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THE WEAK AND STRONG ASYMPTOTIC EQUIVALENCE RELATIONS AND THE GENERALIZED INVERSE*

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Abstract. We discuss the relationship between the weak and strong asymptotic equivalence relations and the generalized inverse in the class $\mathcal{A}$ of all nondecreasing unbounded positive functions on a half-axis $[a, +\infty)$ $(a > 0)$. As a main result, we prove a proper characterization of the functional class $R_\infty \cap \mathcal{A}$, where R_∞ is the class of all rapidly varying functions. Also, we prove a characterization of the functional class $PI^* \cap \mathcal{A}$.

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1 INTRODUCTION

A function $f : [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$ is called *$\mathcal{O}$-regularly varying* in the sense of Karamata if it is measurable and

$$\overline{k}_f(\lambda) := \limsup_{x \to +\infty} \frac{f(\lambda x)}{f(x)} < +\infty \quad (\lambda > 0). \tag{1.1}$$

Condition (1.1) is equivalent with the condition

$$\underline{k}_f(\lambda) := \liminf_{x \to +\infty} \frac{f(\lambda x)}{f(x)} > 0 \quad (\lambda > 0). \tag{1.2}$$

ORV is the class of all $\mathcal{O}$-regularly varying functions defined on some interval $[a, +\infty)$. The class ORV is an important object in asymptotic analysis (see, e.g., [2] and [16]).

A function $f \in ORV$ is called *regularly varying* in the sense of Karamata if $\overline{k}_f(\lambda) = \lambda^\rho$ for all $\lambda > 0$ and some $\rho \in \mathbb{R}$, where ρ is the index of variability of f. The class of all regularly varying functions is denoted by RV. This class is the main object of Karamata theory of regular variability (see, e.g., [15]) and its applications (see also [1, 2] and [16]).

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A function $f \in RV$ is called *slowly varying* in the sense of Karamata (see, e.g., [15]) if its index of variability is $\rho = 0$. We denote by SV the class of all such functions (see [2] and [16]).

A measurable function $f : [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$ is said to belong to the class PI^* if there is $\lambda_0 \geqslant 1$ such that

$$\underline{k}_f(\lambda) > 1 \quad \text{for all } \lambda > \lambda_0.$$

For $\lambda_0 = 1$, we obtain the class ARV (see [14]).

A function $f \in ARV$ is called *rapidly varying* (in the sense of de Haan) of index of variability $+\infty$ (i.e., belongs to the class R_∞) if $\underline{k}_f(\lambda) = +\infty$ for all $\lambda > 1$ (see [2, 8] and [7]). The class PI^* contains, as a proper subclass, the class of regularly varying functions of positive index of variability ρ, but it does not contain any element from the class of slowly varying functions. More information about these classes can be found in [5, 6, 9] and [10].

Let

$$\mathcal{A} = \bigl\{ f : [a, +\infty) \mapsto (0, +\infty)\ (a > 0) \bigm| f \text{ is nondecreasing and unbounded} \bigr\}.$$

If $f \in \mathcal{A}$, consider the set

$$\{f\} = \bigl\{ g \in \mathcal{A} \bigm| f(x) \asymp g(x),\ x \to +\infty \bigr\},$$

where $f(x) \asymp g(x)$, $x \to +\infty$, is the weak asymptotic equivalence relation defined by

$$0 < \liminf_{x \to +\infty} \frac{f(x)}{g(x)} \leqslant \limsup_{x \to +\infty} \frac{f(x)}{g(x)} < +\infty$$

(see, e.g., [2]).

For any function $f \in \mathcal{A}$, define

$$[f]_\sim = \bigl\{ g \in \mathcal{A} \bigm| f(x) \sim g(x),\ x \to +\infty \bigr\},$$

where $f(x) \sim g(x)$, $x \to +\infty$, is the strong asymptotic equivalence relation defined by

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1.$$

For any $f \in \mathcal{A}$, the function

$$f^{\leftarrow}(x) = \inf\bigl\{ y \geqslant a \bigm| f(y) > x \bigr\} \quad \bigl(x \geqslant f(a)\bigr)$$

is its *generalized inverse* (see, e.g., [2]).

If $f \in \mathcal{A}$ is continuous and strictly increasing function, then $f^{\leftarrow}(x) = f^{-1}(x)$ for $x \geqslant f(a)$. Besides, $f^{\leftarrow} \in \mathcal{A}$ whenever $f \in \mathcal{A}$. For any right continuous function $g \in \mathcal{A}$, there is $f \in \mathcal{A}$ $(f(x) = g^{\leftarrow}(x)$, $x \geqslant g(a))$ such that $g = f^{\leftarrow}$.

The next theorem is a modified combination of some results from [1] (see also [2, p. 190, 14(ii), (iii)]).

Theorem A. *Let $f, g \in \mathcal{A}$ and assume that f is a regularly varying function of index of variability $\rho > 0$. If $g \in [f]_\sim$, then $g^{\leftarrow} \in [f^{\leftarrow}]_\sim$.*

Some extensions of Theorem A can be found in [11, 14] and [10]. Also, some modifications of this theorem are given in [3] and [4], where another asymptotic relation is considered, in fact, the process (operator) of inversion of functions.

An extension of Theorem A for the weak asymptotic equivalence relation for continuous and strictly increasing functions from the class $\mathcal{A}$ is proved in [12].

2 MAIN RESULTS

In the following proposition, (a) gives an extension of Theorem A by observing the weaker condition $g \in \{f\}$ instead of $g \in [f]_{\sim}$, while (b) determines the maximal class $R_\infty \cap \mathcal{A}$ for which (a) holds.

Proposition 1. (a) *Let $f, g \in \mathcal{A}$ and $f \in R_\infty$. If, additionally, $g \in \{f\}$, then $g^{\leftarrow} \in [f^{\leftarrow}]_{\sim}$.*
(b) *Let $f, g \in \mathcal{A}$. If $g^{\leftarrow} \in [f^{\leftarrow}]_{\sim}$ for every $g \in \{f\}$, then $f \in R_\infty$ ($g \in R_\infty$).*

Proof. (a) From the fact that $f \in \mathcal{A} \cap R_\infty$ and by [2], we have that $f^{\leftarrow} \in SV$. Since $g \in \{f\}$, there is $m > 0$ such that $g(x)m \leqslant f(x)$ for sufficiently large x. Furthermore, for the same m and x large enough, $g^{\leftarrow}(x) \geqslant f^{\leftarrow}(mx)$, and we have

$$\liminf_{x\to+\infty} \frac{g^{\leftarrow}(x)}{f^{\leftarrow}(x)} \geqslant \lim_{x\to+\infty} \frac{f^{\leftarrow}(mx)}{f^{\leftarrow}(x)} = 1.$$

On the other hand, there is $M > 0$ such that $f(x) \leqslant g(x)M$ for sufficiently large x. Then, for the same M and sufficiently large x, we have $g^{\leftarrow}(x) \leqslant f^{\leftarrow}(Mx)$ and, thus,

$$\limsup_{x\to+\infty} \frac{g^{\leftarrow}(x)}{f^{\leftarrow}(x)} \leqslant \lim_{x\to+\infty} \frac{f^{\leftarrow}(Mx)}{f^{\leftarrow}(x)} = 1.$$

Hence, we obtain

$$\lim_{x\to+\infty} \frac{g^{\leftarrow}(x)}{f^{\leftarrow}(x)} = \liminf_{x\to+\infty} \frac{g^{\leftarrow}(x)}{f^{\leftarrow}(x)} = \limsup_{x\to+\infty} \frac{g^{\leftarrow}(x)}{f^{\leftarrow}(x)} = 1,$$

so that $g^{\leftarrow} \in [f^{\leftarrow}]_{\sim}$.

(b) Let $f \in \mathcal{A}$, and let $g(x) = \lambda f(x)$ for $x \geqslant a$, where λ is an arbitrary fixed positive number. Then, we have $g \in \mathcal{A}$ and $g \in \{f\}$, so that $g^{\leftarrow} \in [f^{\leftarrow}]_{\sim}$. From this we obtain that $g^{\leftarrow}(x) = f^{\leftarrow}(\frac{1}{\lambda}x)$ for the same λ and sufficiently large x. Now we get

$$\lim_{x\to+\infty} \frac{f^{\leftarrow}(\frac{1}{\lambda}x)}{f^{\leftarrow}(x)} = \lim_{x\to+\infty} \frac{g^{\leftarrow}(x)}{f^{\leftarrow}(x)} = 1.$$

For every $\alpha > 0$, we have $\overline{k}_{f^{\leftarrow}}(\alpha) = 1$, because λ is an arbitrary fixed positive number. Hence, we have $f^{\leftarrow} \in SV$ and $f^{\leftarrow} \in \mathcal{A}$. According to results from [13], we obtain that $f \in R_\infty$. Now, for an arbitrary function $g \in \mathcal{A}$ such that $g \in \{f\}$, there are $m > 0$ and $M > 0$ such that $f(x) = r(x)g(x)$ for $x \geqslant a$, where the function $r(x)$ is defined for $x \geqslant a$, and the condition $m \leqslant r(x) \leqslant M$ is satisfied for sufficiently large x. Therefore, we get

$$\liminf_{x\to+\infty} \frac{g(\lambda x)}{g(x)} \geqslant \frac{m}{M} \liminf_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = +\infty$$

for every $\lambda > 1$, i.e., $g \in R_\infty$. □

In the next proposition, (a) yields an extension of Theorem A by considering the weaker condition $g^{\leftarrow} \in \{f^{\leftarrow}\}$ instead of $g^{\leftarrow} \in [f^{\leftarrow}]_{\sim}$, while (b) shows that $PI^* \cap \mathcal{A}$ is the maximal class for which (a) holds.

Proposition 2. (a) *Let $f, g \in \mathcal{A}$ and $f \in PI^*$. If, additionally, $g \in [f]_{\sim}$, then $g^{\leftarrow} \in \{f^{\leftarrow}\}$.*
(b) *Let $f, g \in \mathcal{A}$. If $g^{\leftarrow} \in \{f^{\leftarrow}\}$ for every $g \in [f]_{\sim}$, then $f \in PI^*$ ($g \in PI^*$).*

Proof. (a) This statement is a direct corollary of Proposition 1 from [10]. We can obtain the proof of (a) by applying a methodology analogous to the one used in the proof of Proposition 1(a).

(b) Let $f \in \mathcal{A}$, and let $g_1(x) = (1 - \frac{1}{x})f(x)$ for $x \geqslant a$ (without loss of generality, we assume that $a > 1$). Then, we obtain that $g_1 \in [f]_\sim$ and g_1 is a strictly increasing function from $\mathcal{A}$. This implies that $g_1^{\leftarrow} \in \{f^{\leftarrow}\}$ and $g_1^{\leftarrow}$ is a continuous function from $\mathcal{A}$. Hence, for every strictly increasing function $g \in \mathcal{A}$ for which $g \in [g_1]_\sim$, we obtain $g^{\leftarrow} \in \{g_1^{\leftarrow}\}$. Therefore, there is $M_g \in (0, +\infty)$, associated to g, such that

$$\limsup_{x\to+\infty} \frac{g_1^{\leftarrow}(g(x))}{g_1^{\leftarrow}(g_1(x))} = \limsup_{x\to+\infty} \frac{g_1^{\leftarrow}(g(x))}{x} = \limsup_{x\to+\infty} \frac{g_1^{\leftarrow}(g(x))}{g^{\leftarrow}(g(x))} \leqslant \limsup_{x\to+\infty} \frac{g_1^{\leftarrow}(x)}{g^{\leftarrow}(x)} = M_g < +\infty.$$

Let $\alpha(x)$ be a continuous function for $x \geqslant a$ such that $\alpha(x) \geqslant 1$ for $x \geqslant a$ and $\alpha(x) \to 1$ as $x \to +\infty$. Consider the function $r(x) = \max_{a \leqslant t \leqslant x} h(t)$ for $x \geqslant a$, where the function $h(x)$ is defined by $h(x) = x\alpha(x)$ for $x \geqslant a$. We see that $r(x)$ is continuous, nondecreasing, and $r(x) \to +\infty$ as $x \to +\infty$. The inequality $r(x) \geqslant \alpha(x)x$ for $x \geqslant a$ is also satisfied. Now, we will prove that $r(x) \sim x$. For this purpose, take $\varepsilon > 0$. There is $x_1 = x_1(\varepsilon) \geqslant a$ such that $1 \leqslant \frac{h(x)}{x} < 1 + \varepsilon$ for every $x \geqslant x_1$, and there is $x_2 = x_2(\varepsilon) \geqslant x_1$ such that $h(x) \geqslant \max_{a \leqslant u \leqslant x_1} h(u)$ for every $x \geqslant x_2$. Hence, for every $x \geqslant x_2$, there is a function $v(x)$ with values in $[x_1, x]$ such that

$$1 \leqslant \frac{r(x)}{x} = \frac{1}{x} \max_{a \leqslant u \leqslant x} h(u) = \frac{1}{x} \max_{x_1 \leqslant u \leqslant x} h(u) = \frac{1}{x} h\bigl(v(x)\bigr) \leqslant \frac{h(v(x))}{v(x)} < 1 + \varepsilon,$$

which means that $r(x) \sim x$. Define now the function $r_1(x) = 1 - \frac{1}{x} + r(x)$ for $x \geqslant a$ (without loss of generality, we assume that $a > 1$). Then, r_1 is a strictly increasing and continuous function from the class $\mathcal{A}$ such that $r_1(x) \sim x$. From this we get

$$1 \leqslant \liminf_{x\to+\infty} \frac{g_1^{\leftarrow}(\alpha(x)x)}{g_1^{\leftarrow}(x)} \leqslant \limsup_{x\to+\infty} \frac{g_1^{\leftarrow}(\alpha(x)x)}{g_1^{\leftarrow}(x)} \leqslant \limsup_{x\to+\infty} \frac{g_1^{\leftarrow}(r_1(x))}{g_1^{\leftarrow}(x)}.$$

From $\limsup_{x\to+\infty} \frac{g_1^{\leftarrow}(g(x))}{g_1^{\leftarrow}(g_1(x))} \leqslant M_g < +\infty$ (see above), for sufficiently large x, we obtain $\frac{g_1^{\leftarrow}(r_1(g_1(x)))}{g_1^{\leftarrow}(g_1(x))} \leqslant M_{r_1 \circ g_1} < +\infty$, where $M_{r_1 \circ g_1}$ is a positive real number that corresponds to the composition $r_1 \circ g_1$ in the same way as M_g was associated to g. Using the previous facts, one can prove that $\frac{g_1^{\leftarrow}(r_1(x))}{g_1^{\leftarrow}(x)} \leqslant M_{r_1 \circ g_1} < +\infty$ for sufficiently large x. Finally, we get $\limsup_{x\to+\infty} \frac{g_1^{\leftarrow}(\alpha(x)x)}{g_1^{\leftarrow}(x)} < +\infty$.

Now, we will assume for a moment that the next two sequences exist: (i) a sequence (λ_n) such that $\lambda_n \geqslant 1$ for every $n \in \mathbb{N}$ and $\lambda_n \to 1$ as $n \to +\infty$, and (ii) an increasing sequence (x_n) such that $x_n \geqslant a$ for every $n \in \mathbb{N}$ and $x_n \to +\infty$ for $n \to +\infty$ such that $\lim_{n\to+\infty} \frac{g_1^{\leftarrow}(\lambda_n x_n)}{g_1^{\leftarrow}(x_n)} = +\infty$. Consider a function $\alpha(x)$, $x \geqslant a$, such that $\alpha(x_n) = \lambda_n$ for $n \in \mathbb{N}$, $\alpha(x)$ is linear and continuous for $x \in [x_n, x_{n+1}]$, $n \in \mathbb{N}$, and $\alpha(x) = \lambda_1$ for $x \in [a, x_1]$. This function $\alpha : [a, +\infty) \to [1, +\infty)$ is continuous, and $\lim_{x\to+\infty} \alpha(x) = 1$. From the definition of α it follows that

$$\limsup_{n\to+\infty} \frac{g_1^{\leftarrow}(\alpha(x_n)x_n)}{g_1^{\leftarrow}(x_n)} = \lim_{x\to+\infty} \frac{g_1^{\leftarrow}(\lambda_n x_n)}{g_1^{\leftarrow}(x_n)} = +\infty.$$

This contradicts the fact $\limsup_{x\to+\infty} \frac{g_1^{\leftarrow}(\alpha(x)x)}{g_1^{\leftarrow}(x)} < +\infty$ shown above.

So, we obtain that $\limsup_{x\to+\infty, \lambda\to1} \frac{g_1^{\leftarrow}(\lambda x)}{g_1^{\leftarrow}(x)} = A$ for some $A \in (0, +\infty)$, i.e., for every $\varepsilon > 0$, there are $x_0 \geqslant a$ and $\delta > 0$ such that $1 \leqslant \frac{g_1^{\leftarrow}(\lambda x)}{g_1^{\leftarrow}(x)} \leqslant A + \varepsilon$ for every $x \geqslant x_0$ and every $\lambda \in [1, 1 + \delta]$. Hence, for every $\lambda \in (0, 1 + \delta]$, we have $\overline{k}_{g_1^{\leftarrow}}(\lambda) \leqslant A + \varepsilon < +\infty$. On the other hand, from the fact that the function $g_1^{\leftarrow}$ belongs to the class $\mathcal{A}$ (and is nondecreasing), we have that $\overline{k}_{g_1^{\leftarrow}}(\lambda) < +\infty$ for every $\lambda > 0$ (see [16]). Finally, we obtain that $g_1^{\leftarrow} \in ORV$, and by a result from [10], it follows that $g_1 \in PI^*$. Furthermore, as $g_1(x) = (1 - \frac{1}{x})f(x)$ for every $x \geqslant a$, we have $\underline{k}_f(\lambda) \geqslant \underline{k}_{g_1}(\lambda) \liminf_{x\to+\infty} \frac{\lambda x(x-1)}{(\lambda x-1)x} = \underline{k}_{g_1}(\lambda)$ for every $\lambda > 0$, i.e., we obtain that $f \in PI^*$.

Similarly, we can prove that $g \in PI^*$ for every $g \in \mathcal{A}$ such that $g \in [f]_\sim$. $\square$

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