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Some properties of rapidly varying functions[☆]

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ABSTRACT

In this paper some characterizations of the class of rapidly varying functions using the notions of the lower and upper generalized inverses will be proved. The important properties of this class that are related to two classical integral transformations will be proved, also.

1. Introduction and results

A measurable function $f : [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$ is called slowly varying in the sense of Karamata (see e.g. [10]) if it satisfies the following condition:

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = 1 \tag{1}$$

for every $\lambda > 0$. The class of all these functions (denoted by SV) is the main object in Karamata's theory (see e.g. [1]).

A measurable function $f : [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$ is called rapidly varying in the sense of de Haan with the index of variability $+\infty$ (see e.g. [3]) if it satisfies the following condition:

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = +\infty \tag{2}$$

for every $\lambda > 1$.

This functional class is denoted by R_∞. The theory of rapid variability (with its generalizations) is an important part of asymptotical analysis and game theory (see e.g. [1,4,7,5,9,11]).

Remark 1.1. In this paper we will consider an elements from classes SV and R_∞ defined in $(0, +\infty)$, without loss of generality.

Let $F^{(\infty)}$ be the set of all functions $f : (0, +\infty) \mapsto (0, +\infty)$ which are bounded in $(0, \alpha)$ for every $\alpha \in (0, +\infty)$, and for which $\limsup_{x\to+\infty} f(x) = +\infty$ (see [2]).

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For any $f \in F^{(\infty)}$, the positive, nondecreasing and unbounded function

$$f^{\leftarrow}(y) = \inf\{x > 0 \mid f(x) > y\} \tag{3}$$

defined in $(b, +\infty)$ for every $y > b = \inf\big\{f(t) \mid t \in (0, +\infty)\big\} \geq 0$ is its generalized inverse (see e.g. [1]). It is a very important object in asymptotic analysis (see e.g. [7,8,6,5]) and it can be used to characterize relationships between functional classes SV and R_∞ (see e.g. [1,7]).

Let $F^\infty = \big\{f \in F^{(\infty)} \mid \liminf_{t\to+\infty} f(t) = +\infty\big\}$. Now, for any $f \in F^\infty$, we will consider the following two positive and nondecreasing functions:

1° $f^{\leftarrow l}(y) = \inf\big\{x > 0 \mid f(x) \geq y\big\}$, and
2° $f^{\leftarrow u}(y) = \sup\big\{x > 0 \mid f(x) \leq y\big\}$,

for every $y > b$. These functions are important generalizations of the generalized inverse for elements from F^∞ (see e.g. [2]). Actually, $f^{\leftarrow l}(y) \leq f^{\leftarrow}(y) \leq f^{\leftarrow u}(y)$ is satisfied for any $f \in F^\infty$, and every $y \in (b, +\infty)$. Furthermore, if f is a continuous and strictly increasing function, the previous inequalities become equalities and the observed value is equal to $f^{-1}(y)$, where f^{-1} is the inverse of a function f. It can be proved that $f^{\leftarrow l}\big(f(x)\big) \leq x \leq f^{\leftarrow u}\big(f(x)\big)$ is satisfied for any $f \in F^\infty$ and every $x > 0$. Also, it can be proved that the following two assertions:

1. $x < f^{\leftarrow l}(y)$ if and only if $\overline{f}(x) < y$ and
2. $f^{\leftarrow u}(y) < x$ if and only if $\underline{f}(x) > y$

are satisfied for $x > 0$ and $y > b$, where

3° $\overline{f}(x) = \sup\big\{f(t) \mid t \leq x\big\}$ and
4° $\underline{f}(x) = \inf\big\{f(t) \mid t \geq x\big\}$,

for every $x > 0$. More about functions $\overline{f}$ and $\underline{f}$, for $f \in F^\infty$, can be found in [1]. Here we note that these functions are positive and nondecreasing and it holds that $\underline{f}(x) \leq f(x) \leq \overline{f}(x)$ for every $x > 0$. Thus, it can be concluded that $f^{\leftarrow l}(y) = \sup\big\{x > 0 \mid \overline{f}(x) < y\big\}$ and $f^{\leftarrow u}(y) = \inf\big\{x > 0 \mid y < \underline{f}(x)\big\}$, for any $f \in F^\infty$ and every $y > b$.

In the following theorem we give multiple characterizations of functions from the class R_∞ which belong to the class F^∞.

Theorem 1.1. *Let $f \in F^\infty$ be a measurable function. The following assertions are mutually equivalent:*

(a) *a function f belongs to R_∞;*
(b) $\lim_{x\to+\infty} \inf_{\lambda\geq L} \frac{f(x)}{f(\frac{x}{\lambda})} = +\infty$ *for every $L > 1$;*
(c) $\lim_{x\to+\infty} \frac{\underline{f}(\lambda x)}{\overline{f}(x)} = +\infty$ *for every $\lambda > 1$;*
(d) $\lim_{y\to+\infty} \frac{f^{\leftarrow u}(\lambda y)}{f^{\leftarrow l}(y)} = 1$ *for every $\lambda > 1$;*
(e) *let $g\colon (b, +\infty) \mapsto (0, +\infty)$ be a measurable function such that $f^{\leftarrow l}(y) \leq g(y) \leq f^{\leftarrow u}(y)$ for every $y > b$; then the function $g \in SV$;*
(f) *a function $f^{\leftarrow l}$ belongs to SV and $f^{\leftarrow l}(y) \sim f^{\leftarrow u}(y)$ for $y \to +\infty$ (where $\sim$ is the strong asymptotic equivalence relation (see e.g. [1])).*

In the following theorem we give some properties for elements from the class $SV \cap F^\infty$ (which are analogous to properties given in the previous theorem (assertions (d) and (e)) for elements from the class $R_\infty \cap F^\infty$).

Theorem 1.2. *Let $g \in SV \cap F^\infty$. Then the following assertions hold:*

(a) $\lim_{y\to+\infty} \frac{g^{\leftarrow l}(\lambda y)}{g^{\leftarrow u}(y)} = +\infty$ *for every $\lambda > 1$;*
(b) *every measurable function $f\colon [b, +\infty) \mapsto (0, +\infty)$ such that $g^{\leftarrow l}(y) \leq f(y) \leq g^{\leftarrow u}(y)$ for every $y > b$ belongs to the class R_∞.*

Now, we consider an interesting equivalence relation for the class R_∞. Let f and g be positive functions in $(0, +\infty)$. For these functions we say that they are mutually rapidly equivalent (denoted by $f(x) \overset{r}{\sim} g(x)$ for $x \to +\infty$) if the condition

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{g(x)} = \lim_{x\to+\infty} \frac{g(\lambda x)}{f(x)} = +\infty, \tag{4}$$

is satisfied for every $\lambda > 1$. More about relation (4) can be found in [1,8,6].

Theorem 1.3. *Let f and g be a positive functions in $(0, +\infty)$. Then the following assertions hold:*

(a) *if f and g are measurable functions such that $f(x) \overset{r}{\sim} g(x)$ for $x \to +\infty$, then f, g belong to R_∞;*
(b) *the relation $\overset{r}{\sim}$ is an equivalence relation in the class R_∞;*

(c) *let $f \in R_\infty$ and let $f(x) \asymp g(x)$ for $x \to +\infty$ (where $\asymp$ is the weak asymptotic equivalence relation (see e.g. [1])); then $f(x) \overset{r}{\sim} g(x)$ for $x \to +\infty$;*

(d) *let f be a measurable function; a function f belongs to R_∞ if and only if $\underline{f}(x) \overset{r}{\sim} \overline{f}(x)$ for $x \to +\infty$;*

(e) *let f belong to R_∞ and let $\underline{f}(x) \le g(x) \le \overline{f}(x)$ for $x \ge x_0 > 0$; if g is a measurable function then g belongs to R_∞.*

Now, we will consider two integral transformations in the class R_∞.

For a measurable function $f: (0, +\infty) \mapsto (0, +\infty)$ such that $f(x)$ is a bounded function in $(0, x)$ for every $x > 0$, we define the transformation

$$\underset{\sim}{f}(x) = \frac{1}{x}\int_0^x f(t)dt = \int_0^1 f(xu)du \tag{5}$$

for $x > 0$. On the other hand, for a measurable function $f: (0, +\infty) \mapsto (0, +\infty)$ such that $\frac{1}{f(x)}$ is a bounded function in $(0, x)$ for every $x > 0$, we define the transformation

$$\widetilde{f}(x) = \frac{1}{x\int_x^{+\infty}\frac{dt}{t^2 f(t)}} = \frac{1}{\int_0^1 \frac{du}{f(\frac{x}{u})}} \tag{6}$$

for $x > 0$. More about transformations (5) and (6) can be found in [1].

Theorem 1.4. *Let $f \in R_\infty$ be a bounded function in $(0, x)$ for every $x > 0$. Also, let $\frac{1}{f(x)}$ be a bounded function for every $x > 0$. Then the following assertions hold:*

(a) *$\underset{\sim}{f}(x) \overset{r}{\sim} f(x) \overset{r}{\sim} \widetilde{f}(x)$ for $x \to +\infty$;*

(b) *functions $\underset{\sim}{f}, \widetilde{f} \in R_\infty$;*

(c) *$\underset{\sim}{f}(x) \sim \int_{1-\varepsilon}^1 f(xu)du$ for $x \to +\infty$ and every $\varepsilon \in (0, 1]$, and $\frac{1}{\widetilde{f}(x)} \sim \int_{1-\varepsilon}^1 \frac{du}{f(\frac{x}{u})}$ for $x \to +\infty$ and every $\varepsilon \in (0, 1]$ (where $\sim$ is the strong asymptotic equivalence relation (see e.g. [1]));*

(d) *$\underset{\sim}{f}(x) = o\big(\overline{f}(x)\big)$ for $x \to +\infty$, and $\underline{f}(x) = o\big(\widetilde{f}(x)\big)$ for $x \to +\infty$ (where o is the Landau symbol (see e.g. [1])).*

Corollary 1.1. (a) *Let $f \in R_\infty$ be a function with the same properties as in Theorem 1.4. Then it holds that the function $\int_0^x f(t)dt \in R_\infty$ for $x > 0$, and the function $\frac{1}{\int_x^{+\infty}\frac{dt}{f(t)}} \in R_\infty$ for $x > 0$. Also, it holds that $\int_0^x f(t)dt \sim \int_{\frac{x}{\lambda}}^x f(t)dt$ for $x \to +\infty$ and every $\lambda > 1$, and $\int_x^{+\infty}\frac{dt}{f(t)} \sim \int_x^{\lambda x}\frac{dt}{f(t)}$ for $x \to +\infty$ and every $\lambda > 1$.*

(b) *Let $f \in R_\infty$ be a nondecreasing function. Then $\underset{\sim}{f}(x) = o\big(f(x)\big)$ for $x \to +\infty$, and $f(x) = o\big(\widetilde{f}(x)\big)$ for $x \to +\infty$.*

Theorem 1.5. *Let $f : (0, +\infty) \mapsto (0, +\infty)$ be a nondecreasing function.*

(a) *If $\underset{\sim}{f}(x) = o\big(f(x)\big)$ for $x \to +\infty$, then $f \in R_\infty$.*

(b) *If $f(x) = o\big(\widetilde{f}(x)\big)$ for $x \to +\infty$, then $f \in R_\infty$.*

The set of all measurable functions $f : (0, +\infty) \mapsto (0, +\infty)$ whose lower Matuszewska index is equal to $+\infty$ is denoted by MR_∞ (see e.g. [1]). It is well-known that $MR_\infty \subsetneq R_\infty$. Hence, if we observe a function from the set MR_∞ in Theorem 1.4(d), we will get stronger conclusions than we got for a function from the set R_∞. We discuss this in the following theorem.

Theorem 1.6. *Let $f \in MR_\infty$ be a function with the same properties as in the assumptions of Theorem 1.4. Then $\underset{\sim}{f}(x) = o\big(f(x)\big)$ for $x \to +\infty$, and $f(x) = o\big(\widetilde{f}(x)\big)$ for $x \to +\infty$.*

Example 1.1. (1) Let

$$f(x) = \begin{cases} e^x, & \text{for } x \in (0, +\infty) \setminus \mathbb{N}; \\ xe^x, & \text{for } x \in \mathbb{N}. \end{cases}$$

Then $f \in R_\infty \setminus MR_\infty$ and $\underset{\sim}{f}(x) \ne o(f(x))$ for $x \to +\infty$.

(2) Let

$$f(x) = \begin{cases} \dfrac{e^x}{x^2}, & \text{for } x \in (0, +\infty) \setminus \mathbb{N}; \\ \dfrac{e^x}{x}, & \text{for } x \in \mathbb{N}. \end{cases}$$

Then $f \in R_\infty \setminus MR_\infty$ and $f(x) \ne o(\widetilde{f}(x))$ for $x \to +\infty$.

2. Proofs

Proof of Theorem 1.1. $\big((a) \Rightarrow (b)\big)$ Let $M > 1$. On the basis of the theorem of uniform convergence for the rapidly varying functions (see e.g. [1]), for every $L > 1$ there exists $x_0 > 0$ such that for every $x \ge x_0$ and every $\lambda > L$ it holds that $\frac{f(\lambda x)}{f(x)} > M$. Now, there is an $x_1 > 0$ such that $f(x) \ge M \cdot \sup\big\{f(t) \mid t \in (0, x_0]\big\}$ is satisfied for every $x \ge x_1$. Then the following assertions hold:

1. if $\frac{x}{\lambda} \le x_0$, then $\frac{f(x)}{f(\frac{x}{\lambda})} \ge \frac{f(x)}{\sup\left\{f(t) \mid t \in (0,x_0]\right\}} \ge M$;
2. if $\frac{x}{\lambda} \ge x_0$, then $\frac{f(x)}{f(\frac{x}{\lambda})} = \frac{f(\lambda \frac{x}{\lambda})}{f(\frac{x}{\lambda})} \ge M$;

for every $\lambda > L$ and $x > x_1$. In any case, it holds that $\inf\left\{\frac{f(x)}{f(\frac{x}{\lambda})} \mid \lambda \ge L, x \ge x_1\right\} \ge M$, and we obtain $\lim_{x\to+\infty} \inf_{\lambda \ge L} \frac{f(x)}{f(\frac{x}{\lambda})} = +\infty$.

$\big((b) \Rightarrow (c)\big)$ Let $\lambda > 1$. Then for $x > 0$ it holds that

$$\begin{aligned}\frac{\underline{f}(\lambda x)}{\overline{f}(x)} &= \frac{\inf\big\{f(t) \mid t \ge \lambda x\big\}}{\sup\big\{f(s) \mid s \in (0, x]\big\}} \\ &= \inf\left\{\frac{f(t)}{f(s)} \mid s \in (0,x], t \ge \lambda x\right\} \\ &\ge \inf\left\{\frac{f(t)}{f\left(\frac{t}{\mu}\right)} \mid \mu \ge \lambda, t \ge \lambda x\right\}.\end{aligned}$$

Hence, we obtain $\liminf_{x\to+\infty} \frac{\underline{f}(\lambda x)}{\overline{f}(x)} \ge \lim_{t\to+\infty} \inf_{\mu\ge\lambda} \frac{f(t)}{f(\frac{t}{\mu})} = +\infty$.

$\big((c) \Rightarrow (a)\big)$ Let $\lambda > 1$ and $x > 0$. Then from $\frac{f(\lambda x)}{f(x)} \ge \frac{\underline{f}(\lambda x)}{\overline{f}(x)}$, we obtain $\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = +\infty$, i.e. $f \in R_\infty$.

$\big((a) \Rightarrow (d)\big)$ Let $\lambda > 1$. Then it holds that

$$\begin{aligned}\frac{f^{\leftarrow u}(\lambda x)}{f^{\leftarrow l}(x)} &= \frac{\sup\{s \mid f(s) \le \lambda x\}}{\inf\{t \mid f(t) \ge x\}} \\ &= \sup\left\{\frac{s}{t} \mid f(s) \le \lambda x, f(t) \ge x\right\} \\ &\le \sup\left\{\mu > 0 \mid \frac{f(\mu t)}{f(t)} \le \lambda, f(t) \ge x\right\}.\end{aligned}$$

From results obtained in [9] it follows that

$$\limsup_{x\to+\infty} \frac{f^{\leftarrow u}(\lambda x)}{f^{\leftarrow l}(x)} \le \limsup_{t\to+\infty}\ \sup\left\{\mu > 0 \mid \frac{f(\mu t)}{f(t)} \le \lambda\right\} \le 1.$$

On the other hand, it is obvious that

$$\frac{f^{\leftarrow u}(\lambda x)}{f^{\leftarrow l}(x)} \ge \frac{f^{\leftarrow u}(x)}{f^{\leftarrow l}(x)} \ge 1,$$

is satisfied for the same x and λ. Hence, $\lim_{x\to+\infty} \frac{f^{\leftarrow u}(\lambda x)}{f^{\leftarrow l}(x)} = 1$ for $\lambda > 1$.

$\big((d) \Rightarrow (a)\big)$ Let assertion (d) hold. We will assume that $f \notin R_\infty$. Then from results obtained in [9] there is a sequence of real numbers (x_n) such that $\lim_{n\to+\infty} x_n = +\infty$, and there is a sequence of positive real numbers (λ_n) such that $\liminf_{n\to+\infty} \lambda_n > \lambda > 1$, and for those two sequences it holds that $\limsup_{n\to+\infty} \frac{f(\lambda_n x_n)}{f(x_n)} < \mu < +\infty$ for some $\lambda > 1$ and $\mu > 1$. Furthermore, inequalities $f^{\leftarrow l}\big(f(x_n)\big) \le x_n$ and $f^{\leftarrow u}\big(\mu f(x_n)\big) = \sup\big\{x > 0 \mid f(x) \le \mu \cdot f(x_n)\big\} \ge \lambda_n \cdot x_n > \lambda \cdot x_n$ are satisfied for sufficiently large n (because for sufficiently large n it holds that $f(\lambda_n x_n) \le \mu \cdot f(x_n)$) which implies that

$$\limsup_{y\to+\infty} \frac{f^{\leftarrow u}(\mu y)}{f^{\leftarrow l}(y)} \ge \limsup_{n\to+\infty} \frac{f^{\leftarrow u}\big(\mu f(x_n)\big)}{f^{\leftarrow l}\big(f(x_n)\big)} \ge \limsup_{n\to+\infty} \frac{\lambda \cdot x_n}{x_n} > 1.$$

The last inequality is a contradiction to assertion (d). Hence, $f \in R_\infty$.

$\big((\mathrm{d}) \Rightarrow (\mathrm{e})\big)$ From $f^{\leftarrow l}(y) \le g(y) \le f^{\leftarrow u}(y)$ for every $y \in (b, +\infty)$, we have that

$$F(\lambda, y) = \frac{f^{\leftarrow u}(\lambda y)}{f^{\leftarrow l}(y)} \ge \frac{g(\lambda y)}{g(y)} \ge \frac{f^{\leftarrow l}(\lambda y)}{f^{\leftarrow u}(y)} \ge \frac{f^{\leftarrow l}(y)}{f^{\leftarrow u}(\lambda y)} = \frac{1}{F(\lambda, y)}$$

is satisfied for the same y and every $\lambda > 1$. Hence, $\lim_{y\to+\infty} \frac{g(\lambda y)}{g(y)} = 1$ for the same λ, and we obtain that $g \in SV$, because g is a measurable function.

$\big((\mathrm{e}) \Rightarrow (\mathrm{f})\big)$ Trivially, on the basis of assertion (e) it holds that $f^{\leftarrow l} \in SV$ (similarly, $f^{\leftarrow u} \in SV$). Now, we will assume that

$$g(y) = \begin{cases} f^{\leftarrow l}(y), & \text{for } y \ge b, y \in \mathbb{Q}, \\ f^{\leftarrow u}(y), & \text{for } y \ge b, y \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $g \in SV$, and it holds that

$$1 \le \lim_{y\to+\infty} \frac{f^{\leftarrow u}(\lambda y)}{f^{\leftarrow l}(y)} \le \lim_{y\to+\infty} \sup_{\frac{1}{\lambda} \le \mu \le \lambda} \frac{g(\mu y)}{g(y)} = 1$$

for $\lambda > 1$. Hence, we obtain $f^{\leftarrow l}(y) \sim f^{\leftarrow u}(y)$ for $y \to +\infty$.

$\big((\mathrm{f}) \Rightarrow (\mathrm{d})\big)$ Let $\lambda > 1$. On the basis of assertion (f) it follows that

$$\lim_{y\to+\infty} \frac{f^{\leftarrow u}(\lambda y)}{f^{\leftarrow l}(y)} = \lim_{y\to+\infty} \frac{f^{\leftarrow u}(\lambda y)}{f^{\leftarrow l}(\lambda y)} \cdot \lim_{y\to+\infty} \frac{f^{\leftarrow l}(\lambda y)}{f^{\leftarrow l}(y)} = 1.$$

Finally, assertion (d) holds. □

Proof of Theorem 1.2. (a) Let $\lambda > 1$. Then it holds that

$$\begin{aligned} \frac{g^{\leftarrow l}(\lambda x)}{g^{\leftarrow u}(x)} &= \frac{\inf\{s \mid g(s) \ge \lambda x\}}{\sup\{t \mid g(t) \le x\}} \\ &= \inf\left\{\frac{s}{t} \mid g(t) \le x, g(s) \ge \lambda x\right\} \\ &\ge \inf\left\{\mu > 0 \mid \frac{g(\mu t)}{g(t)} \ge \lambda, g(t) \le x\right\} \end{aligned}$$

for $x \ge b$. Since $g \in SV$, we obtain that

$$\liminf_{x\to+\infty} \frac{g^{\leftarrow l}(\lambda x)}{g^{\leftarrow u}(x)} \ge \liminf_{t\to+\infty} \inf\left\{\mu > 0 \mid \frac{g(\mu t)}{g(g)} \ge \lambda\right\} = +\infty.$$

This completes the proof of assertion (a).

(b) Let $\lambda > 1$. Since f is a measurable function we have that

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} \ge \lim_{x\to+\infty} \frac{g^{\leftarrow l}(\lambda x)}{g^{\leftarrow u}(x)} = +\infty$$

is satisfied. Hence, $f \in R_\infty$. □

Proof of Theorem 1.3. (a) Let f, g be a measurable functions such that $f(x) \overset{r}{\sim} g(x)$ for $x \to +\infty$. Then

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = \lim_{x\to+\infty} \frac{f(\lambda x)}{g(\sqrt{\lambda}\, x)} \cdot \lim_{x\to+\infty} \frac{g(\sqrt{\lambda}\, x)}{f(x)} = +\infty,$$

is satisfied for $\lambda > 1$. Therefore, $f \in R_\infty$ and similarly $g \in R_\infty$.

(b) We will prove only that relation $\overset{r}{\sim}$ is the transitive relation (reflexivity and symmetry of this relation are obvious). Let $f(x) \overset{r}{\sim} g(x)$ for $x \to +\infty$, and $g(x) \overset{r}{\sim} h(x)$ for $x \to +\infty$. Therefore, we have that the equalities

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{h(x)} = \lim_{x\to+\infty} \frac{f(\lambda x)}{g(\sqrt{\lambda}\, x)} \cdot \lim_{x\to+\infty} \frac{g(\sqrt{\lambda}\, x)}{h(x)} = +\infty$$

and

$$\lim_{x\to+\infty} \frac{h(\lambda x)}{f(x)} = \lim_{x\to+\infty} \frac{h(\lambda x)}{g(\sqrt{\lambda}\, x)} \cdot \lim_{x\to+\infty} \frac{g(\sqrt{\lambda}\, x)}{f(x)} = +\infty$$

are satisfied, for $\lambda > 1$. Using assertion (a) we obtain that this assertion holds.

(c) We have that

$$\liminf_{x\to+\infty}\frac{f(\lambda x)}{g(x)} \geq \liminf_{x\to+\infty}\frac{f(\lambda x)}{f(x)} \cdot \liminf_{x\to+\infty}\frac{f(x)}{g(x)} = +\infty$$

for $\lambda > 1$. Since $f(x) \asymp g(x)$ for $x \to +\infty$, it follows that

$$\liminf_{x\to+\infty}\frac{g(\lambda x)}{f(x)} \geq \liminf_{x\to+\infty}\frac{g(\lambda x)}{f(\lambda x)} \cdot \liminf_{x\to+\infty}\frac{f(\lambda x)}{f(x)} = +\infty$$

for $\lambda > 1$. Hence, $f(x) \overset{r}{\sim} g(x)$ for $x \to +\infty$.

(d) ($\Rightarrow$) Let $f \in R_\infty$. Then, from Theorem 1.1 it holds that $\lim_{x\to+\infty}\frac{\underline{f}(\lambda x)}{\overline{f}(x)} = +\infty$ for $\lambda > 1$. From previous computations, it follows that $\lim_{x\to+\infty}\frac{\overline{f}(\lambda x)}{\underline{f}(x)} = +\infty$, i.e. $\overline{f}(x) \overset{r}{\sim} \underline{f}(x)$, for $x \to +\infty$.

($\Leftarrow$) Let $\overline{f}(x) \overset{r}{\sim} \underline{f}(x)$ for $x \to +\infty$. Then, it follows that $\lim_{x\to+\infty}\frac{\underline{f}(\lambda x)}{\overline{f}(x)} = +\infty$ for $\lambda > 1$, and applying Theorem 1.1 we obtain $f \in R_\infty$.

(e) Let $f \in R_\infty$. Then, it holds that

$$\liminf_{x\to+\infty}\frac{g(\lambda x)}{g(x)} \geq \liminf_{x\to+\infty}\frac{\underline{f}(\lambda x)}{\overline{f}(x)} = +\infty$$

for $\lambda > 1$. Applying Theorem 1.1 and previous computations, we obtain $g \in R_\infty$. $\square$

Proof of Theorem 1.4. (a) Let $\lambda > 1$. It is sufficient to prove that

$$\lim_{x\to+\infty}\frac{\underset{\sim}{f}(x)}{f(\lambda x)} = \lim_{x\to+\infty}\frac{f(x)}{\widetilde{f}(\lambda x)} = \lim_{x\to+\infty}\frac{\widetilde{f}(x)}{\underset{\sim}{f}(\lambda x)} = 0.$$

We have that

$$\frac{\underset{\sim}{f}(x)}{f(\lambda x)} = \int_0^1 \frac{f(ux)}{f(\lambda x)}du \longrightarrow 0$$

is satisfied for $x \to +\infty$, because it holds that $\lim_{x\to+\infty}\frac{f(ux)}{f(\lambda x)} = 0$ for $\lambda > 1$ and $u \in (0, 1]$, and $\sup_{0<u\leq 1}\left|\frac{f(ux)}{f(\lambda x)}\right| \leq 1$ for sufficiently large x is satisfied (based on Theorem 1.1(b)). Now, we can apply Lebesgue's theorem of dominant convergence. According to this theorem, it holds that

$$\frac{f(x)}{\widetilde{f}(\lambda x)} = \int_0^1 \frac{f(x)}{f\left(\frac{\lambda x}{u}\right)}du \longrightarrow 0 \tag{7}$$

for $x \to +\infty$. Actually, condition (7) is satisfied because it holds that $\lim_{x\to+\infty}\frac{f(x)}{f\left(\frac{\lambda x}{u}\right)} = 0$ for $\lambda > 1$ and $u \in (0, 1]$, and $\sup_{0<u\leq 1}\left|\frac{f(x)}{f\left(\frac{\lambda x}{u}\right)}\right| \leq 1$ for sufficiently large x is satisfied (on the basis of Theorem 1.1(b)).

Now, let $\lambda > 1$ and $M \in (0, +\infty)$. We have that $\frac{\lambda u}{\frac{1}{u}} > \sqrt{\lambda}$ is satisfied for $u \in (\lambda^{-\frac{1}{4}}, 1)$. Hence, it holds that $\frac{f(\lambda ux)}{f(\frac{x}{u})} > \frac{M}{\left(1-\lambda^{-\frac{1}{4}}\right)^2}$ for sufficiently large x. Applying the Cauchy–Schwarz inequality we obtain that it holds that

$$\begin{aligned}\frac{\underset{\sim}{f}(\lambda x)}{\widetilde{f}(x)} &= \int_0^1 f(\lambda ux)du \cdot \int_0^1 \frac{du}{f\left(\frac{x}{u}\right)} \geq \left(\int_0^1 \sqrt{\frac{f(\lambda ux)}{f\left(\frac{x}{u}\right)}}\,du\right)^2 \\ &\geq \left(\int_{\lambda^{-\frac{1}{4}}}^1 \sqrt{\frac{f(\lambda ux)}{f\left(\frac{x}{u}\right)}}\,du\right)^2 \geq M\end{aligned}$$

for sufficiently large x. Hence, $\lim_{x\to+\infty}\frac{\widetilde{f}(x)}{\underset{\sim}{f}(\lambda x)} = 0$.

(b) Let $\lambda > 1$. Then from assertion (a), it holds that

$$\lim_{x\to+\infty}\frac{\underset{\sim}{f}(\lambda x)}{\underset{\sim}{f}(x)} = \lim_{x\to+\infty}\left(\frac{\underset{\sim}{f}(\lambda x)}{\widetilde{f}(\lambda^{\frac{2}{3}}x)} \cdot \frac{\widetilde{f}(\lambda^{\frac{2}{3}}x)}{f(\lambda^{\frac{1}{3}}x)} \cdot \frac{f(\lambda^{\frac{1}{3}}x)}{\underset{\sim}{f}(x)}\right) = +\infty.$$

Furthermore, $\underset{\sim}{f}$ is a measurable and positive function in $(0, +\infty)$ which implies that $\underset{\sim}{f} \in R_\infty$. Analogously, it can be shown that $\widetilde{f} \in R_\infty$. This assertion can be proved by combining the results obtained in Theorems 1.4(a) and 1.3(a).

(c) Let $\varepsilon > 0$ and $\lambda = \frac{1}{1-\varepsilon} > 1$. Since $\underset{\sim}{f} \in R_\infty$, it follows that $\lim_{x\to+\infty} \frac{\underset{\sim}{f}(\frac{x}{\lambda})}{\underset{\sim}{f}(x)} = 0$. Furthermore, it holds that

$$\int_0^{1-\varepsilon} f(ux)du = \lambda^{-1}\int_0^1 f\left(\frac{t}{\lambda}x\right)dt = o\left(\int_0^1 f(tx)dt\right)$$

for $x \to +\infty$. Hence, $\int_{1-\varepsilon}^1 f(ux)du \sim \underset{\sim}{f}(x)$ for $x \to +\infty$. Analogously, it can be shown that $\frac{1}{\widetilde{f}(x)} \sim \int_{1-\varepsilon}^1 \frac{du}{f(\frac{x}{u})}$ for $x \to +\infty$.

(d) We have that $\lim_{x\to+\infty} \frac{f(ux)}{\widetilde{f}(x)} = 0$ is satisfied for $u \in (0, 1)$ (because $f(x) \overset{r}{\sim} \widetilde{f}(x)$ for $x \to +\infty$). Also, $\sup_{0<u\le 1}\left|\frac{f(ux)}{\widetilde{f}(x)}\right| = 1$ is satisfied for $x > 0$. Then, it follows that $\lim_{x\to+\infty} \frac{\underset{\sim}{f}(x)}{\widetilde{f}(x)} = \lim_{x\to+\infty}\int_0^1 \frac{f(ux)}{\widetilde{f}(x)}du = 0$ according to Lebesgue's theorem of dominant convergence. According to this theorem, we have that $\lim_{x\to+\infty} \frac{\underset{\sim}{f}(x)}{\widetilde{f}(x)} = \lim_{x\to+\infty}\int_0^1 \frac{\underset{\sim}{f}(x)}{f(\frac{x}{u})}du = 0$ is satisfied, because $\lim_{x\to+\infty} \frac{\underset{\sim}{f}(x)}{f(\frac{x}{u})} = 0$ is satisfied for $u \in (0, 1)$ ($f(x) \overset{r}{\sim} \underset{\sim}{f}(x)$ for $x \to +\infty$), and also $\sup_{0<u\le 1}\left|\frac{\underset{\sim}{f}(x)}{f(\frac{x}{u})}\right| = 1$ is satisfied for $x > 0$. $\square$

Proof of Theorem 1.5. (a) Let us assume that $\underset{\sim}{f}(x) = o\big(f(x)\big)$ for $x \to +\infty$ and $f \notin R_\infty$. Then, there is a sequence (x_n) (of positive numbers) such that $x_n \to +\infty$ for $n \to +\infty$, and there is a $\lambda > 1$ such that $\frac{f(\lambda x_n)}{f(x_n)} < M < +\infty$ for some $M \in (0, +\infty)$ and every $n \in \mathbb{N}$. Also, it holds that

$$\int_{x_n}^{\lambda x_n} f(t)dt \le \int_0^{\lambda x_n} f(t)dt = o\big(\lambda \cdot x_n \cdot f(\lambda x_n)\big)$$

for $n \to +\infty$. From the foregoing, it follows that

$$\int_{x_n}^{\lambda x_n} f(t)dt = o\big(x_n \cdot f(x_n)\big)$$

for $n \to +\infty$, and from $\int_{x_n}^{\lambda x_n} f(t)dt \ge (\lambda - 1)\cdot x_n \cdot f(x_n)$ for $n \in \mathbb{N}$ we obtain that assumption $f \notin R_\infty$ is wrong.

(b) Let us assume that $f(x) = o\big(\widetilde{f}(x)\big)$ for $x \to +\infty$ and $f \notin R_\infty$. Then there are a sequence (x_n) and a number λ with the same properties as in the proof of assertion (a). Also, the following two inequalities hold:

$$\int_{x_n}^{\lambda x_n} \frac{dt}{t^2 f(t)} \le \int_{x_n}^{+\infty} \frac{dt}{t^2 f(t)} = o\left(\frac{1}{x_n \cdot f(x_n)}\right)$$

for $n \to +\infty$, and

$$\int_{x_n}^{\lambda x_n} \frac{dt}{t^2 f(t)} \ge \frac{(\lambda - 1)x_n}{(\lambda x_n)^2 f(\lambda x_n)} \ge \frac{\lambda - 1}{\lambda^2 M}\cdot\frac{1}{x_n f(x_n)}$$

for $n \in \mathbb{N}$. Finally, we obtain that assumption $f \notin R_\infty$ is wrong. $\square$

Proof of Theorem 1.6. We will prove this theorem using Lebesgue's theorem of dominant convergence. According to this theorem, we have that

$$\lim_{x\to+\infty} \frac{\underset{\sim}{f}(x)}{f(x)} = \lim_{x\to+\infty}\int_0^1 \frac{f(ux)}{f(x)}du = 0,$$

is satisfied, because it holds that $\lim_{x\to+\infty} \frac{f(ux)}{f(x)}du = 0$ for $u \in (0, 1)$, and $\sup_{0<u\le 1}\left|\frac{f(ux)}{f(x)}\right| < M$ is satisfied for some $M \in (0, +\infty)$ and sufficiently large x (the last inequality holds because $\beta(f) = +\infty$, where $\beta(f)$ is the lower Matuszewska index (see e.g. [1]), and therefore $\alpha(f) = +\infty$, where $\alpha(f)$ is upper Matuszewska index).

Similarly, we have that

$$\lim_{x\to+\infty} \frac{f(x)}{\widetilde{f}(x)} = \lim_{x\to+\infty}\int_0^1 \frac{f(x)}{f\left(\frac{x}{u}\right)}du = 0$$

is satisfied, because it holds that $\lim_{x\to+\infty} \frac{f(x)}{f(\frac{x}{u})}du = 0$ for $u \in (0, 1)$, and $\inf_{\lambda>1}\left|\frac{f(\lambda x)}{f(x)}\right| \ge \frac{1}{M}$ is satisfied for some $M \in (0, +\infty)$ and sufficiently large x (this inequality holds because $\beta(f) = +\infty$). Hence, $\sup_{0<u<1}\left|\frac{f(x)}{f(\frac{x}{u})}\right| \le M$. $\square$

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