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Selection principles and double sequences II

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Dedicated to Prof. Ljubiša Kočinac on the occasion of his 65th birthday

Abstract

This paper is a continuation of the research on selection properties of certain classes of double sequences of positive real numbers that was began in [6].

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1. Introduction

In recent years a number of papers concerning relations between selection principles theory and the theory of convergence/divergence of sequences of positive real numbers appeared in the literature [1, 2, 3, 4, 7]. Special attention have been paid to connections between α_i selection principles [12] and classes of sequences important in Karamata's theory of regular variation (see the papers [1, 3] and references therein, and also the papers [13, 17] for important applications). On the other hand, in [6] the authors introduced modified α_i selection properties for double real sequences and gave their relations with Pringsheim's convergence of double sequences (see [14] and also [9, 10, 15]).

In this note we continue investigation began in [6] and extend results from this paper considering the class of translationally rapidly varying double sequences following some ideas from [3, 16].

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Recall definitions of two selection principles that we consider in this note. If $\mathcal{A}$ and $\mathcal{B}$ are families of subsets of an infinite set X, then:

(1) $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$.

(2) $\alpha_2(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis that for each sequence $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that for each n, $B \cap A_n$ is infinite (see [12]).

For more details on selection principles see [11].

2. Results

Given $a \in \mathbb{R}$, by c_2^a we denote the set of double sequences of real numbers which converge to a in the sense of Pringsheim [14]. Let

$$c_{2,+}^a := \{\mathbf{x} = (x_{m,n})_{m,n\in\mathbb{N}} \in c_2^a : x_{m,n} > 0 \text{ for all } m,n \in \mathbb{N}\}.$$

We say that a positive double sequence $\mathbf{x} = (x_{m,n})_{m,n\in\mathbb{N}}$ belongs to the class $\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2})$ of *translationally rapidly varying double sequences* if

$$\lim_{\min\{m,n\}\to\infty} \frac{x_{[m+\alpha],[n+\beta]}}{x_{m,n}} = 0$$

for each $\alpha \geq 0$ and each $\beta \geq 0$ such that $\max\{\alpha,\beta\} \geq 1$. Here $[x]$ denotes the integer part of $x \in \mathbb{R}$.

Notice that the class $\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2})$ is nonempty, because it contains the double sequence $(x_{m,n})$ defined by

$$x_{m,n} = \frac{1}{(m+n)!},\ m \in \mathbb{N}, n \in \mathbb{N}.$$

2.1. Theorem. $\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2}) \subsetneqq c_{2,+}^0$.

Proof. Let $\mathbf{x} = (x_{m,n})_{m,n\in\mathbb{N}} \in \mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2})$. Let $\varepsilon = \frac{1}{2}$ and $\alpha = \beta = 1$. There is $N_0 = N_0(1/2,1,1) \in \mathbb{N}$ such that

$$\frac{x_{m+1,n+1}}{x_{m,n}} \leq \frac{1}{2}$$

for all $m,n \geq N_0$. Therefore, for $m = n \geq N_0$ we have $x_{n+1,n+1} \leq \frac{1}{2}x_{n,n}$, and it follows that $\lim_{n\to\infty} x_{n,n} = 0$. Similarly, for $\varepsilon = \frac{1}{2}$ and $\alpha = 1$, $\beta = 0$, there is $N_1 = N_1(1/2,1,0) \in \mathbb{N}$ such that $x_{m+1,n} \leq \frac{1}{2}x_{m,n}$ for all $m,n \geq N_1$. It implies that for $n \geq N_1$ we have $\lim_{m\to\infty} x_{m,n} = 0$. Finally for $\varepsilon = \frac{1}{2}$, $\alpha = 0$, $\beta = 1$ there is $N_2 = N_2(1/2,0,1) \in \mathbb{N}$ such that $x_{m,n+1} \leq \frac{1}{2}x_{m,n}$ for all $m,n \geq N_2$. From here we obtain $\lim_{n\to\infty} x_{m,n} = 0$, for each $m \geq N_2$.

Let $\varepsilon > 0$ be arbitrary (and fixed). Then there is $n_\varepsilon \in \mathbb{N}$ such that $x_{n,n} \leq \varepsilon$ for each $n \geq n_\varepsilon$. Set $n_* = \max\{n_\varepsilon, N_1, N_2\}$. Then $x_{m,n} \leq \varepsilon$ for each $m,n \geq n_*$, which means that $\mathbf{x} \in c_{2,+}^0$.

The double sequence $\mathbf{x} = (x_{m,n})_{m,n\in\mathbb{N}}$ defined by

$$x_{m,n} = \begin{cases} 1/m & \text{for } m \in \mathbb{N},\ n \in \{1,2,\cdots,m\}, \\ 1/n & \text{for } n \in \mathbb{N},\ m \in \{1,2,\cdots,n\}. \end{cases}$$

evidently belongs to the class $c_{2,+}^0$, but it does not belong to $\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2})$ because for $\alpha = \beta = 1$ and $m = n$ we have

$$\lim_{n\to\infty} \frac{x_{m+1,n+1}}{x_{m,n}} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

$\square$

In what follows we need two definitions from [6].

Let $\mathcal{A}$ and $\mathcal{B}$ be as above. Then:

(a) $\mathsf{S}_1^{(\mathsf{d})}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: for each double sequence $(A_{m,n} : m,n\in\mathbb{N})$ of elements of $\mathcal{A}$ there are elements $a_{m,n}\in A_{m,n}$, $m,n\in\mathbb{N}$, such that the double sequence $(a_{m,n})_{m,n\in\mathbb{N}}$ belongs to $\mathcal{B}$.

(b) $\alpha_2^{(d)}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: for each double sequence $(A_{m,n} : m,n\in\mathbb{N})$ of elements of $\mathcal{A}$ there is an element B in $\mathcal{B}$ such that $B\cap A_{m,n}$ is infinite for all $(m,n)\in\mathbb{N}\times\mathbb{N}$.

2.2. Theorem. *The selection principle* $\mathsf{S}_1^{(d)}(c^0_{2,+},\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2}))$ *is satisfied.*

Proof. Let $(x_{m,n,j,k})$ be a double sequence of double sequences such that for a fixed $(j_0,k_0)\in\mathbb{N}\times\mathbb{N}$, $(x_{m,n,j_0,k_0})\in c^0_{2,+}$. We create a new double sequence $\mathbf{y}=(y_{j,k})_{j,k\in\mathbb{N}}$ in the following way.

1^0. $y_{1,1}=x_{m,n,1,1}$ for an arbitrary fixed $(m,n)\in\mathbb{N}\times\mathbb{N}$;

2^0. $y_{1,2}=x_{m,n,1,2}$ such that $y_{1,2}<\frac{1}{2}y_{1,1}$, $y_{2,1}=x_{m,n,2,1}$ such that $y_{2,1}<\frac{1}{2}y_{1,1}$, and $y_{2,2}=x_{m,n,2,2}$ such that $y_{2,2}<\frac{1}{2}\min\{y_{1,2},y_{2,1}\}$.

p^0, $p\geq 3$. Choose $y_{p,1}=x_{m,n,p,1}$ so that $y_{p,1}<\left(\frac{1}{2}\right)^p y_{p-1,1}$. For $\ell\in\{2,3,\cdots,p-1\}$ pick $y_{p,\ell}=x_{m,n,p,\ell}$ such that $y_{p,\ell}<\left(\frac{1}{2}\right)^p y_{p,\ell-1}$ and $y_{p,\ell}<\left(\frac{1}{2}\right)^p y_{p-1,\ell}$. Similarly, let $y_{1,p}=x_{m,n,1,p}$ be such that $y_{1,p}<\left(\frac{1}{2}\right)^p y_{1,p-1}$. Choose also $y_{\ell,p}=x_{m,n,\ell,p}$ such that $y_{\ell,p}<\left(\frac{1}{2}\right)^p y_{\ell-1,p}$ and $y_{\ell,p}<\left(\frac{1}{2}\right)^p y_{\ell,p-1}$. Finally, take $y_{p,p}$ to be some $x_{m,n,p,p}$ such that $y_{p,p}<\left(\frac{1}{2}\right)^p\min\{y_{p,p-1},y_{p-1,p}\}$.

We prove that $\mathbf{y}\in\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2})$. Let $\varepsilon>0$ and $\alpha,\beta\geq 0$ with $\max\{\alpha,\beta\}\geq 1$ be given. Set $h=h(\alpha,\beta)=[\alpha]+[\beta]$. There is $s_0\in\mathbb{N}$ such that $(\frac{1}{2})^s\leq\varepsilon$ for each $s\geq s_0$. For $j\geq s_0$, $k\geq s_0$ we have

$$\frac{y_{j+1,k}}{y_{j,k}}\leq\left(\frac{1}{2}\right)^{s_0+1}\quad\text{and}\quad\frac{y_{j,k+1}}{y_{j,k}}\leq\left(\frac{1}{2}\right)^{s_0+1},$$

and thus we have

$$\frac{y_{[j+\alpha],[k+\beta]}}{y_{j,k}}=\frac{y_{j+[\alpha],k+[\beta]}}{y_{j,k}}\leq\left(\frac{1}{2}\right)^{(s_0+1)h}\leq\left(\frac{1}{2}\right)^{s_0}\leq\varepsilon,$$

which means that $\mathbf{y}\in\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2})$. $\square$

2.3. Theorem. *The selection principle* $\alpha_2^{(d)}(c^0_{2,+},\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2}))$ *is satisfied.*

Proof. Let $(x_{m,n,j,k})$ be a double sequence of double sequences such that for a fixed $(j_0,k_0)\in\mathbb{N}\times\mathbb{N}$, $(x_{m,n,j_0,k_0})\in c^0_{2,+}$. Form a double sequence $\mathbf{y}=(y_{p,t})_{p,t\in\mathbb{N}}$ as follows.

Step 1. Using some standard method arrange the given double sequence of double sequences in a sequence $(x_{n,m,r})$ of double sequences, where for each $r_0\in\mathbb{N}$ the double sequence (x_{m,n,r_0}) belongs to $c^0_{2,+}$.

Step 2. Consider the sequence of sequences $(x_{n,n,r})$, $r\in\mathbb{N}$. Observe that for each $r_0\in\mathbb{N}$ it holds $(x_{n,n,r_0})\in\mathbb{S}_0$, where $\mathbb{S}_0$ denotes the set of all sequences of positive real numbers converging to 0 (see, for instance, [7]). Let $J=\{(a_n)_{n\in\mathbb{N}}\in\mathbb{S}: a_1>0, a_{n+1}\leq\frac{a_n}{n+1}\}$, where $\mathbb{S}$ is the set of all sequences of positive real numbers. It holds $J\subsetneqq\mathbb{S}_0$ and the selection principle $\mathsf{S}_1(\mathbb{S}_0,J)$ is satisfied.

Step 3. (In this part of the proof we use some techniques from [2]) Take an increasing sequence $(p_i)_{i\in\mathbb{N}}$ of prime numbers, $(p_1=2)$, and a fixed $r\in\mathbb{N}$. Consider subsequences $(x_{p_i^n,p_i^n,r})$, $i\in\mathbb{N}$, of the sequence $(x_{n,n,r})$. These subsequences are in the class $\mathbb{S}_0$. Varying i and r in $\mathbb{N}$, arrange those subsequences in a sequence of sequences of $\mathbb{S}_0$.

Applying $\mathsf{S}_1(\mathbb{S}_0, J)$ one finds a sequence $(z_j) \in J$ such that (z_j) has infinitely many elements with the sequence $(x_{n,n,r})$ for each $r \in \mathbb{N}$. In other words, we conclude that the selection principle $\alpha_2(\mathbb{S}_0, J)$ is true.

Let now $y_{j,j} = z_j$, $j \in \mathbb{N}$. For $j \geq 2$ we choose $y_{s,j} = \sqrt{s+1} \cdot y_{s+1,j}$ for $s \in \{1, 2, \cdots, j-1\}$, and $y_{j,s} = \sqrt{s+1} \cdot y_{j,s+1}$. It is easy to see that the double sequence $\mathbf{y} = (y_{p,t})$ obtained in this way has infinitely many common elements with each double sequence $(x_{m,n,j.k})$ for arbitrary and fixed $(j,k) \in \mathbb{N} \times \mathbb{N}$.

It remains to prove $\mathbf{y} \in \mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2})$. Let $\varepsilon > 0$ and $\alpha \geq 0$, $\beta \geq 0$ with $\max\{\alpha, \beta\} \geq 1$, be given. Set $h = [\alpha] + [\beta]$. There is $N_0 \in \mathbb{N}$ such that $\left(\frac{1}{\sqrt{N+1}}\right)^h \leq \varepsilon$ for each $N \in \mathbb{N}$ with $N \geq N_0$ ($N_0 \geq \varepsilon^{-(2/h)} - 1$). For $p, t \geq N_0$ we have

$$\frac{y_{p+1,t}}{y_{p,t}} \leq \frac{1}{\sqrt{N_0+1}} \quad \text{and} \quad \frac{y_{p,t+1}}{y_{p,t}} \leq \frac{1}{\sqrt{N_0+1}}.$$

So we have

$$\frac{y_{[p+\alpha],[t+\beta]}}{y_{p,t}} = \frac{y_{p+[\alpha],t+[\beta]}}{y_{p,t}} \left(\frac{1}{\sqrt{N_0+1}}\right)^h \leq \varepsilon,$$

i.e. $\mathbf{y} \in \mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2})$.

□

2.4. Remark. (1) The selection principles $\alpha_i^{(d)}(c^0_{2,+}, \mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2}))$, $i = 3, 4$, are also satisfied; see the papers [8, 6] in connection with these selection principles.

(2) From the proof of Theorem 2.3 it follows that selection principles $\alpha_i(c^0_{2,+}, \mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2}))$, $i \in \{2, 3, 4\}$, are true; see [12] for these selection properties.

(3) From the proof of Theorem 2.2 one concludes that this theorem remains true if the first coordinate $c^0_{2,+}$ in it is replaced by the class of double sequences of positive real numbers which possesses at least one Pringsheim's limit point equal to 0 (see, for instance, [6]).

(4) Similarly, Theorem 2.3 remains true if the first coordinate $c^0_{2,+}$ is replaced by the class of double sequences $(x_{m,n})$ having property that the sequence $(x_{n,n})$ contains a subsequence converging to 0.

For a double sequence $\mathbf{x} = (x_{m,n})$ we define

$$\omega_n(\mathbf{x}) := \sup\{|x_{j,k} - x_{r,s}| : j \geq n, k \geq n, r \geq n, s \geq n\}, \; n \in \mathbb{N}.$$

The sequence $(\omega_n(\mathbf{x}))$ is called the *Landau-Hurwicz sequence* of $\mathbf{x}$ (compare with [4]).

2.5. Proposition. *A double sequence $\mathbf{x} = (x_{m,n})$ belongs to the class c^a_2, $a \in \mathbb{R}$, if and only if* $\lim_{n\to\infty} \omega_n(\mathbf{x}) = 0$.

Proof. ($\Rightarrow$) Assume that $\mathbf{x} = (x_{m,n})$ is a double sequence from c^a_2 for some arbitrary and fixed $a \in \mathbb{R}$. Let $\varepsilon > 0$ be given. There is $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $|x_{j,k} - a| \leq \varepsilon/2$ for each $j \geq n_0$ and each $k \geq n_0$. Therefore we have

$$|x_{j,k} - x_{r,s}| = |x_{j,k} - a + a - x_{r,s}| \leq |x_{j,k} - a| + |x_{r,s} - a| \leq \varepsilon/2 + \varepsilon/2$$

for all $j, k, r, s \geq n_0$. This implies that for each $n \geq n_0$ we have

$$0 \leq \omega_n(\mathbf{x}) \leq \sup\{|x_{j,k} - x_{r,s}| : j \geq n_0, k \geq n_0, r \geq n_0, s \geq n_0\} \leq \varepsilon,$$

i.e. $\lim_{n\to\infty} \omega_n(\mathbf{x}) = 0$.

($\Leftarrow$) Let $\mathbf{x} = (x_{m,n})$ be a double sequence with $\lim_{n\to\infty} \omega_n(\mathbf{x}) = 0$. For a given $\varepsilon > 0$, there is $n_1 = n_1(\varepsilon) \in \mathbb{N}$ such that $0 \leq |x_{j,k} - x_{r,s}| \leq \varepsilon/2$ for $j \geq n_1$, $k \geq n_1$, $r \geq n_1$, $s \geq n_1$, because

$$0 \leq \omega_n(\mathbf{x}) = \sup\{|x_{j,k} - x_{r,s}| : j \geq n_1, k \geq n_1, r \geq n_1, s \geq n_1\} \leq \varepsilon/2$$

for $n \geq n_1$. Since for all $j, r \geq n_1$ it holds $|x_{j,j} - x_{r,r}| \leq \varepsilon/2$, it follows that the sequence $(x_{t,t})$ is convergent (as a Cauchy sequence), i.e. there is $A \in \mathbb{R}$ such that $\lim_{t\to\infty} x_{t,t} = A$. This implies there is $n_2 = n_2(\varepsilon) \in \mathbb{N}$ such that $|x_{t,t} - A| \leq \varepsilon/2$ for each $t \geq n_2$. Therefore, for $n_0 = \max\{n_1, n_2\}$ and all $j, k \geq n_0$ we have

$$|x_{j,k} - A| \leq |x_{j,k} - x_{j,j}| + |x_{j,j} - A| \leq \varepsilon.$$

□

For $a \in \mathbb{R}$ we define

$$c^a_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}}),2} := \{\mathbf{x} \in c^a_2 : (\omega_n(\mathbf{x})) \in \mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})\}.$$

(For the definition of $\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})$ see [3].)

2.6. Example. Given $a \in \mathbb{R}$, consider the double sequence $\mathbf{x} = (x_{j,k})$ defined by

$$x_{j,k} = \begin{cases} a & \text{for } j \neq k, \\ a + 1/j & \text{for } j = k. \end{cases}$$

It is clear that $\mathbf{x} \in c^a_2$. However, $\mathbf{x} \notin c^a_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}}),2}$ because $\omega_n(\mathbf{x}) = 1/n$ for each $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{\omega_{n+1}(\mathbf{x})}{\omega_n(\mathbf{x})} = 1.$$

2.7. Theorem. *The following selection principles are satisfied:*

(1) $\mathsf{S}^{(d)}_1(c^0_{2,+}, c^0_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}}),2,+})$;

(2) $\alpha^{(d)}_2(c^0_{2,+}, c^0_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}}),2,+})$.

Proof. (1) Consider the double sequence $\mathbf{y} = (y_{j,k})$ which was the selector in the proof of Theorem 2.2. We have

$$\omega_n(\mathbf{y}) = \sup\{|y_{j,k} - y_{r,s}| : j \geq n, k \geq n, r \geq n, s \geq n\} = y_{n,n},\ \ n \in \mathbb{N},$$

which implies

$$\lim_{n\to\infty} \frac{\omega_{n+1}(\mathbf{y})}{\omega_n(\mathbf{y})} = \lim_{n\to\infty} \frac{y_{n+1,n+1}}{y_{n,n}} \leq \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n+1} = 0,$$

i.e. (1) is true.

(2) Consider the double sequence $\mathbf{y} = (y_{j,k})$ which was the selector in the proof of Theorem 2.3. For this double sequence we have $\omega_n(\mathbf{y}) = y_{n,n},\ \ n \in \mathbb{N}$. Since

$$\lim_{n\to\infty} \frac{\omega_{n+1}(\mathbf{y})}{\omega_n(\mathbf{y})} = \lim_{n\to\infty} \frac{y_{n+1,n+1}}{y_{n,n}} \leq \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

one concludes that (2) is satisfied. □

We recall a definition from [6]. Let $\mathcal{A}$ and $\mathcal{B}$ be as in Introduction. Then $\mathsf{S}^{\varphi}_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence (A_t) of elements from $\mathcal{A}$ there is an element $B = (b_{j,k}) \in \mathcal{B}$ such that $b_{j,k} \in A_t$ for $t = \varphi(j,k)$, where $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a given bijection.

2.8. Theorem. *The selection principle* $\mathsf{S}^{\varphi}_1(c^0_{2,+}, \mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}_2}))$ *is satisfied.*

Proof. Suppose that (A_t) is a sequence of double sequences $A_t = (x_{m,n,t})$ in $c^0_{2,+}$. Let us consider the double sequence of double sequences $(x_{m,n,j,k})$ (constructed from the sequence (A_t)), where $(j,k) = (j(t), k(t)) = \varphi^{-1}(t)$, $t \in \mathbb{N}$. To this double sequence of double sequences apply the procedure from the proof of Theorem 2.2 to obtain the double sequence $\mathbf{y} = (y_{j,k})$ which will witness that the theorem is true. □

From the proof of Theorem 2.8 and Theorem 2.7(1) we have the following corollary.

2.9. Corollary. *The selection principle* $\mathsf{S}_1^{\varphi}(c^0_{2,+}, c^0_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}}),2,+})$ *is satisfied.*

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