



## Fractional Order Convergence Rate Estimate of Finite-Difference Method for the Heat Equation with Concentrated Capacity

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**Abstract.** The convergence of difference scheme for initial-boundary value problem for the heat equation with concentrated capacity and time-dependent coefficient of the space derivatives, is considered. Fractional order convergence rate estimate in a special discrete Sobolev norms, compatible with the smoothness of the coefficient and solution, is proved.

### 1. Introduction

The finite-difference method is one of the basic tools for the numerical solution of partial differential equations. In the case of problems with discontinuous coefficients and concentrated factors (Dirac delta functions, free boundaries, etc.) the solution has weak global regularity and it is impossible to establish convergence of finite difference schemes using the classical Taylor series expansion. Often, the Bramble-Hilbert lemma takes the role of the Taylor formula for functions from the Sobolev spaces [4], [6], [10].

Following Lazarov et al. [10], a convergence rate estimate of the form

$$\|u - v\|_{W_{2,h}^k} \leq Ch^{s-k} \|u\|_{W_2^s}, \quad s > k, \quad (1)$$

is called **compatible** with the smoothness (regularity) of the solution  $u$  of the boundary-value problem. Here  $v$  is the solution of the discrete problem,  $h$  is the spatial mesh step,  $W_2^s$  and  $W_{2,h}^k$  are Sobolev spaces of functions with continuous and discrete argument, respectively,  $C$  is a constant which doesn't depend on  $u$  and  $h$ . For the parabolic case typical estimates are of the form

$$\|u - v\|_{W_{2,h\tau}^{k,k/2}} \leq C(h + \sqrt{\tau})^{s-k} \|u\|_{W_2^{s/2}}, \quad s > k, \quad (2)$$

where  $\tau$  is the time step.

In the case of equations with variable coefficients the constant  $C$  in the error bounds depends on norms of the coefficients (see, for example, [6], [13])

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One interesting class of parabolic problems model processes in heat-conducting media with concentrated capacity in which the heat capacity coefficient contains a Dirac delta function, or equivalently, the jump of the heat flow in the singular point is proportional to the time-derivative of the temperature [11]. Such problems are nonstandard and the classical tools of the theory of finite difference schemes are difficult to apply to their convergence analysis.

Finite difference approximation of initial-boundary value problems for the heat equation with concentrated capacity is considered in [8], [9], [1], [2]. Integer order convergence rate estimates are proved in [8], [1], [2]. Fractional order convergence rate estimate is derived in [9], for the problem with variable (but not time-dependent) coefficients. In the present paper problem with time-dependent coefficient is considered. Fractional order convergence rate estimate is proved.

## 2. Differential problem and its approximation

Let us consider the initial-boundary-value problem for the heat equation in the presence of a concentrated capacity at the interior point  $x = \xi$  :

$$\begin{aligned} (1 + K\delta(x - \xi)) \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial u}{\partial x} \right) + b(x, t)u &= f(x, t), \quad \text{in } Q = \Omega \times (0, T), \\ u(0, t) = 0, \quad u(1, t) = 0, \quad 0 < t < T, \\ u(x, 0) = u_0(x), \quad x \in (0, 1), \end{aligned} \tag{3}$$

where  $\delta(x)$  is the Dirac delta function,  $K > 0$  , and  $\Omega = (0, 1)$ . We shall assume that

$$a \in W_2^{s-1, (s-1)/2}(Q^L) \cap W_2^{s-1, (s-1)/2}(Q^R), \quad b \in W_2^{s-2, s/2-1}(Q^L) \cap W_2^{s-2, s/2-1}(Q^R), \quad a > 0, \quad b > 0 \tag{4}$$

$$f \in W^{s-2, s/2-1}(Q), \quad u_0(x) \in W_2^{s-1}(\Omega^L) \cap W_2^{s-1}(\Omega^R), \tag{5}$$

$$u \in W_2^{s, s/2}(Q^L) \cap W_2^{s, s/2}(Q^R), \tag{6}$$

for  $5/2 < s \leq 3$  , where  $\Omega^L = (0, \xi)$  ,  $\Omega^R = (\xi, 1)$  and  $Q^L = \Omega^L \times (0, T)$  ,  $Q^R = \Omega^R \times (0, T)$ . Note that conditions (4)-(5) express the minimal smoothness requirements on the data under which the solution  $u$  of (3) may belong to the function space stated in (6) (see [9]). To guarantee that such  $u$  really exists, we also need some additional compatibility conditions at the corners of  $Q$  (see [5]).

Let  $w_h$ - uniform mesh with step size  $h$  on  $(0, 1)$ ,  $\omega_h^- = \omega_h \cup \{0\}$  and  $\bar{\omega}_h = \omega_h \cup \{0, 1\}$ . Suppose that  $\xi$  is a rational number. Then one can choose the step  $h$  so that  $\xi \in \omega_h$ . Let  $\omega_\tau$  be an uniform mesh on  $(0, T)$  with the step size  $\tau = T/m$ ,  $\omega_\tau^+ = \omega_\tau \cup \{T\}$  and  $\bar{\omega}_\tau = \omega_\tau \cup \{0, T\}$ . Also we assume that the condition  $c_1 h^2 \leq \tau \leq c_2 h^2$  is satisfied. Define finite differences in the usual way [12]:

$$\begin{aligned} v_{\bar{x}}(x, t) &= \frac{v(x, t) - v(x - h, t)}{h} = v_x(x - h, t), \\ v_{x\bar{x}}(x, t) &= (v_x(x, t))_{\bar{x}} = \frac{v(x + h, t) - 2v(x, t) + v(x - h, t)}{h^2}, \\ v_{\bar{t}}(x, t) &= \frac{v(x, t) - v(x, t - \tau)}{\tau} = v_t(x, t - \tau). \end{aligned}$$

The problem (3) can be approximated on the mesh  $\bar{Q}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$  by the following difference scheme with averaged right-hand side (see [7]):

$$\begin{aligned} (1 + K\delta_h(x - \xi))v_{\bar{t}} - \frac{1}{2}((av_x)_{\bar{x}} + (av_{\bar{x}})_x) + (T_x^2 T_t^- b)v &= T_x^2 T_t^- f, \quad (x, t) \in \omega_h \times \omega_\tau^+, \\ v(0, t) = 0, \quad v(1, t) = 0, \quad t \in \omega_\tau^+, \\ v(x, 0) = u_0(x), \quad x \in \bar{\omega}_h, \end{aligned} \tag{7}$$

where

$$\delta_h(x - \xi) = \begin{cases} 0, & x \in \omega_h \setminus \{\xi\}, \\ 1/h, & x = \xi \end{cases}$$

is the mesh Dirac function, and  $T_x, T_t^-$  are Steklov averaging operators defined as follows:

$$T_x f(x, t) = T_x^\pm f(x \mp h/2, t) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(x', t) dx',$$

$$T_t^- f(x, t) = T_t^+ f(x, t - \tau) = \frac{1}{\tau} \int_{t-\tau}^t f(x, t') dt'.$$

Notice that these operators are commute and map derivatives into finite differences, for example:

$$T_x^2 \frac{\partial^2 u}{\partial x^2} = u_{x\bar{x}}, \quad T_t^- \frac{\partial u}{\partial t} = u_{\bar{t}}.$$

We also define operators

$$T_x^{2-} f(x, t) = \frac{2}{h} \int_{x-h}^x \left(1 + \frac{x' - x}{h}\right) f(x', t) dx', \quad T_x^{2+} f(x, t) = \frac{2}{h} \int_x^{x+h} \left(1 - \frac{x' - x}{h}\right) f(x', t) dx'.$$

We define the following discrete inner products, norms and seminorms:

$$(v, u)_{L_2(\omega_h)} = h \sum_{x \in \omega_h} v(x)u(x), \quad \|v\|_{L_2(\omega_h)} = (v, v)_{L_2(\omega_h)}^{1/2},$$

$$(v, u)_{L_2(\omega_h^-)} = h \sum_{x \in \omega_h^-} v(x)u(x), \quad \|v\|_{L_2(\omega_h^-)} = (v, v)_{L_2(\omega_h^-)}^{1/2},$$

$$\|v\|_{W_2^1(\omega_h)}^2 = \|v_x\|_{L_2(\omega_h^-)}^2 + \|v\|_{L_2(\omega_h)}^2,$$

$$\|v\|_{L_2(Q_{h\tau})}^2 = \tau \sum_{t \in \bar{\omega}_\tau^+} \|v(\cdot, t)\|_{L_2(\omega_h)}^2, \quad \|v\|_{L_2(\xi \times \omega_\tau)}^2 = \tau \sum_{t \in \bar{\omega}_\tau^+} |v(\xi, t)|^2,$$

$$|v|_{W_2^{1/2}(\omega_\tau; L_2(\omega_h))}^2 = \tau \sum_{t \in \bar{\omega}_\tau} \tau \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|v(\cdot, t) - v(\cdot, t')\|_{L_2(\omega_h)}^2}{|t - t'|^2},$$

$$|v|_{W_2^{1/2}(\omega_\tau, \xi)}^2 = \tau \sum_{t \in \bar{\omega}_\tau} \tau \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{|v(\xi, t) - v(\xi, t')|^2}{|t - t'|^2},$$

$$\|v\|_{W_2^{1/2}(\omega_\tau; L_2(\omega_h))}^2 = |v|_{W_2^{1/2}(\omega_\tau; L_2(\omega_h))}^2 + \tau \sum_{t \in \bar{\omega}_\tau} \left(\frac{1}{t + \tau} + \frac{1}{T - t + \tau}\right) \|v(\cdot, t)\|_{L_2(\omega_h)}^2,$$

$$\|v\|_{W_2^{1/2}(\omega_\tau; \xi)}^2 = |v|_{W_2^{1/2}(\omega_\tau; \xi)}^2 + \tau \sum_{t \in \bar{\omega}_\tau} \left(\frac{1}{t + \tau} + \frac{1}{T - t + \tau}\right) |v(\xi, t)|^2,$$

$$\|v\|_{W_2^{1,1/2}(Q_{h\tau})}^2 = \tau \sum_{t \in \bar{\omega}_\tau^+} \|v(\cdot, t)\|_{W_2^1(\omega_h)}^2 + |v|_{W_2^{1/2}(\omega_\tau; L_2(\omega_h))}^2 + |v|_{W_2^{1/2}(\omega_\tau; \xi)}^2.$$

### 3. Convergence of the difference scheme

In the section we shall prove the convergence of the difference scheme (7) in the  $\widetilde{W}_2^{1,1/2}(Q_{h\tau})$  norm.

Let  $u$  be the solution of the boundary-value problem (3) and  $v$  the solution of the difference problem (7). The error  $z = u - v$  satisfies the finite difference scheme

$$\begin{aligned} (1 + K\delta_h(x - \xi))z_{\bar{i}} - \frac{1}{2}((az_x)_{\bar{x}} + (az_{\bar{x}})_x) + (T_x^2 T_t^- b)z &= \eta_{\bar{x}} + \phi_{\bar{i}} + \mu \quad \text{on } Q_{h\tau}, \\ z(0, t) = z(1, t) &= 0, \quad t \in \omega_{\tau}^+, \\ z(x, 0) &= 0, \quad x \in \bar{\omega}_h, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \eta &= T_x^+ T_t^- (a \frac{\partial u}{\partial x}) - \frac{1}{2}(a + a^+)u_x, \\ \phi &= u - T_x^2 u, \quad \mu = (T_x^2 T_t^- b)u - T_x^2 T_t^- (bu). \end{aligned}$$

and  $a^+(x, t) = a(x + h, t)$ . Let us set  $\phi = \tilde{\phi} + \delta_h \hat{\phi}, \mu = \tilde{\mu} + \delta_h \hat{\mu}$  where

$$\begin{aligned} \hat{\phi} &= \frac{h^2}{6} \left[ \frac{\partial u}{\partial x} \right]_{\xi}, \\ \hat{\mu} &= -\frac{h^2}{3} \left[ (T_t^- b) \left( T_t^- \frac{\partial u}{\partial x} \right) \right]_{\xi} \end{aligned}$$

and  $[u]_{\xi} = u(\xi + 0, t) - u(\xi - 0, t)$ .

The following a priori estimate for the solution of the difference scheme (8) is valid (see [9], [14]):

$$\|z\|_{\widetilde{W}_2^{1,1/2}(Q_{h\tau})} \leq C(\|\eta\|_{L_2(Q_{h\tau})} + \|\tilde{\phi}\|_{\widetilde{W}_2^{1/2}(\omega_{\tau}; L_2(\omega_h))} + \|\hat{\phi}\|_{\widetilde{W}_2^{1/2}(\omega_{\tau}; \xi)} + \|\tilde{\mu}\|_{L_2(Q_{h\tau})} + \|\hat{\mu}\|_{L_2(\xi \times \omega_{\tau})}) \tag{9}$$

Therefore, in order to estimate the rate of convergence of the difference scheme (7), it is sufficient to estimate the right-hand side of the inequality (9).

Let us estimate the term  $\eta$ . We decompose term  $\eta = \eta_1 + \eta_2 + \eta_3$  where

$$\begin{aligned} \eta_1 &= T_x^+ T_t^- (a \frac{\partial u}{\partial x}) - (T_x^+ T_t^- a)(T_x^+ T_t^- \frac{\partial u}{\partial x}), \\ \eta_2 &= (T_x^+ T_t^- a - 0.5(a + a^+))(T_x^+ T_t^- \frac{\partial u}{\partial x}), \\ \eta_3 &= 0.5(a + a^+)(T_x^+ T_t^- \frac{\partial u}{\partial x} - u_x). \end{aligned} \tag{10}$$

The first, we define rectangles  $e = e(x, t) = \{(\xi, v) : \xi \in (x, x + h), v \in (t - \tau, t)\}$ . Using linear transformations  $\xi = x + hx^*, v = t + \tau t^*$ , a single-sided mapping between  $e$  and rectangle  $E = \{(x^*, t^*) : 0 < x^* < 1, -1 < t^* < 0\}$  is established. Also, let's mark  $u^*(x^*, t^*) = u(x + hx^*, t + \tau t^*)$ ,  $a^*(x^*, t^*) = a(x + hx^*, t + \tau t^*)$ .

Using Cauchy-Schwarz and Hölder's inequality, we obtain the following estimate

$$|\eta_1(x, t)| \leq \frac{C}{h} \|a^*\|_{W_q^{\lambda, \lambda/2}(E)} \|u^*\|_{W_{2q/(q-2)}^{\mu, \mu/2}(E)}, \quad \lambda \geq 0, \mu \geq 1, q > 2.$$

Further,  $\eta_1 = 0$  whenever  $a^*$  is a constant or  $u^*$  is a polynomial of degree one in  $x^*$  and a constant in  $t^*$ . Applying Bramble-Hilbert lemma [4] we get:

$$|\eta_1(x, t)| \leq \frac{C}{h} |a^*|_{W_q^{\lambda, \lambda/2}(E)} |u^*|_{W_{2q/(q-2)}^{\mu, \mu/2}(E)}, \quad 0 \leq \lambda \leq 1, 1 \leq \mu \leq 2, q > 2.$$

Going back to the old variables, using the condition  $c_1 h^2 \leq \tau \leq c_2 h^2$ , we have:

$$|a^*|_{W_q^{\lambda, \lambda/2}(E)} \leq Ch^{\lambda - \frac{3}{q}} |a|_{W_q^{\lambda, \lambda/2}(e)} \quad \text{and}$$

$$|u^*|_{W_{2q/(q-2)}^{\mu, \mu/2}(E)} \leq Ch^{\mu - \frac{3(q-2)}{2q}} |u|_{W_{2q/(q-2)}^{\mu, \mu/2}(e)}.$$

Now, we have

$$|\eta_1(x, t)| \leq Ch^{\lambda + \mu - 5/2} |a|_{W_q^{\lambda, \lambda/2}(e)} |u|_{W_{2q/(q-2)}^{\mu, \mu/2}(e)}, \quad 0 \leq \lambda \leq 1, 1 \leq \mu \leq 2, q > 2. \tag{11}$$

After summation over the mesh  $Q_{h\tau}^+ = \omega_h \times \omega_\tau^+$  we get

$$\|\eta_1(x, t)\|_{L_2(Q_{h\tau})} \leq Ch^{\lambda + \mu - 1} \|a\|_{W_q^{\lambda, \lambda/2}(Q)} \|u\|_{W_{2q/(q-2)}^{\mu, \mu/2}(Q)}, \quad 0 \leq \lambda \leq 1, 1 \leq \mu \leq 2, q > 2.$$

Further, using imbeddings

$$W_2^{\lambda + \mu, (\lambda + \mu)/2}(Q) \subset W_{2q/(q-2)}^{\mu, \mu/2}(Q), \quad \lambda \geq 3/q,$$

$$W_2^{\lambda + \mu - 1, (\lambda + \mu - 1)/2}(Q) \subset W_q^{\lambda, \lambda/2}(Q), \quad \mu \geq 5/2 - 3/q$$

and taking  $\lambda = 1, q = 3, \mu = s - 1$  we have

$$\|\eta_1\|_{L_2(Q_{h\tau})} \leq Ch^{s-1} (\|a\|_{W_2^{s-1, (s-1)/2}(Q^L)} \|u\|_{W_2^{s, s/2}(Q^L)} + \|a\|_{W_2^{s-1, (s-1)/2}(Q^R)} \|u\|_{W_2^{s, s/2}(Q^R)}), \quad 2, 5 < s \leq 3. \tag{12}$$

Further,

$$\eta_2 = (T_x^+ T_t^- a - 0.5(a + a^+))(T_x^+ T_t^- \frac{\partial u}{\partial x}) = \eta_{21}(T_x^+ T_t^- \frac{\partial u}{\partial x}),$$

where  $\eta_{21} = T_x^+ T_t^- a - 0.5(a + a^+)$ . Using the continuity of the coefficient, we have

$$|\eta_{21}(x, t)| \leq C \|a^*\|_{W_2^{\lambda, \lambda/2}(E)}, \quad \lambda > 3/2.$$

Further,  $\eta_{21}(x, t) = 0$  whenever  $a^*$  is a polynomial of degree one in  $x^*$  and a constant in  $t$ . Applying Bramble-Hilbert lemma we get:

$$|\eta_{21}(x, t)| \leq C |a^*|_{W_2^{\lambda, \lambda/2}(E)}, \quad 3/2 < \lambda \leq 2.$$

Going back to the old variables, we have

$$|a^*|_{W_2^{\lambda, \lambda/2}(E)} \leq Ch^{\lambda - \frac{3}{2}} |a|_{W_2^{\lambda, \lambda/2}(e)},$$

and

$$|\eta_{21}(x, t)| \leq Ch^{\lambda - \frac{3}{2}} |a|_{W_2^{\lambda, \lambda/2}(e)}.$$

It's further

$$|\eta_2(x, t)| \leq Ch^{\lambda - 3/2} |a|_{W_2^{\lambda, \lambda/2}(e)} \left\| \frac{\partial u}{\partial x} \right\|_{C(\overline{Q^L})},$$

and analogous estimate is valid on  $Q^R$ . After summation over the mesh  $Q_{h\tau}^+$  we get

$$\|\eta_2\|_{L_2(Q_{h\tau})} \leq Ch^\lambda \left( \|a\|_{W_2^{\lambda, \lambda/2}(Q^L)} \left\| \frac{\partial u}{\partial x} \right\|_{C(\overline{Q^L})} + \|a\|_{W_2^{\lambda, \lambda/2}(Q^R)} \left\| \frac{\partial u}{\partial x} \right\|_{C(\overline{Q^R})} \right).$$

Further, using imbedding

$$\frac{\partial u}{\partial x} \in W_2^{s-1,(s-1)/2}(Q^L) \subset C(\overline{Q^L}), \quad s > 5/2,$$

analogous imbedding on  $Q^R$  and taking  $\lambda = s - 1, 5/2 < s \leq 3$ , finally obtain

$$\|\eta_2\|_{L_2(Q_{ht})} \leq Ch^{s-1}(\|a\|_{W_2^{s-1,(s-1)/2}(Q^L)}\|u\|_{W_2^{s,s/2}(Q^L)} + \|a\|_{W_2^{s-1,(s-1)/2}(Q^R)}\|u\|_{W_2^{s,s/2}(Q^R)}), \quad 5/2 < s \leq 3. \quad (13)$$

We have

$$\eta_3 = 0.5(a + a^+)(T_x^+ T_t^- \frac{\partial u}{\partial x} - u_x) = 0.5(a + a^+)\eta_{31},$$

where  $\eta_{31} = T_x^+ T_t^- \frac{\partial u}{\partial x} - u_x$ . The following estimate is valid:

$$|\eta_{31}(x, t)| \leq \frac{C}{h} \|u^*\|_{W_2^{s,s/2}(E)}, \quad s > 5/2.$$

Further,  $\eta_{31}(x, t) = 0$  whenever  $u^*$  is a polynomial of degree two in  $x^*$ . Applying Bramble-Hilbert lemma we get:

$$|\eta_{31}(x, t)| \leq \frac{C}{h} |u^*|_{W_2^{s,s/2}(E)}, \quad 5/2 < s \leq 3.$$

Going back to the old variables, we have

$$|u^*|_{W_2^{s,s/2}(E)} \leq Ch^{s-\frac{3}{2}} |u|_{W_2^{s,s/2}(e)},$$

and

$$|\eta_{31}(x, t)| \leq Ch^{s-\frac{5}{2}} |u|_{W_2^{s,s/2}(e)}.$$

It's further

$$|\eta_3(x, t)| \leq Ch^{s-5/2} \|a\|_{C(\overline{Q^L})} |u|_{W_2^{s,s/2}(e)}$$

and analogous estimate is valid on  $Q^R$ . After summation over the mesh  $Q_{ht}^+$  we get

$$\|\eta_3\|_{L_2(Q_{ht})} \leq Ch^{s-1}(\|a\|_{C(\overline{Q^L})}\|u\|_{W_2^{s,s/2}(Q^L)} + \|a\|_{C(\overline{Q^R})}\|u\|_{W_2^{s,s/2}(Q^R)}).$$

Using imbeddings  $W_2^{s-1,(s-1)/2}(Q^L) \subset C(\overline{Q^L}), W_2^{s-1,(s-1)/2}(Q^R) \subset C(\overline{Q^R}), s > 5/2$  finally obtain

$$\|\eta_3\|_{L_2(Q_{ht})} \leq Ch^{s-1}(\|a\|_{W_2^{s-1,(s-1)/2}(Q^L)}\|u\|_{W_2^{s,s/2}(Q^L)} + \|a\|_{W_2^{s-1,(s-1)/2}(Q^R)}\|u\|_{W_2^{s,s/2}(Q^R)}), \quad 5/2 < s \leq 3. \quad (14)$$

From estimates (12)- (14) we have

$$\|\eta\|_{L_2(Q_{ht})} \leq Ch^{s-1}(\|a\|_{W_2^{s-1,(s-1)/2}(Q^L)}\|u\|_{W_2^{s,s/2}(Q^L)} + \|a\|_{W_2^{s-1,(s-1)/2}(Q^R)}\|u\|_{W_2^{s,s/2}(Q^R)}), \quad 5/2 < s \leq 3. \quad (15)$$

Now we estimates the term  $\tilde{\mu}$ . At the point  $x \neq \xi$  we have  $\tilde{\mu} = \mu$ . We decompose term  $\mu = \bar{\mu}_1 + \bar{\mu}_2$  where

$$\begin{aligned} \bar{\mu}_1 &= (T_x^2 T_t^- b)(u - T_x^2 T_t^- u), \\ \bar{\mu}_2 &= (T_x^2 T_t^- b)(T_x^2 T_t^- u) - T_x^2 T_t^- (bu). \end{aligned}$$

At the point  $x = \xi$  we decompose term  $\tilde{\mu} = \mu_{(1)}^+ + \mu_{(1)}^- + \mu_{(2)}^+ + \mu_{(2)}^-$  where

$$\mu_{(1)}^\pm = (T_x^{2\pm} T_t^- b) \left[ u - (T_x^{2\pm} T_t^- u) \pm \frac{h}{3} \left( T_t^- \frac{\partial u}{\partial x} \right) \right] \Big|_{x=\xi \pm 0},$$

$$\mu_{(2)}^\pm = \left[ (T_x^{2\pm} T_t^- b)(T_x^{2\pm} T_t^- u) - T_x^{2\pm} T_t^- (bu) \pm \frac{h}{3} ((T_t^- b) - (T_x^{2\pm} T_t^- b) \left( T_t^- \frac{\partial u}{\partial x} \right)) \right] \Big|_{x=\xi \pm 0}.$$

Using the same technique as for terms  $\eta_i$  we can derive estimates of the form (11) for terms  $\bar{\mu}_i$  and  $\mu_{(i)}^\pm$ . On that way we get

$$\|\tilde{\mu}\|_{L_2(Q_{h\tau})} \leq Ch^{s-1} (\|b\|_{W_2^{s-2, s/2-1}(Q^L)} \|u\|_{W_2^{s, s/2}(Q^L)} + \|b\|_{W_2^{s-2, s/2-1}(Q^R)} \|u\|_{W_2^{s, s/2}(Q^R)}), \quad 5/2 < s \leq 3. \tag{16}$$

Let us estimate the term  $\hat{\mu}$ . At the point  $(x, t) \in \xi \times \omega_\tau$  we decompose term  $\hat{\mu} = \hat{\mu}^+ - \hat{\mu}^-$ , where

$$\hat{\mu}^\pm = -\frac{h^2}{3} (T_t^- b(\xi \pm 0, t)) \left( T_t^- \frac{\partial u}{\partial x}(\xi \pm 0, t) \right).$$

The following estimate is valid:

$$|\hat{\mu}^+(\xi + 0, t)| \leq Ch \|b(\xi + 0, \cdot)\|_{L_2(t-\tau, t)} \left\| \frac{\partial u}{\partial x} \right\|_{C(\bar{Q}^R)}.$$

After summation over the mesh  $\omega_\tau$  and using obvious imbeddings we have

$$\|\hat{\mu}^+\|_{L_2(\xi \times \omega_\tau)} \leq Ch^2 \|b(\xi + 0, \cdot)\|_{W_2^{\lambda-1/4}(0, T)} \|u\|_{W_2^{s, s/2}(Q^R)}$$

for  $\lambda > 1/4$  and  $s > 5/2$ . Further, using imbedding  $W_2^{2\lambda, \lambda}(Q^R) \subset W_2^{\lambda-1/4}(0, T)$  and taking  $\lambda = s/2 - 1$ , we obtain

$$\|\hat{\mu}^+\|_{L_2(\xi \times \omega_\tau)} \leq Ch^2 \|b\|_{W_2^{s-2, s/2-1}(Q^R)} \|u\|_{W_2^{s, s/2}(Q^R)}. \tag{17}$$

Analogous estimate is valid for the term  $\hat{\mu}^-$ :

$$\|\hat{\mu}^-\|_{L_2(\xi \times \omega_\tau)} \leq Ch^2 \|b\|_{W_2^{s-2, s/2-1}(Q^L)} \|u\|_{W_2^{s, s/2}(Q^L)}. \tag{18}$$

From estimates (17)-(18) we have

$$\|\hat{\mu}\|_{L_2(\xi \times \omega_\tau)} \leq Ch^2 (\|b\|_{W_2^{s-2, s/2-1}(Q^L)} \|u\|_{W_2^{s, s/2}(Q^L)} + \|b\|_{W_2^{s-2, s/2-1}(Q^R)} \|u\|_{W_2^{s, s/2}(Q^R)}). \tag{19}$$

Also, the following estimates are valid (see [9]):

$$\|\widehat{\phi}\|_{\widetilde{W}_2^{1/2}(\omega_\tau, L_2(\omega_h))} \leq Ch^{s-1} \sqrt{\log \frac{1}{h}} (\|u\|_{W_2^{s, s/2}(Q^L)} + \|u\|_{W_2^{s, s/2}(Q^R)}), \quad 5/2 < s \leq 3, \tag{20}$$

$$\|\widehat{\phi}\|_{\widetilde{W}_2^{1/2}(\omega_\tau, \xi)} \leq Ch^2 \sqrt{\log \frac{1}{h}} (\|u\|_{W_2^{s, s/2}(Q^L)} + \|u\|_{W_2^{s, s/2}(Q^R)}), \quad 5/2 < s \leq 3. \tag{21}$$

Finally from (9) and (15)- (21) we obtain the following result.

**Theorem 3.1.** *Suppose that solution and coefficients of the differential problem (3) satisfy conditions (4)-(6). Then, the solution of the difference scheme (7) converges in  $\widetilde{W}_2^{1, 1/2}(Q_{h\tau})$  to the solution of the differential problem (3) and, assuming that  $\tau \asymp h^2$ , the following estimate is valid:*

$$\|u - v\|_{\widetilde{W}_2^{1, 1/2}(Q_{h\tau})} \leq Ch^{s-1} (\|a\|_{W_2^{s-1, (s-1)/2}(Q^L)} + \|a\|_{W_2^{s-1, (s-1)/2}(Q^R)} + \|b\|_{W_2^{s-2, s/2-1}(Q^L)} + \|b\|_{W_2^{s-2, s/2-1}(Q^R)} + l(h)) \times (\|u\|_{W_2^{s, s/2}(Q^L)} + \|u\|_{W_2^{s, s/2}(Q^R)}), \quad 5/2 < s \leq 3, \tag{22}$$

where  $l(h) = \sqrt{\log 1/h}$ .

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