Exponents of Convergence and Games

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Abstract

The notion of exponents of convergence has many applications in different mathematical disciplines. In this paper we consider exponents of convergence of sequences of positive real numbers in connection with games and selection principles.

1 Introduction

Let $\mathbb{S}$ denote the set of all sequences of positive real numbers and let

$$
\begin{aligned}
\mathbb{S}_0 &= \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{S} : \lim_{n\to\infty} x_n = 0 \right\}, \\
\mathbb{S}_{\underline{0}} &= \left\{ (x_n)_{n\in\mathbb{N}} \in \mathbb{S} : \liminf_{n\to\infty} x_n = 0 \right\}.
\end{aligned}
$$

A real number λ is said to be the *numerical exponent of convergence* of a sequence $a = (a_n)_{n \in \mathbb{N}} \in \mathbb{S}_0$ if for every $\varepsilon > 0$, the series $\sum_{n=1}^{\infty} a_n^{\lambda+\varepsilon}$ converges while the series $\sum_{n=1}^{\infty} a_n^{\lambda-\varepsilon}$ diverges. If for every $\varepsilon > 0$, the series $\sum_{n=1}^{\infty} a_n^{\varepsilon}$ diverges, then we say that the sequence a has the numerical exponent of convergence ∞. It is evident that every sequence in $\mathbb{S}_0$ has exactly one numerical exponent of convergence $\lambda \in [0, \infty]$. This notion was (implicitly) introduced in [17] (see also [4, 16]). For a sequence $a = (a_n)_{n \in \mathbb{N}} \in \mathbb{S}_0$ we use the following notation:

- $e(a) = \rho$ denotes that ρ is the numerical exponent of convergence of a;

- $\mathbb{S}_{0,\rho} = \{a \in \mathbb{S}_0 : e(a) = \rho\}$.

M. Petrović in [15] defined the following: A sequence $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{S}$ is said to be a *sequence of exponents of convergence* of a sequence $a = (a_n)_{n \in \mathbb{N}} \in \mathbb{S}_0$ if for every $\varepsilon > 0$, the series $\sum_{n=1}^{\infty} a_n^{\lambda_n(1+\varepsilon)}$ converges and the series $\sum_{n=1}^{\infty} a_n^{\lambda_n(1-\varepsilon)}$ diverges. In this article, the sequence of exponents of convergence will be called the *exponent of convergence*. Observe that for each $a = (a_n)_{n \in \mathbb{N}} \in \mathbb{S}_0$, there is an exponent of convergence; it is the sequence $\lambda_s = (\lambda_n)_{n \in \mathbb{N}}$ defined by

$$\lambda_n = -\frac{\ln n}{\ln a_n},$$

called the *standard exponent of convergence* of a. In [1] it was shown that for a given $a \in \mathbb{S}_0$, there are infinitely many exponents of convergence of a. Some nice results and characterizations in connection with exponents of convergence and numerical exponents of convergence have been obtained in [1, 12, 15, 18]. Important generalizations of these concepts have been studied in [7, 8]. Also, we mention the paper [19] in which a natural and essential generalization of exponents of convergence was given by using a sequence of parameters of convergence. Exponents of convergence have many applications in several branches of mathematics, e.g., in complex analysis [14], differential equations (see [2, 3, 13] and references therein), and so on.

In this paper we investigate relationships among exponents of convergence of sequences, game theory, and the theory of selection principles, a growing field of mathematics which has nice relations with different areas of mathematics.

2   Terminology and Notation

For a sequence $a = (a_n)_{n \in \mathbb{N}} \in \mathbb{S}_0$, we denote:

- $\mathcal{E}(a) = \{\lambda = (\lambda_n)_{n \in \mathbb{N}} \in \mathbb{S} : \lambda \text{ is the exponent of convergence of } a\}$;

- If $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in \mathcal{E}(a)$, then we write $\lambda = \lambda(a)$ (and $\lambda_n = \lambda_n(a)$, $n \in \mathbb{N}$);

- $[\lambda(a)] = \{\delta = (\delta_n)_{n\in\mathbb{N}} \in \mathbb{S} : \delta_n \sim \lambda_n, n \to \infty\}$, where $\sim$ is the sequence asymptotic equivalence on $\mathbb{S}$;

- $[\lambda(a)]_{\mathcal{E}(a)} = [\lambda(a)] \cap \mathcal{E}(a)$;

- If $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in \mathbb{S}$, then $\mathcal{E}^{\leftarrow}(\lambda) = \{a = (a_n)_{n\in\mathbb{N}} \in \mathbb{S}_0 : \lambda \in \mathcal{E}(a)\}$;

- If $a \in \mathcal{E}^{\leftarrow}(\lambda)$, then we write $a = a^{(\lambda)} = (a_n^{(\lambda)})_{n\in\mathbb{N}}$;

- $[a^{(\lambda)}] = \{b = (b_n)_{n\in\mathbb{N}} \in \mathbb{S}_0 : b_n \sim a_n^{(\lambda)}, n \to \infty\}$;

- $[a^{(\lambda)}]_{\mathcal{E}^{\leftarrow}(\lambda)} = [a^{(\lambda)}] \cap \mathcal{E}^{\leftarrow}(\lambda)$.

Let us recall some facts about selection principles and games which will be considered in this paper. For more information about selection principles and games, see [9–11]. Because we are interested here in classes of sequences, we formulate selection principles for subclasses $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{S}$. We identify a sequence x and its image $\mathrm{Im}(x)$.

The selection hypothesis $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ states that for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. There is a two-person game corresponding to $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$. The symbol $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, who play a round for each positive integer. In the n-th round, ONE chooses some $A_n \in \mathcal{A}$ and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $A_1, b_1; \ldots; A_n, b_n; \ldots$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins. Clearly, if ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$, then the selection hypothesis $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ is true. The converse implication is not always true.

$\alpha_i(\mathcal{A}, \mathcal{B})$, $i = 2, 3, 4$, denotes the selection hypothesis that for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is an element $B \in \mathcal{B}$ such that (see [10, 11]):

- $\alpha_2(\mathcal{A}, \mathcal{B})$: for each $n \in \mathbb{N}$, the set $A_n \cap B$ is infinite;

- $\alpha_3(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$, the set $A_n \cap B$ is infinite;

- $\alpha_4(\mathcal{A}, \mathcal{B})$: for infinitely many $n \in \mathbb{N}$, the set $A_n \cap B$ is nonempty.

Evidently, $\alpha_2(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_3(\mathcal{A}, \mathcal{B}) \Rightarrow \alpha_4(\mathcal{A}, \mathcal{B})$. The following game $\mathsf{G}_{\alpha_2}(\mathcal{A}, \mathcal{B})$ corresponds to $\alpha_2(\mathcal{A}, \mathcal{B})$: In the first round, ONE plays a sequence $x_1 = (x_{1,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$ and TWO responds by choosing a subsequence $y_{k_1} = (x_{1,k_1(j)})$ of x_1. In the n-th round, $n \geq 2$, ONE chooses a sequence $x_n = (x_{n,j})_{j\in\mathbb{N}}$ from $\mathcal{A}$ and TWO responds by choosing a subsequence $y_{k_n} = (x_{n,k_n(j)})$ of x_n so that $\mathrm{Im}(k_n(j)) \cap \mathrm{Im}(k_p(j)) = \emptyset$ for each $p \leq n - 1$. The player TWO wins a play $x_1, y_{k_1}; \ldots; x_n, y_{k_n}; \ldots$ if and only if elements of $Y = \bigcup_{n\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} x_{n,k_n(j)}$, with respect to the second index, form a subsequence $y = (y_m)_{m\in\mathbb{N}}$ of some element from $\mathcal{B}$. It is understood that if TWO has a winning strategy in this game, then the selection principle $\alpha_2(\mathcal{A}, \mathcal{B})$ holds.

3 Results

Theorem 3.1. *The player* TWO *has a winning strategy in the game* $\mathsf{G}_1(\mathbb{S}_{\underline{0}}, \mathbb{S}_{0,0})$.

Proof. Suppose that in the first round, ONE plays a sequence $x_1 = (x_{1,m})_{m \in \mathbb{N}} \in \mathbb{S}_{\underline{0}}$. TWO responds by choosing an arbitrary element $y_1 = x_{1,m_1}$. If in the n-th round, $n \geq 2$, ONE chooses a sequence $x_n = (x_{n,m})_{m \in \mathbb{N}} \in \mathbb{S}_{\underline{0}}$, then TWO picks an element $y_n = x_{n,m_n} \in x_n$ such that $2^n y_n \leq y_{n-1}$. We prove that the sequence $y = (y_n)_{n \in \mathbb{N}}$ of moves of TWO belongs to $\mathbb{S}_{0,0}$. Obviously, from the definition, the sequence y is a strictly decreasing sequence of positive real numbers, and thus it converges to a number $\ell \geq 0$. Since

$$\limsup_{n \to \infty} \frac{y_{n+1}}{y_n} \leq \lim_{n \to \infty} \frac{1}{2^{n+1}} = 0,$$

it follows $\ell = 0$, i.e., $y \in \mathbb{S}_0$. Therefore, for each $\varepsilon > 0$ the sequence $(y_n^{-\varepsilon})_{n \in \mathbb{N}}$ tends to ∞ as $n \to \infty$, so that the series $\sum_{n=1}^{\infty} y_n^{-\varepsilon}$ diverges. On the other hand, for the same ε we have

$$\limsup_{n \to \infty} \frac{y_{n+1}^\varepsilon}{y_n^\varepsilon} \leq \lim_{n \to \infty} 2^{-\varepsilon(n+1)} = 0,$$

and by the d'Alembert criterion, the series $\sum_{n=1}^{\infty} y_n^\varepsilon$ converges. So, the numerical exponent of convergence of y is 0, i.e., $y \in \mathbb{S}_{0,0}$. $\square$

Corollary 3.2. *The selection principle* $\mathsf{S}_1(\mathbb{S}_{\underline{0}}, \mathbb{S}_{0,0})$ *is true.*

Remark 3.3. By using standard techniques from [5], one can prove that selection principles $\alpha_i(\mathbb{S}_{\underline{0}}, \mathbb{S}_{0,0})$, $i = 2, 3, 4$, are also satisfied.

For the proof of the following theorem, we employ the (modified) techniques used in [5, 6].

Theorem 3.4. TWO *has a winning strategy in the game* $\mathsf{G}_{\alpha_2}([\lambda(a)], [\lambda(a)]_{\mathcal{E}(a)})$ *for any* $a \in \mathbb{S}_0$.

Proof. Fix $a = (a_n)_{n \in \mathbb{N}} \in \mathbb{S}_0$ and let $\lambda(a) = (\lambda_n(a))_{n \in \mathbb{N}} \in \mathcal{E}(a)$. Let σ be a strategy for TWO. Suppose that the first move of ONE is a sequence $x_1 = (x_{1,j})_{j \in \mathbb{N}} \in [\lambda(a)]$. Then TWO chooses a subsequence $\sigma(x_1) = y_{k_1} = (x_{1,k_1(j)})_{j \in \mathbb{N}}$ of x_1 so that $\text{Im}(k_1)$ is the set of natural numbers which are divisible by 2, and not divisible by 2^2. If in the i-th round, $i \geq 2$, ONE has played a sequence $x_i = (x_{i,j})_{j \in \mathbb{N}} \in [\lambda(a)]$, then TWO argues in the following way. There is $j_i \in \mathbb{N}$ such that

$$1 - \frac{1}{2^i} \leq \frac{x_{1,j}}{x_{i,j}} \leq 1 + \frac{1}{2^i} \text{ for each } j \geq j_i.$$

TWO plays a subsequence $\sigma(x_i) = (x_{i,k_i(j)})_{j \in \mathbb{N}}$ of x_i so that $\text{Im}(k_i)$ is the set of natural numbers $\geq j_i$ which are divisible by 2^i and not divisible by 2^{i+1}. The set $Y = \bigcup_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} x_{i,k_i(j)}$ is a set of positive real numbers indexed by two indices, and the elements of Y can be enumerate in such a way to form a subsequence of the sequence $y = (y_m)_{m \in \mathbb{N}}$, where

$$y_m = \begin{cases} x_{i,k_i(j)}, & \text{if } m = k_i(j) \text{ for some } i, j \in \mathbb{N}; \\ x_{1,m}, & \text{otherwise.} \end{cases}$$

By construction of y, one concludes that $y \in \mathbb{S}$ and that its intersection with each x_i, $i \in \mathbb{N}$, is infinite.

Claim 1. $y \in [\lambda(a)]$. It suffices to prove that $y_m \sim x_{1,m}$, $m \to \infty$. Let $\varepsilon \in (0,1)$. Pick the smallest $i \in \mathbb{N}$ such that $\frac{1}{2^i} < \varepsilon$. For each $k \in \{1, 2, \ldots, i-1\}$, there is $j_k^* \in \mathbb{N}$ such that $1 - \varepsilon \leq \frac{x_{1,j}}{x_{k,j}} \leq 1 + \varepsilon$ for all $j \geq j_k^*$. Set $j^* = \max\{j_1^*, \ldots, j_{i-1}^*\}$. Then $1 - \varepsilon \leq \frac{x_{1,m}}{y_m} \leq 1 + \varepsilon$ for each $m \geq j^*$. Because ε was an arbitrary element of $(0,1)$ we conclude $y \in [\lambda(a)]$.

Claim 2. $y \in \mathcal{E}(a)$. (Here we use techniques which are modified techniques from [1].) Let $\varepsilon > 0$ be given, and let $\delta_1 = \frac{\varepsilon}{2(1+\varepsilon)}$. For this $\delta_1 > 0$, there is $n_0 = n_0(\delta_1) \in \mathbb{N}$ such that

$$(1 - \delta_1)x_{1,n} \leq y_n \leq (1 + \delta_1)x_{1,n} \quad \text{for each } n \geq n_0.$$

For each $n \geq n_0$, we also have $(1 - \delta_1)(1 + \varepsilon) > 1$ and $(1 + \varepsilon)y_n \geq (1 + \varepsilon)(1 - \delta_1)x_{1,n}$. Since $a \in \mathbb{S}_0$, for $n \geq n_0$ it holds

$$a_n^{y_n(1+\varepsilon)} \leq a_n^{x_{1,n}(1+\varepsilon)(1-\delta_1)} = a_n^{x_{1,n}(1+\frac{\varepsilon}{2})}.$$

Choose now $\delta_2 = \frac{\varepsilon}{2(1-\varepsilon)}$ for $\varepsilon \in (0,1)$. For this $\delta_2 > 0$, there is $n_1 = n_1(\delta_2) \in \mathbb{N}$ such that for each $n \geq n_1$ it holds

$$(1 - \delta_2)x_{1,n} \leq y_n \leq (1 + \delta_2)x_{1,n}.$$

For $n \geq n_1$, we also have

$$(1 + \delta_2)(1 - \varepsilon) < 1 \quad \text{and} \quad (1 - \varepsilon)y_n \leq (1 - \varepsilon)(1 + \delta_2)x_{1,n}.$$

As $a \in \mathbb{S}_0$, we have that for $n \geq n_1$,

$$a_n^{y_n(1-\varepsilon)} \geq a_n^{x_{1,n}(1-\varepsilon)(1+\delta_2)} = a_n^{x_{1,n}(1-\frac{\varepsilon}{2})}.$$

Since for $\varepsilon \geq 1$, the series $\sum_{n=1}^{\infty} a_n^{y_n(1-\varepsilon)}$ diverges, we get $y \in \mathcal{E}(a)$. So, $y \in [\lambda(a)]_{\mathcal{E}(a)}$ which completes the proof. $\square$

Corollary 3.5. *For each $a \in \mathbb{S}_0$, the selection principles $\alpha_i([\lambda(a)], [\lambda(a)]_{\mathcal{E}(a)})$, $i = 2, 3, 4$, are satisfied.*

Theorem 3.6. *TWO has a winning strategy in the game $\mathsf{G}_{\alpha_2}\left([a^{(\lambda)}], [a^{(\lambda)}]_{\mathcal{E}^{\leftarrow}(a)}\right)$ for each $\lambda \in \mathbb{S}$.*

Proof. Let $\lambda = (\lambda_n)_{n \in \mathbb{N}}$ be a fixed element in $\mathbb{S}$. A strategy φ of the player TWO in the game $\mathsf{G}_{\alpha_2}\left([a^{(\lambda)}, [a^{(\lambda)}]_{\mathcal{E}^{\leftarrow}(a)}\right)$ is similar to the strategy σ from the proof of Theorem 3.4: If in the i-th round, $i \in \mathbb{N}$, ONE has played a sequence $x_i = (x_{i,j})_{j \in \mathbb{N}} \in [a^{(\lambda)}]$, then TWO's response is a subsequence $(x_{i,k_i(j)})_{j \in \mathbb{N}}$ of x_i such to that $\mathrm{Im}(k_i)$ is the set of natural numbers $\geq j_i$ which are divisible by 2^i and not divisible by 2^{i+1}. Using φ, TWO actually creates a sequence $y = (y_m)_{m \in \mathbb{N}}$ which belongs to $[a^{(\lambda)}]$ and intersects each x_i in infinitely many elements.

It remains to prove that y belongs to $\mathcal{E}^{\leftarrow}(\lambda)$, i.e., that $\lambda = (\lambda_m)_{m \in \mathbb{N}}$ is the exponent of convergence of y. Since $y \in [a^{(\lambda)}]$, it holds $y_m = a_m p_m$ for each $m \in \mathbb{N}$, where $a^{(\lambda)} = (a_m)_{m \in \mathbb{N}}$ and $\lim_{m \to \infty} p_m = 1$. Let $\varepsilon > 0$ be given. Then for sufficiently large m,

$$y_m^{\lambda_m(1+\varepsilon)} = e^{\lambda_m(1+\varepsilon)\ln a_m\left(1+\frac{\ln p_m}{\ln a_m}\right)}$$

so that $a_m < 1$. Since for sufficiently large $m \in \mathbb{N}$, $\dfrac{\ln p_m}{\ln a_m} > -\dfrac{\varepsilon}{2(1+\varepsilon)}$, for the same m we have

$$y_m^{\lambda_m(1+\varepsilon)} \leq a_m^{\lambda_m\left(1+\frac{\varepsilon}{2}\right)},$$

which implies that the series $\sum_{m=1}^{\infty} y_m^{\lambda_m(1+\varepsilon)}$ converges. Because $y \in [a^{(\lambda)}]$ and $a^{(\lambda)} \in \mathbb{S}_0$, we have $y \in \mathbb{S}_0$, and thus the series $\sum_{m=1}^{\infty} y_m^{\lambda_m(1-\varepsilon)}$ diverges for $\varepsilon \geq 1$ (as $\lim_{m \to \infty} y_m^{\lambda_m(1-\varepsilon)} \neq 0$). Let $\varepsilon \in (0, 1)$. Then for m large enough, we have

$$y_m^{\lambda_m(1-\varepsilon)} = e^{\lambda_m(1-\varepsilon)\ln a_m\left(1+\frac{\ln p_m}{\ln a_m}\right)},$$

so that $a_m < 1$ for each such m. Since for m large enough, it holds

$$\frac{\ln p_m}{\ln a_m} < \frac{\varepsilon}{2(1-\varepsilon)},$$

for the same m, we have

$$y_m^{\lambda_m(1-\varepsilon)} \geq a_m^{\lambda_m\left(1-\frac{\varepsilon}{2}\right)},$$

which implies that the series $\sum_{m=1}^{\infty} y_m^{\lambda_m(1-\varepsilon)}$ diverges. This means $y \in \mathcal{E}^{\leftarrow}(\lambda)$ and thus $y \in [a^{(\lambda)}]_{\mathcal{E}^{\leftarrow}(\lambda)}$. $\qquad\square$

Corollary 3.7. *The selection principle $\alpha_2\left([a^{(\lambda)}], [a^{(\lambda)}]_{\mathcal{E}^{\leftarrow}(a)}\right)$ is satisfied for each $\lambda \in \mathbb{S}$.*

Remark 3.8. The winning strategy of the player TWO in Theorems 3.4 and 3.6 can be realized in infinitely many manners.

Acknowledgment

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