THE WEAK ASYMPTOTIC EQUIVALENCE AND THE GENERALIZED INVERSE

D. Djurčić, R. Nikolić, A. Torgašev

Abstract. In this paper we discuss the relationship between the weak asymptotic equivalence relation and the generalized inverse in the class $\mathcal{A}$ of all nondecreasing and unbounded functions, defined and positive on a half-axis $[a,+\infty)$ $(a>0)$. In the main theorem, we prove a proper characterization of the functional class $ORV \cap \mathcal{A}$, where ORV is the class of all $\mathcal{O}$–regularly varying functions (in the sense of Karamata).

AMS *Subject Classification*: Primary 26 A 12.

Key words: Regular variation, generalized inverse, weak asymptotic equivalence.

1. Introduction

A function $f:[a,+\infty)\mapsto(0,+\infty)$ $(a>0)$ is called *$\mathcal{O}$–regularly varying in the sense of Karamata* if it is measurable and if

(1) $$\overline{k}_f(\lambda) := \overline{\lim_{x\to+\infty}} \frac{f(\lambda x)}{f(x)} < +\infty \quad (\lambda>0).$$

Condition (1) is equivalent with condition

(2) $$\underline{k}_f(\lambda) := \underline{\lim}_{x\to+\infty} \frac{f(\lambda x)}{f(x)} > 0 \quad (\lambda>0).$$

Function $\overline{k}_f(\lambda)$ $(\lambda>0)$ is called *index function* of f, and function $\underline{k}_f(\lambda)$ $(\lambda>0)$ is called *auxiliary index function* of f. ORV is the class of all $\mathcal{O}$–regularly varying functions defined on some interval $[a,+\infty)$.

The class ORV is an important object in the qualitative analysis of divergent processes (see e.g. [1] and [7]).

The Tauberian condition generated by condition (1) or (2) is an important convergence condition in the theory of Tauberian theorems (see [3] and [17]), and also in the asymptotic analysis in general (see [7]).

A measurable function $f:[a,+\infty)\mapsto(0,+\infty)$ $(a>0)$ is said to belong the class αRV if there is a $\lambda_0\geq 1$ such that

(3) $$\underline{k}_f(\lambda) > 1$$

for every $\lambda>\lambda_0$.

The class αRV contains as proper subclasses: class of regularly varying functions (denoted by RV) whose Karamata index of variability ρ is positive (e.g. see [21]), class of rapidly varying functions (denoted by R_∞) whose de Haan index is $+\infty$, (e.g. see [16]), the class ARV (e.g. see [15]), but it does not contain any

This paper is supported by the Ministry of Science of Republic of Serbia, Grant No. 144031.

element from the class of slowly varying Karamata functions (denoted by SV) (e.g. see [18]).

Since the class αRV is first introduced and investigated in papers [19], [8], [9] and [10], we shall call the functions from the class αRV *the Buldygin functions*. The denotation αRV is inspired by its very important proper subclass ARV, where "A" should associate to V. Avakumović (1919–1990), the known serbian mathematician who worked in asymptotic analysis (see e.g. [4], [5] and [2]).

Let $\mathcal{A} = \{f : [a, +\infty) \mapsto (0, +\infty)(a > 0) \mid f \text{ is nondecreasing and unbounded}\}$. If $f \in \mathcal{A}$, consider the set $\{f\} = \{g \in \mathcal{A} \,|\, f(x) \asymp g(x), x \to +\infty\}$, where $f(x) \asymp g(x), x \to +\infty$ is the weak asymptotic equivalence relation defined by

$$0 < \varliminf_{x \to +\infty} \frac{f(x)}{g(x)} \leq \varlimsup_{x \to +\infty} \frac{f(x)}{g(x)} < +\infty$$

(e.g. see [7]).

For any $f \in \mathcal{A}$, the function $f^i(x) = \inf\{y \geq a \mid f(y) > x\}$ $(x \geq f(a))$ is its *generalized invers* (e.g. see [7]).

If $f \in \mathcal{A}$ is continuous and strictly increasing, then $f^i(x) = f^{-1}(x)$, for $x \geq f(a)$. Besides, $f^i \in \mathcal{A}$ whenever $f \in \mathcal{A}$. For any right continuous function $g \in \mathcal{A}$ there is an $f \in \mathcal{A}$ $(f(x) = g^i(x), x \geq g(a))$ such that $g = f^i$.

A function $f \in ORV$ is called *regularly varying* in Karamata sense if $\overline{k}_\rho(\lambda) = \lambda^\rho$ holds for all $\lambda > 0$ and some $\rho \in \mathbb{R}$, where ρ is general index of variability of f. The class of all regularly varying functions is denoted by RV. This class is the main object of the Karamata theory of regular variability (e.g. see [21]), and its variations and applications (see also [7], [11] and [20]).

For any function $f \in \mathcal{A}$ define $[f] = \{g \in \mathcal{A} \mid f(x) \sim g(x), x \to +\infty\}$, where $f(x) \sim g(x)$, $x \to +\infty$ is the strong asymptotic equivalence relation defined by $\lim\limits_{x \to +\infty} \dfrac{f(x)}{g(x)} = 1$.

The next theorem is a modified combination of some results from [6] (see also [7, p. 190, 14 (ii), (iii)]).

THEOREM A. *Let $f, g \in \mathcal{A}$ and assume that f is a regularly varying function whose index $\rho > 0$.*

(a) *If $g \in [f]$, then $g^i \in [f^i]$.*
(b) *If $g^i \in [f^i]$, then $g \in [f]$.*

Some extensions of the Theorem A can be found in [12], [8] and [15]. Also, some modifications of this theorem are given in [14] and [9], where another asymptotic relation is considered, in fact the process (operator) of inverting the functions.

2. The main results

An extension of Theorem A in the sense of the weak asymptotic equivalence relation for continuous and strictly increasing functions from the class $\mathcal{A}$ is proved in [13]. In the next statements we shall extend these results and establish a complete relationship between weak asymptotic equivalence relation and generalized inverse ([7]) in the functional class $\mathcal{A}$.

PROPOSITION 1. *Let $f, g \in \mathcal{A}$ and $f^i \in ORV$. If aditionally $g \in \{f\}$, then $g^i \in \{f^i\}$.*

Proof. If $f, g \in \mathcal{A}$ and $f^i \in ORV$, $g \in \{f\}$, then there are some constants $m, M \in \mathbb{R}^+$ $(m \le M)$ such that $m \le g(x)/f(x) \le M$, $x \ge x_0$. Hence $g(x) \le Mf(x)$ for $x \ge x_0$. Therefore $g^i(x) \ge f^i(x/M)$, $x \ge x_0$, i.e. $g^i(x)/f^i(x) \ge f^i(x/M)/f^i(x)$, $x \ge x_0$. We also have $g(x) \ge mf(x)$, $x \ge x_0$. Hence $g^i(x) \le f^i(x/m)$ holds for $x \ge x_0$, so that $g^i(x)/f^i(x) \le f^i(x/m)/f^i(x)$ for such x. Therefore, for $x \ge x_0$ holds

$$\frac{f^i(x/M)}{f^i(x)} \le \frac{g^i(x)}{f^i(x)} \le \frac{f^i(x/m)}{f^i(x)}.$$

Hence, we have

$$\underline{k}_{f^i}(1/M) \le \varliminf_{x\to+\infty} \frac{g^i(x)}{f^i(x)} \le \underline{k}_{f^i}(1/m),$$

and

$$\overline{k}_{f^i}(1/M) \le \varlimsup_{x\to+\infty} \frac{g^i(x)}{f^i(x)} \le \overline{k}_{f^i}(1/m).$$

Therefore, it follows

$$0 < \underline{k}_{f^i}(1/M) \le \varliminf_{x\to+\infty} \frac{g^i(x)}{f^i(x)} \le \varlimsup_{x\to+\infty} \frac{g^i(x)}{f^i(x)} \le \overline{k}_{f^i}(1/m),$$

hence $g^i \in \{f^i\}$. $\square$

EXAMPLE 1. In general case, if $g^i \in \{f^i\}$ then it is not necessarily $g \in \{f\}$. Take for example $f(x) = \mathrm{e}^x, x \ge 1$ and $g(x) = \dfrac{1}{2}\mathrm{e}^x, x \ge 1$.

PROPOSITION 2. *Let $f, g \in \mathcal{A}$. If $g^i \in \{f^i\}$ for every $g \in \{f\}$, then $f^i \in ORV$ $(g^i \in \{ORV\})$.*

Proof. Let $f \in \mathcal{A}$ and for every $g \in \mathcal{A}$ holds $g^i \in \{f^i\}$ when $g \in \{f\}$. For arbitrary and fixed $\lambda > 0$, consider the function $g_1(x) = \lambda f(x)$, $x \ge a$. Then $g_1 \in \{f\}$, so it follows $g_1^i \in \{f^i\}$. Since $g_1^i(x) = f^i(x/\lambda)$, $x \ge \lambda a$, we have

$$+\infty > M(\lambda) \ge \varlimsup_{x\to+\infty} \frac{f^i(x)}{f^i(x/\lambda)} = \varlimsup_{t\to+\infty} \frac{f^i(\lambda t)}{f^i(t)} = \overline{k}_{f^i}(\lambda).$$

Therefore $f^i \in ORV$.

Next, for every $g \in \{f\}$ we have $g^i \in \{f^i\}$, so it follows $g^i(x) = h(x)f^i(x)$, $x \ge f(a)$, where

$$0 < \frac{1}{A(g)} \le h(x) \le A(g) < +\infty,$$

for all $x \ge f(a)$. Hence,

$$\overline{k}_{g^i}(\lambda) \le \overline{k}_{f^i}(\lambda) A^2(g) < +\infty, \quad \lambda > 0,$$

so it follows $g^i \in ORV$. $\square$

PROPOSITION 3. *Let $f \in \mathcal{A}$. Then $f \in \alpha RV$ if and only if $f^i \in ORV$.*

Proof. First assume $f \in \mathcal{A} \cap \alpha RV$. Then for some $\lambda_0 \geq 1$ and for some $\lambda > \lambda_0$ holds $f(\lambda x) \geq c(\lambda) f(x)$, $x \geq x_0 = x_0(\lambda)$, where $c(\lambda) = c_f(\lambda) > 1$ for $\lambda > \lambda_0$. Hence, for this λ and x we have $f(\lambda x)/c(\lambda) \geq f(x)$, so it follows $f^i(c(\lambda)x)/\lambda \leq f^i(x)$. Therefore, for this λ we have $\overline{k}_{f^i}(c(\lambda)) \leq \lambda < +\infty$. Thus, $f^i \in ORV$.

Next, assume that $f^i \in ORV \cap \mathcal{A}$. Then by [1] we have

$$\varlimsup_{x\to+\infty} \sup_{\lambda\in[1,2]} \frac{f^i(\lambda x)}{f^i(x)} = \varlimsup_{x\to+\infty} \frac{f^i(2x)}{f^i(x)} = \overline{k}_{f^i}(2) \geq 1.$$

For every $\varepsilon > 0$ there is an $x_0 = x_0(\varepsilon) > 0$ such that

$$\sup_{\lambda\in[1,2]} \frac{f^i(\lambda x)}{f^i(x)} \leq \overline{k}_{f^i}(2) + \varepsilon = M(\varepsilon), \quad x \geq x_0,$$

so that for every $x \geq x_0$ and every $\lambda \in [1,2]$ we have

$$f^i(\lambda x)/f^i(x) \leq M(\varepsilon).$$

Now it follows:

$$\begin{aligned}
&\frac{f^i(\lambda x)}{M(\varepsilon)} \leq f^i(x),\\
\Rightarrow\quad &\left(\left(\frac{f(M(\varepsilon)x)}{\lambda}\right)^i\right)^i \geq \left(f^i(x)\right)^i,\\
\Rightarrow\quad &f(x) \leq \frac{f(M^2(\varepsilon)x)}{\lambda},\\
\Rightarrow\quad &\frac{f(M^2(\varepsilon)x)}{f(x)} \geq \lambda,\\
\Rightarrow\quad &\frac{f(M^2(\varepsilon)x)}{f(x)} \geq 2 > 1,\\
\Rightarrow\quad &\varliminf_{x\to+\infty} \frac{f(M^2(\varepsilon)x)}{f(x)} = \underline{k}_f(M^2(\varepsilon)) \geq 2 > 1.
\end{aligned}$$

Since $\underline{k}_f(s)$ is nondecreasing for $s > 0$, we find that $\underline{k}_f(\lambda) > 1$, for $\lambda > M^2(\varepsilon) > 1$. Hence, $f \in \alpha RV \cap \mathcal{A}$.

This completes the proof. $\square$

COROLLARY 1. *Let $f \in \mathcal{A}$. Then $f, f^i \in ORV$ if and only if $f \in ORV \cap \alpha RV$ ($f^i \in ORV \cap \alpha RV$).*

PROPOSITION 4. *Let $f \in \mathcal{A}$. Then $f \in ORV$ if and only if $f^i \in \alpha RV$.*

Proof. First assume $f \in \mathcal{A} \cap ORV$. Then by [1]

$$\varlimsup_{x\to+\infty} \sup_{\lambda\in[1,2]} \frac{f(\lambda x)}{f(x)} = \varlimsup_{x\to+\infty} \frac{f(2x)}{f(x)} = \overline{k}_f(2) \geq 1.$$

For every $\varepsilon > 0$, there is an $x_0 = x_0(\varepsilon) > 0$ such that

$$\sup_{\lambda\in[1,2]} \frac{f(\lambda x)}{f(x)} \leq \overline{k}_f(2) + \varepsilon = m(\varepsilon), \quad \text{for all } x \geq x_0,$$

so for the same x it and for every $\lambda \in [1,2]$ we have $f(\lambda x)/f(x) \leq m(\varepsilon)$. Therefore, $f(\lambda x)/m(\varepsilon) \leq f(x)$, and it follows

$$\begin{aligned}
& \frac{f^i(m(\varepsilon)x)}{\lambda} \geq f^i(x),\\
\Rightarrow\quad & f^i(m(\varepsilon)x) \geq \lambda f^i(x),\\
\Rightarrow\quad & f^i(m(\varepsilon)x) \geq 2f^i(x),\\
\Rightarrow\quad & \frac{f^i(m(\varepsilon)x)}{f^i(x)} \geq 2 > 1,\\
\Rightarrow\quad & \underline{\lim}_{x\to+\infty} \frac{f^i(m(\varepsilon)x)}{f^i(x)} \geq 2 > 1,\\
\Rightarrow\quad & \underline{k}_{f^i}(m(\varepsilon)) < 1.
\end{aligned}$$

Hence, $\underline{k}_{f^i}(\lambda) > 1$ for $\lambda > m(\varepsilon) = \lambda_0 \geq 1$, so it follows $f^i \in \alpha RV$.

Next, assume that $f^i \in \alpha RV \cap \mathcal{A}$. Then for some $\lambda_0 \geq 1$ and all $\lambda > \lambda_0$ we have $f^i(\lambda x) \geq c(\lambda) f^i(x)$, for all $x \geq x_0 = x_0(\lambda)$, where $c(\lambda) = c_f(\lambda) > 1$, $\lambda > \lambda_0$. Hence, for that λ and x we have $f^i(\lambda x)/c(\lambda) \geq f^i(x)$, so that $\big(f(c(\lambda)x)/\lambda)\big)^i \geq f^i(x)$. Then, similarly as in the previous proof, we have $f(c(\lambda)x)/\lambda \leq f(\sqrt{c(\lambda)}x)$. Therefore, $f(c(\lambda)x)/f(\sqrt{c(\lambda)}x) \leq \lambda$, and consequently, for a fixed $\lambda > \lambda_0$, we obtain $\overline{k}_f(\sqrt{c(\lambda)}) \leq \lambda < +\infty$. In other words, $f \in ORV$. $\square$

COROLLARY 2. *Let $f \in \mathcal{A}$. Then $f, f^i \in \alpha RV$ if and only if $f \in ORV \cap \alpha RV$ ($f^i \in ORV \cap \alpha RV$).*

PROPOSITION 5. *If $f \in \mathcal{A} \cap ORV$, then $f(x) \asymp (f^i(x))^i$, for $x \to +\infty$, i.e. $f \in \{(f^i)^i\}$.*

Proof. We have that for $x \geq 0$ and $\beta > 1$, $f(x) \leq (f^i(x))^i \leq f(\beta x)$, so that

$$1 \leq \frac{(f^i(x))^i}{f(x)} \leq \frac{f(\beta x)}{f(x)}.$$

Therefore,

$$0 < 1 \leq \underline{\lim}_{x\to+\infty} \frac{(f^i(x))^i}{f(x)} \leq \overline{\lim}_{x\to+\infty} \frac{f^i(x))^i}{f(x)} \leq \overline{k}_f(\beta) < +\infty.$$

Hence, $f(x) \asymp (f^i(x))^i$, for $x \to +\infty$. $\square$

PROPOSITION 6. *Let $f \in \mathcal{A} \cap ORV$ and $g \in \mathcal{A}$. Then $f(x) \asymp g(x)$, $x \to +\infty$, if and only if $(f^i(x))^i \asymp (g^i(x))^i$, $x \to +\infty$.*

Proof. Let $f \in \mathcal{A} \cap ORV$ and $g \in \mathcal{A}$. If $f(x) \asymp g(x)$, $x \to +\infty$, then $g \in ORV$, so that by Proposition 5 we have $f(x) \asymp (f^i(x))^i$, $x \to +\infty$ and $g(x) \asymp (g^i(x))^i$, $x \to +\infty$. Hence, $(f^i(x))^i \asymp (g^i(x))^i$, $x \to +\infty$.

Conversely, assume that $f \in \mathcal{A} \cap ORV$, $g \in \mathcal{A}$ and $(f^i(x))^i \asymp (g^i(x))^i$, $x \to +\infty$. Then, by Proposition 5, we have $f(x) \asymp (g^i(x))^i$ as $x \to +\infty$, because $f(x) \asymp (f^i(x))^i$, $x \to +\infty$. Hence, $(g^i(x))^i$, $x \geq a$ belongs to $ORV \cap \mathcal{A}$. By Proposition 3 we have $g^i \in \alpha RV$, while by Proposition 4 it follows $g \in ORV$. Hence, by Proposition 5 we have that $g(x) \asymp (g^i(x))^i$, $x \to +\infty$, so that $f(x) \asymp g(x)$, $x \to +\infty$.
□

PROPOSITION 7. (a) *Let $f, g \in \mathcal{A}$ and $f \in \alpha RV \cap ORV$. Then $g \in \{f\}$ if and only if $g^i \in \{f^i\}$.*

(b) *Let $f \in \mathcal{A}$ and $f \notin ORV \cap \alpha RV$. Then there is a $g \in \mathcal{A}$ such that $g \in \{f\}$ and $g^i \notin \{f^i\}$, or there is a $g \in \mathcal{A}$ such that $g^i \in \{f^i\}$ and $g \notin \{f\}$.*

Proof. (a) Let $f, g \in \mathcal{A}$ and $f \in \alpha RV \cap ORV$.

First assume that $g \in \{f\}$. Since $f \in \alpha RV$, by Proposition 3 we have $f^i \in ORV$, so by Proposition 1 we get $g^i \in \{f^i\}$.

Conversely, assume that $g^i \in \{f^i\}$. Since $f \in ORV$, by Proposition 5 we have $f(x) \asymp (f^i(x))^i$, $x \to +\infty$. Therefore, $(f^i(x))^i$, $x \geq a$ belongs to $ORV \cap \mathcal{A}$. By Proposition 1 it follows $(f^i(x))^i \asymp (g^i(x))^i$, $x \to +\infty$, and by Proposition 6 we have $f(x) \asymp g(x)$, $x \to +\infty$. Thus, $g \in \{f\}$.

(b) Assume that $f \in \mathcal{A} \setminus (ORV \cap \alpha RV)$. We shall discuss following three cases.

(1^0) Let $f \in ORV \setminus ARV$. Then by Corollary 1, $f^i \notin ORV$. Hence, by Proposition 2 there is a $g \in \mathcal{A}$ such that $g \in \{f\}$ and $g^i \notin \{f^i\}$.

(2^0) Let $f \notin ORV \cup \alpha RV$. Then by Proposition 3 $f^i \notin ORV$, and by Proposition 4 $f^i \notin \alpha RV$, so by Proposition 2 there is a $g \in \mathcal{A}$ such that $g \in \{f\}$ and $g^i \notin \{f^i\}$.

(3^0) Let $f \in \alpha RV \setminus ORV$. Then there is a $p > 1$ such that

$$\overline{k}_f(\lambda) = \overline{\lim_{x \to +\infty}} \frac{f(\lambda x)}{f(x)} = +\infty, \quad \lambda \geq p.$$

Consider a function $g(x) = f(px)$, for $x \geq a$. Then:

$$1 \leq \underline{\lim_{x \to +\infty}} \frac{g(x)}{f(x)} \leq \overline{\lim_{x \to +\infty}} \frac{g(x)}{f(x)} = +\infty,$$

so that $g \notin \{f\}$. Since

$$g^i(x) = (f(px))^i = \inf\{y > 0 \,|\, f(py) > x\} =$$
$$= \inf\{\frac{t}{p} > 0 \,|\, f(t) > x\} = \frac{f^i(x)}{p},$$

we have $f^i(x) \asymp g^i(x)$, $x \to +\infty$, so there is a $g \in \mathcal{A}$ such that $g^i \in \{f^i\}$ and $g \notin \{f\}$. □

Let $a > 0$. Functions $f, g \colon [a, +\infty) \mapsto (0, +\infty)$ are called mutually “inverse weak asymptotic” in $+\infty$ (which is denoted by $f(x) \overset{*}{\asymp} g(x)$, $x \to +\infty$) if there is a $\lambda_0 \geq 1$ such that for every $\lambda > \lambda_0$ there is an $x_0 = x_0(\lambda) > 0$ so that $f(x/\lambda) \leq g(x) \leq f(\lambda x)$ for all $x \geq x_0$ (see e.g. [6]).

PROPOSITION 8. *Let $f, g \in \mathcal{A}$. Then $f^i(x) \asymp g^i(x)$, $x \to +\infty$ if and only if $f(x) \overset{*}{\asymp} g(x)$, $x \to +\infty$.*

Proof. First assume $f, g \in \mathcal{A}$ and $f(x) \overset{*}{\asymp} g(x)$, $x \to +\infty$. Then there is a fixed $\lambda_0 \geq 1$ such that for every $\lambda > \lambda_0$ we have $f(x/\lambda) \leq g(x) \leq f(\lambda x)$ for $x \geq x_0 = x_0(\lambda) > 0$.

Since for such λ and x we have $g(x) \leq f(\lambda x)$, it follows that $g^i(x) \geq f^i(x)/\lambda$. Consequently, we have $g^i(x)/f^i(x) \geq 1/\lambda$ for such λ and x.

From the previous, $f(x/\lambda) \leq g(x)$ holds for the mentioned λ and x, i.e. $\lambda f^i(x) \geq g^i(x)$ holds for the mentioned λ and sufficiently large x, therefore $g^i(x)/f^i(x) \leq \lambda$. Now it follows $0 < 1/\lambda \leq g^i(x)/f^i(x) \leq \lambda < +\infty$, and hence $g^i(x) \asymp f^i(x)$, $x \to +\infty$.

Conversely, assume that $f, g \in \mathcal{A}$ and $f^i(x) \asymp g^i(x)$, $x \to +\infty$. Then there is an $M > 1$ such that

$$\frac{1}{M} \leq \frac{g^i(x)}{f^i(x)} \leq M, \quad x \geq x_0(M) > 0.$$

First, for such x we have $g^i(x) \leq M f^i(x)$, so it follows $g^i(x) \leq (f(x/M))^i$, and hence $(g^i(x))^i \geq ((f(x/M))^i)^i$. Therefore, for such x and any $\lambda > 1$ we have $f(x/M) \leq g(\lambda x)$. If $\lambda > 1$ and $x \geq x_0$, put $t = \lambda x$. Then for every $t \geq t_0$, we have $f(t/\lambda M) \leq g(t)$.

Next, for the same x, we have that $g^i(x) \geq f^i(x)/M$, so that $g^i(x) \geq (f(Mx))^i$. Therefore, $(g^i(x))^i \leq ((f(Mx))^i)^i$, so it follows $g(x) \leq f(\lambda M x)$ for any $\lambda > 1$ and $x \geq x_0$.

By taking $t = x$ we obtain $g(t) \leq f(\lambda M t)$ for any $\lambda > 1$ and $t \geq x_0$. Thus, if $\lambda > 1$ and $t \geq t_0 = t_0(\lambda)$ it follows $f(t/\lambda M) \leq g(t) \leq f(\lambda M t)$. Taking $s = \lambda M > M > 1$, for every $s > M > 1$ we have $f(t/s) \leq g(t) \leq f(st)$ as $t \geq t_0(s)$. This finally gives $f(x) \overset{*}{\asymp} g(x)$, $x \to +\infty$. $\square$

To prove 9 we need the following lemma.

LEMMA 1. *Let $f \in \alpha RV$. Then there is a $\lambda_0 \geq 1$ and at least one function $c(\lambda) > 1$ for every $\lambda > \lambda_0$, depending on f, such that $\lim_{\lambda \to +\infty} c(\lambda) = +\infty$ and $f(\lambda x) \geq c(\lambda) f(x)$ for every $x \geq x_0(\lambda) > 0$.*

Proof. Let $\underline{k}_f(\lambda) = \varliminf_{x \to +\infty} \frac{f(\lambda x)}{f(x)}$, where $\lambda > \lambda_0 \geq 1$, and let define $c(\lambda) = \frac{1}{2}(\underline{k}_f(\lambda) + 1)$ for that λ. Then $c(\lambda) > 1$ $(\lambda > \lambda_0)$.

Next, let $\lambda_1 > \lambda_0 \geq 1$. If $\lambda \in [\lambda_1^n, \lambda_1^{n+1})$ and $n \geq 2$ we get

$$\underline{k}_f(\lambda) \geq \varliminf_{x \to +\infty} \frac{f(\lambda_1 x)}{f(x)} \cdots \varliminf_{x \to +\infty} \frac{f(\lambda_1^{n-1} x)}{f(\lambda_1^{n-2} x)} \cdot \varliminf_{x \to +\infty} \frac{f(\lambda x)}{f(\lambda_1^{n-1} x)} =$$

$$= \Big(\underline{k}_f(\lambda_1)\Big)^{n-1} \cdot \Big(\underline{k}_f\big(\frac{\lambda}{\lambda_1^{n-1}}\big)\Big),$$

which gives $\lim_{\lambda\to+\infty} \underline{k}_f(\lambda) = +\infty$, so that $\lim_{\lambda\to+\infty} c(\lambda) = +\infty$. $\Box$

PROPOSITION 9. *Let $f, g\colon [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$ and let $f \in \alpha RV$. If $g \in \{f\}$, then $f(x) \overset{*}{\asymp} g(x)$, $x \to +\infty$.*

Proof. Since $f \in \alpha RV$, by Lemma 1, there is a function $c(\lambda) > 1$ $(\lambda > \lambda_0 \geq 1)$, depending on f, such that $f(\lambda x) \geq c(\lambda) f(x)$ for every $\lambda > \lambda_0$ and every $x \geq x_0(\lambda) \geq a > 0$, and such that $\lim_{\lambda\to+\infty} c(\lambda) = +\infty$.

Since $g \in \{f\}$, there is an $M \in \mathbb{R}$, $M > 1$ such that $1/M \leq g(x)/f(x) \leq M$, for every $x \geq x_1(M) = x_1 > 0$. Hence, for $x \geq \max\{x_0, x_1\}$ we have $f(\lambda x) \geq c(\lambda) f(x) \geq c(\lambda) g(x)/M$. Since $c(\lambda)/M \geq 1$ for $\lambda > \lambda'$, we get $f(\lambda x) \geq g(x)$ for such λ and x. Next, since $\overline{k}_f(1/\lambda) = 1/\underline{k}_f(\lambda)$ for $\lambda \geq \lambda_0$, for such λ we also have $\overline{k}_f(1/\lambda) \leq 1/c(\lambda)$. Hence, for such λ and all $x \geq x_2(\lambda) = x_2 > 0$ we have $f(x/\lambda) \leq 2f(x)/(1 + c(\lambda))$.

Therefore, for such λ and $x \geq \max\{x_1, x_2\}$ we have $f(x/\lambda) \leq 2f(x)/(1 + c(\lambda)) \leq 2Mg(x)/(1 + c(\lambda))$.

Since next $2M/(1 + c(\lambda)) \leq 1$ for $\lambda \geq \lambda''$, we have that $f(x/\lambda) \leq g(x)$ for these λ and x. Therefore $f(x/\lambda) \leq g(x) \leq f(\lambda x)$ for $\lambda \geq \max\{\lambda', \lambda''\}$ and $x \geq \max\{x_0, x_1, x_2\}$, which means that $f(x) \overset{*}{\asymp} g(x)$, $x \to +\infty$. $\Box$

PROPOSITION 10. *Let $f, g\colon [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$ and let $f \in ORV$. If $f(x) \overset{*}{\asymp} g(x)$, $x \to +\infty$, then $g \in \{f\}$.*

Proof. From all assumptions of Proposition 10 we have that

$$\frac{f(x/\lambda)}{f(x)} \leq \frac{g(x)}{f(x)} \leq \frac{f(\lambda x)}{f(x)},$$

for every $\lambda > \lambda_0 \geq 1$ and every $x \geq x_0 = x_0(\lambda) \geq a > 0$. Therefore, for $\lambda > \lambda_0$ we have

$$0 < \underline{k}_f\Big(\frac{1}{\lambda}\Big) \leq \varliminf_{x\to+\infty} \frac{g(x)}{f(x)} \leq \varlimsup_{x\to+\infty} \frac{g(x)}{f(x)} \leq \overline{k}_f(\lambda) < +\infty,$$

because of $f \in ORV$ which implies $f(x) \asymp g(x)$ for $x \to \infty$, i.e. $g \in \{f\}$. $\Box$

COROLLARY 3. *Let $f, g\colon [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$ and let $f \in ORV \cap \alpha RV$. Then $g \in \{f\}$ if and only if $f(x) \overset{*}{\asymp} g(x)$ as $x \to +\infty$.*

We remark that the class $ORV \cap \alpha RV$ contains all regularly varying functions whose index is positive as well as all functions from the class ERV (for the definition see [7]) whose down Matuszewska index is positive (see [19]). More generaly, this class contains all functions from the class $IRV \cap ARV$ (see [15]). Also notice that the same class does not contain any slowly varying Karamata functions and any rapidly varying function in the sense of de Haan (see e.g. [16]).

Example 2. The function $f(x) = (2 + \sin x)x$, $x \geq 1$, satisfies

$$f \in (\alpha RV \cap ORV) \setminus (ARV \cap IRV).$$

We end this paper with an open question.

QUESTION. Is the class $ORV \cap \alpha RV$ the largest class for which the Corollary 3 remains true?

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Addresses:

Dragan Djurčić: University of Kragujevac, Technical Faculty, Svetog Save 65, 32000 Čačak, Serbia. E–mail: dragandj@tfc.kg.ac.rs

Rale Nikolić: University of Kragujevac, Technical Faculty, Svetog Save 65, 32000 Čačak, Serbia. E–mail: rale@tfc.kg.ac.rs

Aleksandar Torgašev: University of Belgrade, Faculty of mathematics, Studentski trg 16a, 11000 Belgrade, Serbia. E–mail: torgasev@matf.bg.ac.rs

Fabruary 25, 2009.