initial value x_0 is given by*

$$x_n = \left(\frac{x_0 + \sqrt{x_0^2 + 4}}{2}\right)^{(2k-1)^n} + \left(\frac{x_0 - \sqrt{x_0^2 + 4}}{2}\right)^{-(2k-1)^n}, \tag{30}$$

for $n \in \mathbb{N}_0$.

(b) *If*

$$x_0 = 2i \sin\theta \tag{31}$$

for some $\theta \in [0, 2\pi)$, then the solution to Eq. (29) *with the initial value x_0 is given by*

$$x_n = 2i \sin\bigl((2k-1)^n \theta\bigr), \quad n \in \mathbb{N}_0. \tag{32}$$

Proof (a) First note that for each $x_0 \in \mathbb{C}$ there is a such that the relation

$$x_0 = a - \frac{1}{a} \tag{33}$$

holds.

Indeed, from (33) we have $a^2 - x_0 a - 1 = 0$, from which it follows that

$$a_1 = \frac{x_0 + \sqrt{x_0^2 + 4}}{2} \quad \text{and} \quad a_2 = \frac{x_0 - \sqrt{x_0^2 + 4}}{2}. \tag{34}$$

Let $a := a_1$. Then, from Eq. (29) and by a simple inductive argument, we have

$$x_n = Q_k(x_{n-1}) = Q_k^{[n]}(x_0), \tag{35}$$

for $n \in \mathbb{N}_0$.

By using Eq. (20) in (35) we have

$$x_n = Q_k^{[n]}\bigl(g(a)\bigr) = Q_k^{[n-1]}\bigl(Q_k\bigl(g(a)\bigr)\bigr) = Q_k^{[n-1]}\bigl(g\bigl(a^{2k-1}\bigr)\bigr),$$

for $n \in \mathbb{N}$.

By using a similar argument and the method of mathematical induction it is shown that

$$x_n = g\bigl(a^{(2k-1)^n}\bigr),$$

for $n \in \mathbb{N}$, that is, we have

$$x_n = a^{(2k-1)^n} - \frac{1}{a^{(2k-1)^n}}, \quad n \in \mathbb{N}_0. \tag{36}$$

From (34), (36) and since

$$\frac{1}{a} = -\frac{x_0 - \sqrt{x_0^2 + 4}}{2}$$

we have

$$x_n = \left(\frac{x_0 + \sqrt{x_0^2 + 4}}{2}\right)^{(2k-1)^n} - \left(-\frac{x_0 - \sqrt{x_0^2 + 4}}{2}\right)^{-(2k-1)^n}, \tag{37}$$

for $n \in \mathbb{N}_0$.

From Eq. (37) and since $2k - 1$ is an odd natural number, Eq. (30) easily follows.

(b) Since (31) holds for some $\theta \in [0, 2\pi)$, then we have $a = e^{i\theta}$. Employing this in Eq. (36) we have

$$x_n = e^{i\theta(2k-1)^n} - e^{-i\theta(2k-1)^n}, \quad n \in \mathbb{N}_0,$$

from which Eq. (32) easily follows. □

Remark 3 Unlike the sequence of polynomials $(P_k)_{k\in\mathbb{N}}$ which are closely related to Chebyshev polynomials ([6, 18, 24]), the sequence $(Q_k)_{k\in\mathbb{N}}$ seems much less known to a wider audience. But it might be known to some specialists on polynomials and related topics. Our literature investigation shows that the polynomials $(P_k)_{k\in\mathbb{N}}$ are connected to the polynomials that commute under composition (see, e.g., [36, Problem 251]).

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Author details
[1]Mathematics Department, Faculty of Science, Taif University, P.O. Box 11099, Taif, 21944, Saudi Arabia. [2]Faculty of Electrical Engineering, Belgrade University, Bulevar Kralja Aleksandra 73, 11000, Beograd, Serbia. [3]Faculty of Mechanical and Civil Engineering in Kraljevo, University of Kragujevac, Kraljevo, Serbia. [4]Deptartment of Mathematical Sciences, Appalachian State University, Boone, NC 28608, USA. [5]Mathematical Institute of the Serbian Academy of Sciences and Arts, Knez Mihailova 36/III, 11000, Beograd, Serbia. [6]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, 40402, Taiwan, Republic of China. [7]Brno University of Technology, Department of Mathematics, Faculty of Electrical Engineering and Communication, Technicka 3058/10, CZ-616 00, Brno, Czech Republic.

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