On the class $\mathbb{S}_{\underline{0}}$ of real sequences

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Abstract

We prove that certain classes of sequences of positive real numbers satisfy some selection principles related to a special kind of convergence.

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1 Introduction

Let $\mathbb{S}$ denote the set of sequences of positive real numbers, and let $\mathbb{S}_{\underline{0}}$ be the subset of $\mathbb{S}$ consisting of all sequences $(a_n)_{n\in\mathbb{N}} \in \mathbb{S}$ satisfying

$$\liminf_{n\to\infty} a_n = 0.$$

Recent work by the authors showed that there are nice relations between Karamata's theory of regular variation of sequences [1], [8], [11] from certain subclasses of $\mathbb{S}$ with Selection Principles Theory, Game Theory and Ramsey Theory

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(see [2], [3], [4], [5], [6], [7]). In particular, in [6], [7] we investigated these relations in connection with rates of divergence of sequences. Here, we study similar relations, but in connection with a special sort of convergence of a sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{S}_0$. In difference of the previous study our selection here is controlled in two "parameters": convergence of the series corresponding to $(a_n)_{n\in\mathbb{N}}$ and convergence of $(a_n)_{n\in\mathbb{N}}$.

We begin with the following definition.

Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of $\mathbb{S}$. Then the symbol $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection principle:

> For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that $b_n \in A_n$ for each $n \in \mathbb{N}$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

The following game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ is naturally associated to the previous selection principle.

> Two players, ONE and TWO, play a round for each positive integer. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \cdots; A_n, b_n; \cdots)$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

A strategy of a player is a function σ from the set of all finite sequences of moves of the other player into the set of admissible moves of the strategy owner.

A strategy σ for the player TWO is a *coding strategy* if TWO remembers only the most recent move by ONE and by TWO before his next move. More precisely the moves of TWO are: $b_1 = \sigma(A_1, \emptyset)$; $b_n = \sigma(A_n, b_{n-1})$, $n \geq 2$.

For more information on selection principles and games see the survey papers [9], [10] and references therein.

2 Results

Definition 2.1 A sequence $(a_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is said to belong to the class $\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})$ if for each $\lambda \geq 1$ it satisfies

$$\lim_{n\to\infty} \frac{a_{[n+\lambda]}}{a_n} = 0,$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

Definition 2.2 ([4]) For a sequence $b = (b_n)_{n\in\mathbb{N}} \in \mathbb{S}$, the *Landau-Hurwicz sequence* $w(b) = (w_n(b))_{n\in\mathbb{N}}$ of b is defined by

$$w_n(b) := \sup\{|b_m - b_k| : m \geq n, k \geq n\}, \quad n \in \mathbb{N}.$$

Note. A sequence $b = (b_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is convergent if and only if $w_n(b) \to 0$, as $n \to \infty$.

We also need the following notation.

For a sequence $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{S}$, we define

$$S_n(a) = \sum_{k=1}^{n} a_k,\ n \in \mathbb{N},$$

and write

$$S_a = (S_n(a))_{n\in\mathbb{N}}.$$

Let $\ell^1_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})}$ be the set of all sequences $(a_n)_{n\in\mathbb{N}} \in \mathbb{S}$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $w(S_a) \in \mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})$.

Theorem 2.3 TWO *has a winning coding strategy in the game* $\mathsf{G}_1(\mathbb{S}_{\underline{0}}, \ell^1_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})})$.

Proof. Let σ denote a strategy of TWO. Suppose that in the first round ONE chooses a sequence $a_1 = (a_{1,m})_{m\in\mathbb{N}}$ from $\mathbb{S}_{\underline{0}}$. Then TWO chooses an *arbitrary* $b_1 = a_{1,m_1} \in a_1$; let $\sigma(a_1) = b_1$. If in the second round ONE has chosen $a_2 = (a_{2,m})_{m\in\mathbb{N}} \in \mathbb{S}_{\underline{0}}$, then TWO responds by taking $\sigma(a_2) = b_2 = a_{2,m_2}$ such that $b_2 \leq 2^{-1}b_1$; this is possible by the definition of $\mathbb{S}_{\underline{0}}$. If in the n-th round ONE has played $a_n = (a_{n,m})_{m\in\mathbb{N}}$ from $\mathbb{S}_{\underline{0}}$, then TWO finds $\sigma(a_n) = b_n = a_{n,m_n}$ such that $b_n \leq 2^{1-n}b_{n-1}$. And so on. We prove that the sequence $b = (b_n)_{n\in\mathbb{N}}$ is in $\ell^1_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})}$.

Evidently, the sequence $b = (b_n)_{n\in\mathbb{N}}$ belongs to $\mathbb{S}_{\underline{0}}$. Consider now the sequence $S_b = (S_n(b))_{n\in\mathbb{N}}$. This sequence is strictly increasing, and, by the d'Alembert criterion, it converges to a positive real number $S(b)$ as $n \to \infty$.

Claim 1. $\lim\limits_{n\to\infty} \frac{b_{n+1}}{b_{n+1}+b_{n+2}+\cdots} = 1$.

We have

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_{n+1}+b_{n+2}+\cdots} = \lim_{n\to\infty} \frac{1}{1+\frac{b_{n+2}}{b_{n+1}}+\frac{b_{n+3}}{b_{n+1}}+\cdots},$$

and, clearly, the series

$$\sum_{k=1}^{\infty} \frac{b_{k+1}}{b_k}$$

is convergent. Since

$$\frac{b_{n+2}}{b_{n+1}}+\frac{b_{n+3}}{b_{n+1}}+\cdots = \frac{b_{n+2}}{b_{n+1}}+\frac{b_{n+3}}{b_{n+2}}\cdot\frac{b_{n+2}}{b_{n+1}}+\cdots \leq \frac{b_{n+2}}{b_{n+1}}+\frac{b_{n+3}}{b_{n+2}}+\cdots,$$

for a sufficiently large n, and the right side of the last inequality is the n-th remainder of a convergent series, we conclude

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_{n+1}+b_{n+2}+\cdots} = 1,$$

Claim 2. $w(S_b) = (w_n(S_b))_{n\in\mathbb{N}} \in \mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})$.

Observe first that

$$w_n(S_b) = S(b) - S_n(b),\ n \in \mathbb{N}.$$

By Claim 1, we have

$$\lim_{n\to\infty} \frac{w_{n+1}(S_b)}{w_n(S_b)} = \lim_{n\to\infty} \frac{S(b) - S_{n+1}(b)}{S(b) - S_n(b)} = 1 - \lim_{n\to\infty} \frac{b_{n+1}}{b_{n+1} + b_{n+2} + \cdots} = 0.$$

Next, for $\lambda \geq 1$ it holds

$$\begin{aligned} \lim_{n\to\infty} \frac{w_{[n+\lambda]}(S_b)}{w_n(S_b)} &= \lim_{n\to\infty} \frac{w_{n+[\lambda]}(S_b)}{w_n(S_b)} \\ &= \lim_{n\to\infty} \left(\frac{w_{n+[\lambda]}(S_b)}{w_{n+[\lambda]-1}(S_b)} \cdot \frac{w_{n+[\lambda]-1}(S_b)}{w_{n+[\lambda]-2}(S_b)} \cdots \frac{w_{n+1}(S_b)}{w_n(S_b)} \right) = 0, \end{aligned}$$

because there are $[\lambda]$ factors in the last limit and each of them tends to 0 as $n \to \infty$. This means that $w(S_b) = (w_n(S_b))_{n\in\mathbb{N}}$ belongs to $\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})$. $\triangle$

Corollary 2.4 *The selection principle* $\mathsf{S}_1(\mathbb{S}_{\underline{0}}, \ell^1_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})})$ *is true.*

Remark 2.5 It is worth to mention that a minor modification in the previous proof allows to prove a more general result: for each $1 < p < \infty$ (and even for each $0 < p < 1$) it holds $\mathsf{S}_1(\mathbb{S}_{\underline{0}}, \ell^p_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})})$, where $\ell^p_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})}$ denotes the set of all sequences $a = (a_n)_{n\in\mathbb{N}}$ in $\mathbb{S}$ such that $\sum_{k=1}^{\infty}(a_k)^p < \infty$ and $w(S_{a^p}) \in \mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})$.

Suppose that a sequence $(a_n : n \in \mathbb{N})$ of sequences $(a_{n,m})_{m\in\mathbb{N}}$ from $S_{\underline{0}}$ is given. For each a_n, $n \in \mathbb{N}$, consider a sequence $(b_k^{(n)} : k \in \mathbb{N})$ of pairwise disjoint subsequences of a_n such that each $b_k^{(n)} \in \mathbb{S}_{\underline{0}}$. Apply now Corollary 2.4 to the sequence $(b_k^{(n)} : k, n \in \mathbb{N})$. One obtains that the following is true.

Corollary 2.6 *For each sequence* $(a_n : n \in \mathbb{N})$ *of sequences in* $\mathbb{S}_{\underline{0}}$ *there is a sequence* $b \in \ell^1_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})}$ *such that for each* n *the intersection* $b \cap a_n$ *is infinite.*

In established terminology and notation [10] this corollary states that the selection principle $\alpha_2(\mathbb{S}_{\underline{0}}, \ell^1_{\mathsf{Tr}(\mathsf{R}_{-\infty,\mathsf{s}})})$ is true.

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