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ON INELASTICITY OF DAMAGED QUASI-RATE-INDEPENDENT ORTHOTROPIC MATERIALS

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Abstract. The paper deals with a body having a random 3D-distribution of two-phase inclusions: spheroidal mutually parallel voids as well as differently oriented reinforcing parallel stiff spheroidal short fibers. By the effective field approach the effective stiffness fourth-order tensor is formulated and found numerically. Simultaneous and sequential embeddings of inclusions are compared. Damage evolution is described by modified Vakulenko's approach to endochronic thermodynamics. A brief account of the problem of effective elastic symmetry is given. The results of the theory are applied to the damage-elasto-viscoplastic strain of reactor stainless steel AISI 316H.

1. Introduction

In classical texts devoted to the continuum theory of dislocations as the principal source of residual stresses, incompatibility either of plastic strains or quasi-plastic strains (thermal and some others) is considered. The key point is that if volume elements in the natural state space of the body deform freely, i.e. independently of neighbors, then they cannot be connected without residual stresses. While such an approach (promoted originally by Kondo, Kroener, Stojanovic and others) looked very promising in plasticity based on continuum dislocations, the recent papers mainly use an alternative approach of implantations as proposed by Eshelby in the paper [4]. The Eshelbian approach is especially suitable for the description of composites with particulate phases such as either stiff or soft inclusions (more specifically, voids or cracks).

To achieve this aim, first we briefly review some existing self-consistent theories of elastic composites with multiphase structure. Further, damage development is treated by means of Vakulenko's endochronic thermodynamics (cf. [16]) in its extended version [10]. We assume, as in [6], that the considered composite consists

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of three isotropic phases: solid phase matrix, spheroidal oblate voids and spheroidal stiff prolate inclusions. For simplification of numerical analysis, each class of inclusions contains parallel, but randomly distributed spheroids

1.1. Eshelbian approach to eigenstrains. Constrained implanting strains induced by free strains are often termed as "eigenstrains". Constrained and free strains are connected by the famous Eshelby formula:

$$\boldsymbol{\varepsilon}^{\text{constr}} = \mathbb{S}\,\boldsymbol{\varepsilon}^{\text{free}} \tag{1.1}$$

derived in [4]. In the above formula the unconstrained strain $\boldsymbol{\varepsilon}^{\text{free}}$ is related to the implanting "eigenstrain" by the fourth-rank tensor $\mathbb{S}$. All the details concerning determination of the Eshelby tensor for isotropic and anisotropic materials are given in [14] and elsewhere. It is essential that inclusions be ellipsoidal. For other shapes relation (1.1) and the whole next section do not hold.

2. Effective properties tensors

Let stiffness and its inverse be denoted by $\mathbb{D}_\Lambda$, $\mathbb{M}_\Lambda \equiv \mathbb{D}_\Lambda^{-1}$, $(\Lambda \in \{0, c, f\})$ for matrix, voids and fibers respectively. Then by means of the notation $\delta\mathbb{D}(x) \equiv \mathbb{D}(x) - \mathbb{D}_0$, $\delta\mathbb{M}(x) \equiv \mathbb{M}(x) - \mathbb{M}_0$, and by the characteristic function

$$V(x) = \sum_{k=1}^{N} V_k(x) = \begin{cases} 1, & x \in v, \\ 0, & x \notin v, \end{cases}$$

for N inclusions we have

$$\boldsymbol{\varepsilon}(x) = \boldsymbol{\varepsilon}_0 - \mathbb{K}_0^{\varepsilon} * (\delta\mathbb{D}\boldsymbol{\varepsilon}V), \tag{2.1}$$

and

$$\boldsymbol{\sigma}(x) = \boldsymbol{\sigma}_0 + \mathbb{K}_0^{\sigma} * (\delta\mathbb{M}\boldsymbol{\sigma}V) \tag{2.2}$$

with a compact writing $(\mathbb{K} * \mathcal{A})(x) \equiv K(x-y)A(y)dy$.

The above two kernels are introduced by means of the Green function of the matrix $\mathbf{G}_0$ by Kunin's notation (cf.[8]) $\mathbb{K}_0^{\varepsilon} \equiv -\operatorname{def}\mathbf{G}_0\operatorname{def}$ and $\mathbb{K}_0^{\sigma} = \mathbb{D}_0\mathbb{K}_0^{\varepsilon}\mathbb{D}_0 - \mathbb{D}_0\delta(x)$ where total (linear) strain expressed by the displacement reads $\boldsymbol{\varepsilon} = \operatorname{def}\mathbf{u}$ and $\delta(x)$ is the delta function. Here $(\operatorname{def}\mathbf{u})_{ij} = (\nabla_i u_j + \nabla_j u_i)/2$ is the symmetrized gradient of the displacement vector and ∇_i stands for the covariant derivative.

2.1. Effective stiffness. Effective stiffness tensor is defined by spatial averaging as follows. Micro formulation of Hooke's law in the case of a thermoelastic deformation leads to the definition of the effective stiffness:

$$\langle\boldsymbol{\sigma}\rangle = \mathbb{D}^{\text{eff}}\langle\boldsymbol{\varepsilon}_e\rangle,$$

where $\boldsymbol{\varepsilon}_{ke} = \boldsymbol{\varepsilon}_k - \boldsymbol{\alpha}_k\theta$ for a point $x \in v_k$ and $\langle F\rangle := 1/v\int_v F(x)dx$ due to the ergodic hypothesis.

Introducing the notation

$$\boldsymbol{\varepsilon}_*(x_k) := \boldsymbol{\varepsilon}_0 - \sum_{m\neq k} \mathbb{K}_0^{\varepsilon} * (\delta\mathbb{D}\boldsymbol{\varepsilon}V_m), \tag{2.3}$$

we may write for a k-th inclusion a nonlocal formula which has the same form as the corresponding formula for a continuum with a single inclusion i.e.:

$$\boldsymbol{\varepsilon}(x_k) + \int_{v_k} \mathbb{K}_0^{\varepsilon}(x - x')\delta D(x')\boldsymbol{\varepsilon}(x')dx' = \boldsymbol{\varepsilon}_*(xk). \tag{2.4}$$

The two principal hypotheses of the *effective field method* are:

a) $\boldsymbol{\varepsilon}_*(x) = \text{const}$ for $|x - x_k|$ inside V_k-inclusion and
b) $\boldsymbol{\varepsilon}_*(x_k)$ is statistically independent on the positions and shapes of the other inclusions V_m $(m \neq k)$.

Then, due to linearity of governing equations it is possible to write: $\boldsymbol{\varepsilon}(x) = \mathbb{L}_k(x)\boldsymbol{\varepsilon}_*$, for $x \in v_k$, where, for the k-th inclusion $\mathbb{L}_k = (\mathbb{I} + \mathbb{A}_k\delta\mathbb{D}_k)^{-1}$ with

$$\mathbb{A}_k = \int_{v_k} \mathbb{K}_0^{\varepsilon}(x)dx \equiv \mathbb{S}(a_k)\mathbb{D}_0^{-1}, \tag{2.5}$$

$\mathbb{S}(a_k)$ being Eshelby's fourth-rank tensor for the considered inclusion. Let concentration of inclusions be c. Following [6], let us introduce a *correlation function* by means of:

$$\mathbb{A}^{\Phi} = \int_{v_k} \mathbb{K}_0^{\varepsilon}(x)\Phi(x)dx, \tag{2.6}$$

where the scalar function

$$\Phi(x - x') := 1 - \frac{1}{c}\left\langle \sum_{m \neq k} V_m(x') \mid x \right\rangle, \quad x \in V_k$$

obtained by averaging is defined by the shape of the correlation hole. This function shows how an inclusion situated at a place x' acts on a particle at a place x (see the right side of Figure 1 where the region of acting is limited to $r \in (0, 2)$).

By means of this function we get the effective stiffness for a single family of inclusions (built from the same material):

$$\mathbb{D}^{\text{eff}} = \mathbb{D}_0 + c(\langle\delta\mathbb{D}\mathbb{L}\rangle^{-1} - c\mathbb{A}^{\Phi})^{-1}. \tag{2.7}$$

If all the inclusions are the same (shape, orientation and elastic properties) with $\mathbb{A}_k \equiv \mathbb{A}(a)$ then (2.7) simplifies into:

$$\mathbb{D}^{\text{eff}} = \mathbb{D}_0 + c(\delta\mathbb{D}^{-1} + \mathbb{A}(a) - c\mathbb{A}^{\Phi})^{-1}, \tag{2.8}$$

which in the most special case when $\mathbb{A}(a) = \mathbb{A}^{\Phi}$ (the correlation gap and inclusion have the same aspect ratios) leads to the Mori–Tanaka formula.

Let us note that due to linearity of (2.4) it is possible to solve it by $\boldsymbol{\varepsilon}_{(k)} = \mathbb{F}_k\boldsymbol{\varepsilon}_0$, $k \in 1, \ldots, N$ and get in such a way the effective stiffness tensor $\mathbb{D}^{\text{eff}}$. Details are given in [**12**]. The calculation in [**12**] was done under the hypothesis of ellipsoidal symmetry for the distribution of the inclusions. The integration ellipsoids exclude overlapping of ellipsoidal inclusions. Explicit results that these authors gave hold for spheroidal inclusions immersed into an isotropic matrix.

In the paper [**6**], the authors considered pure elasticity of a composite with isotropic matrix possessing two families of mutually parallel inclusions (spheroids and cylinders). We will use their results to our subject of interest: prolate spheroidal voids and oblate spheroidal fibres (denoted by indices c and f respectively). From their analysis Boolean distribution of inclusions is preferable. They considered two typical cases of generation priority:

- Simultaneous generation of both families when one family has priority. In our case it is logical to take that fibres have priority since voids could not introduce restriction on fibre appearance.
- Sequential generation when the second family of voids is delayed after the Boolean generation of fibres. As the authors in [6] pointed out, such an order is recommended when the fibre concentration c_f is considerably greater than the void concentration c_c.

The correlation hole here has the same form and orientation as the spheroidal inclusion itself. The above-mentioned Φ-correlation functions have a smooth ending at the relative distance $r_C = 2$, which is two times larger than the distance from the centre of the inclusion to its boundary. Thus, outside of the hole for $r > 2$ the considered grain does not interact with other grains (cf. Figure 1).

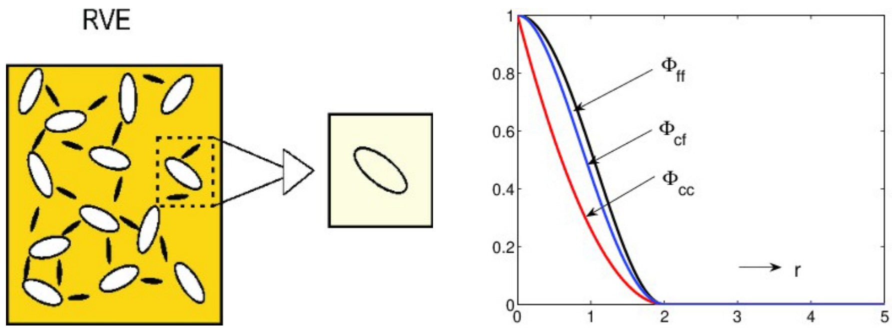

FIGURE 1. Notion of the representative volume element and Kanaun–Jeulin interactions.

2.2. Symmetry groups in presence of ellipsoidal inclusions. The material symmetry group $\aleph$ of an elastic anisotropic material with Hooke's tensor $\mathbb{D}$ is defined by all orthogonal second-order tensors $\mathbf{H}$ satisfying the relationship: $\mathbb{D} = \mathbf{H}\Diamond\mathbb{D}$, $(\mathbf{H} \in \aleph)$, where the Rayleigh product explicitly reads: $(\mathbf{H}\Diamond\mathbb{D})_{klmn} \equiv (\mathbf{H})_{ka}(\mathbf{H})_{lb}(\mathbf{H})_{mc}(\mathbf{H})_{nd}(\mathbb{D})_{abcd}$. A similar relationship holds true for the thermal expansion tensor (cf. [12]). We now state the problem of overall symmetry for a representative volume element.

Overall symmetry definition. Given elastic symmetries of the matrix and N ellipsoidal inclusions (whose semiaxes are defined by the rotation tensors $\mathbf{R}_\Lambda$) are as follows: $\mathbb{D}_\Lambda = \mathbf{H}^e_\Lambda\Diamond\mathbb{D}_\Lambda$, with $\mathbf{H}^e_\Lambda \in \aleph^e_\Lambda$, $(\Lambda \in \{0, 1, \ldots N\})$ find 2-tensor $\mathbf{H}^e_{\text{eff}} \in \aleph^e_{\text{eff}}$ such that

$$\mathbb{D}_{\text{eff}} = \mathbf{H}^e_{\text{eff}}\Diamond\mathbb{D}_{\text{eff}}, \quad \mathbf{H}^e_{\text{eff}} \in \aleph^e_{\text{eff}} \tag{2.9}$$

holds. The group $\aleph^e_{\text{eff}}$ is then called *effective elastic symmetry group.*

Obviously, the real task is to find $\aleph^e_{\text{eff}}$ when $\aleph^e_\Lambda$, $(\Lambda \in \{0, 1, \dots N\})$, is given. An appealing and the simplest approach would be to use the orientation distribution function ω^{ODF} (ODF) by statistical averaging in the following way:

$$\langle F\rangle = \int_{SO(3)} F(\mathbf{R})\omega^{\text{ODF}}(\mathbf{R})d\aleph, \quad F \equiv \langle F\rangle + \delta F,\ F \in \{\mathbb{D}, \sigma, \varepsilon\}. \tag{2.10}$$

Then, using the definition of the effective stiffness tensor, we get the expression for the effective stiffness:

$$\mathbb{D}^{\text{eff}} = \langle \mathbb{D}\rangle + \langle \delta\mathbb{D}\,\delta\varepsilon\rangle\langle\varepsilon\rangle^{-1}. \tag{2.11}$$

Explicit structure of (2.11) depends on the topology and materials of the matrix and inclusions. However, in all the cases we have the dependence

$$\mathbb{D}^{\text{eff}} = \mathbb{D}^{\text{eff}}(\mathbb{D}_0, \omega^{\text{ODF}}(\mathbf{R}), a_1, \dots, a_N, \mathbf{R}_1, \dots, \mathbf{R}_N),$$

as should be expected. For calculation of the above indicated mean values of strain and stiffness the usual procedure is to develop the ODF function into a series over generalized spherical functions (cf. [14]).

Before proceeding with general symmetry issues, let us consider now some characteristic distributions of inclusions immersed into a matrix. While orientations and shapes of inclusions of constituting diverse subgroups are fixed, their translational distributions inside each subgroup are random.

For the sake of a simple estimation of anisotropy degree induced by inclusions, let us recall Schouten's harmonic decomposition (cf. [7])

$$\mathbb{D}^{\text{eff}} = \mathbb{D}^{\text{eff}}_{\text{iso}} + \mathbb{D}^{\text{eff}}_{an}. \tag{2.12}$$

Here the isotropy elasticity tensor $\mathbb{D}^{\text{eff}}_{\text{iso}}$ is the nearest to $\mathbb{D}^{\text{eff}}$ and the anisotropy elasticity tensor $\mathbb{D}^{\text{eff}}_{an}$ is their difference. By means of the identity 2-tensor $\mathbf{I}$ and identity 4-tensor $\mathbb{I}$ (satisfying $\mathbf{IA} = \mathbf{A}$ and $\mathbb{IA} = \mathbb{A}$) we have for the isotropic part (cf. [7]):

$$\mathbb{D}^{\text{eff}}_{\text{iso}} = K^{\text{eff}}\mathbf{I}\otimes\mathbf{I} + 2\mu^{\text{eff}}\mathbb{I}. \tag{2.13}$$

Herein K^{eff} is the effective bulk modulus and μ^{eff} is the effective shear modulus

$$K^{\text{eff}} = \frac{1}{15}(6tr\mathbf{I}\mathbb{D}^{\text{eff}}\mathbf{I} - Tr\mathbb{D}^{\text{eff}}) = \frac{1}{15}(6\mathbb{D}^{\text{eff}}_{aabb} - \mathbb{D}^{\text{eff}}_{abab}),$$
$$\mu^{\text{eff}} = \frac{1}{30}(3Tr\mathbb{D}^{\text{eff}} - tr\mathbf{I}\mathbb{D}^{\text{eff}}\mathbf{I}) = \frac{1}{30}(3\mathbb{D}^{\text{eff}}_{abab} - \mathbb{D}^{\text{eff}}_{aabb}).$$

Let us introduce 6×6 matrices $[\mathbb{D}^{\text{eff}}_{an}]$ and $[\mathbb{D}^{\text{eff}}_{\text{iso}}]$ by Kelvin notation in the way often done in anisotropic elasticity. Now, the simplest way is to introduce the anisotropy factor ζ^{an} by norms of these matrices as follows:

$$\zeta^{an} := \frac{||\mathbb{D}^{\text{eff}}_{an}||}{||\mathbb{D}^{\text{eff}}_{\text{iso}}||}. \tag{2.14}$$

For a detailed analysis of the anisotropic part $\mathbb{D}^{\text{eff}}_{an}$ it is necessary to employ the

Novozhilov second-rank tensors $\boldsymbol{\nu}^1_{ij} = \mathbb{D}^{\mathrm{eff}}_{ijaa}$ and $\boldsymbol{\nu}^2_{ij} = \mathbb{D}^{\mathrm{eff}}_{iaja}$. An alternative way is to apply the so-called spectral decomposition (cf. [7]) by an analysis of eigenvectors and eigenvalues of the 6×6 matrix $[\mathbb{D}^{\mathrm{eff}}]$.

Example 2.1. First, suppose that a matrix is weakened by some identical parallel spheroidal voids with the symmetry axis aligned with a Cartesian axis $z_3 = x_3$. Suppose now that two thirds of voids population are rotated by some angle θ_1 around an axis z_1, whereas the remaining one third is rotated by $\theta_2 = -\theta_1 = \pi/6$ around the same axis. Then concentrations of voids are $c_{c1} = 2c_{c2}$. Aspect ratios are the same: $\gamma_{c1} = a_1/a_3 = \gamma_{c2} = a_2/a_3$. In this way, we obtain a composite with a planar symmetry with a mirror axis $x_1 = z_1$ and one family of voids with two subfamilies of parallel identical voids. Otherwise voids inside each subgroup are randomly distributed.

Clearly, speaking here about two families of inclusions and a simultaneous approach is not necessary since in both subgroups inclusions are identical, but differently oriented. Applying the formulae (2.9)–(2.11) for $N = 2$ with tensors $\mathcal{A}_k(z) = \mathbf{R}_k \diamondsuit \mathcal{A}_k(x)$, $k \in \{c1, c2\}$, we obtain results for the stiffness tensor which are best represented by a monoclinic group. Here, for brevity of notation $(\mathbf{R} \diamond \mathcal{A})_{\alpha\beta\gamma\delta} \equiv (\mathbf{R})_{\alpha a}(\mathbf{R})_{\beta b}(\mathbf{R})_{\gamma c}(\mathbf{R})_{\delta d}(\mathcal{A})_{abcd}$ is introduced. When the axis of reflexive symmetry is $z_1 = x_1$ with invariance to the transformation $x \mapsto x^*$ and $x_1^* = -x_1$, $x_2^* = x_2$, $x_3^* = x_3$, then the corresponding effective stiffness tensor of the monoclinic group is met. In order to show elastic material symmetry in the simplest way, we calculated the effective Young modulus as a function of direction by means of $E^{\mathrm{eff}}(n) = n \diamond \mathbb{D}^{\mathrm{eff}}$. In the following subsection the other smaller angles between the two subgroups of voids are analyzed.

2.3. Elasticity and damage tensors caused by ellipsoidal inclusions. In the majority of papers dealing with composites where the matrix is reinforced either with one or two distinct groups of inclusions, simple distributions are considered. Typical distributions are: (a) random with a constant *orientation distribution function* (ODF) where all orientations are equally probable, (b) parallel (but randomly distributed) ellipsoidal inclusions. Neither is the case in real situations. Namely, if inclusions are randomly distributed thin ellipsoidal voids, then during either compression or tension some of them either close or open in accordance with the direction of applied stress. Thus (mechanical as well as thermal) anisotropy appears. Such dissymmetry effects, ODF, and corresponding correlation holes are analyzed in detail in [11].

Let us consider a body whose Young modulus and Poisson ratio of the virgin matrix are taken to be $E_0 = 100\,\mathrm{GPa}$ and $\nu_0 = 0.3$, whereas the aspect ratio amounts to $1/200$. In the sequel we introduce 6×6 symmetric matrices of stiffness formed by indices equivalence $\{11, 22, 33, 23, 31, 12\} \mapsto \{1, 2, 3, 4, 5, 6\}$.

If instead of $\pi/3$ the angle of disorientation of the two subgroups of voids is much smaller, say (a) $2\pi/36$, (b) $2\pi/216$, (c) $2\pi/1296$, then instead of monoclinic we arrive approximately at orthotropic symmetry with a small error. This can be seen from the following results for effective stiffness of a composite with two families of disoriented voids:

$$[\mathcal{D}]^{\mathrm{ISO}}_{\theta=\pi/3} = \left\{\begin{array}{cccccc} 106.4477 & 63.2435 & 63.2435 & 0 & 0 & 0 \\ & 106.4477 & 63.2435 & 0 & 0 & 0 \\ & & 106.4477 & 0 & 0 & 0 \\ & & & 43.2042 & 0 & 0 \\ & & & & 43.2042 & 0 \\ & & & & & 43.2042 \end{array}\right\},$$

$$[\mathcal{D}]^{\mathrm{eff}}_{\theta=\pi/3} = \left\{\begin{array}{cccccc} 121.5701 & 57.7953 & 58.3116 & -0.1962 & -0.0000 & -0.0000 \\ & 103.6423 & 72.6828 & 5.0261 & 0.0000 & -0.0000 \\ & & 96.0126 & -2.1262 & -0.0000 & 0.0000 \\ & & & 45.9620 & -0.0000 & 0.0000 \\ & & & & 29.3102 & 6.2215 \\ & & & & & 52.4587 \end{array}\right\},$$

$$[\mathcal{D}]^{\mathrm{eff}}_{\theta=\pi/18} = \left\{\begin{array}{cccccc} 121.5702 & 57.5450 & 58.5619 & -0.0393 & -0.0000 & -0.0000 \\ & 120.8868 & 59.1372 & 1.7031 & 0.0000 & -0.0000 \\ & & 105.8594 & -1.1216 & -0.0000 & -0.0000 \\ & & & 18.8711 & -0.0000 & 0.0000 \\ & & & & 18.0879 & 1.2475 \\ & & & & & 63.6813 \end{array}\right\}.$$

These matrices of effective elasticity are calculated by the homogenization procedure of subsection 2.1 taking two groups of very flattened voids (aspect ratio $1/200$) mutually inclined by $\pi/3, \dots \pi/648$.

Comparing these matrices, we can see that in the case of slight disorder D_{14}, D_{24}, D_{34}, D_{56} are nearly equal to zero, which is the feature of the orthotropic stiffness matrix. Moreover, such an orthotropy is characterized also by a reduced number of constants since the axes x_1 and x_2 in this special case become almost indistinguishable.

$$[\mathcal{D}]^{\mathrm{eff}} \approx \left\{\begin{array}{cccccc} D11 & D12 & D13 & 0 & 0 & 0 \\ & D11 & D13 & 0 & 0 & 0 \\ & & D33 & 0 & 0 & 0 \\ & & & D44 & 0 & 0 \\ & & & & D44 & 0 \\ & & & & & D11-D12 \end{array}\right\} \equiv \left\{\begin{array}{cccccc} a & b & e & 0 & 0 & 0 \\ & a & e & 0 & 0 & 0 \\ & & c & 0 & 0 & 0 \\ & & & d & 0 & 0 \\ & & & & d & 0 \\ & & & & & a-b \end{array}\right\}.$$

When these elasticity tensors are calculated, the corresponding anisotropy factors and angles of effective elastic symmetry are determined by the ODF analysis explained above. The results are

$$\begin{array}{llll} \zeta^{an}_{\pi/3} = 0.0806, & \theta^{\mathrm{eff}}_{\pi/3} = -0.0670, & \zeta^{an}_{\pi/18} = 0.108, & \theta^{\mathrm{eff}}_{\pi/18} \approx 0, \\ \zeta^{an}_{\pi/108} = 0.1093, & \theta^{\mathrm{eff}}_{\pi/108} \approx 0, & \zeta^{an}_{\pi/648} = 0.1093, & \theta^{\mathrm{eff}}_{\pi/648} \approx 0. \end{array}$$

Let us comment on damage measures. Comparing the above matrices with the stiffness matrix of the isotropic matrix, we can see that the most natural definition of damage by means of:

$$\Omega_{\Gamma\Lambda} := 1 - \frac{D^{\text{eff}}_{\Gamma\Lambda}}{D^{0}_{\Gamma\Lambda}}, \quad \Gamma \in \{1, 2, \ldots 6\} \tag{2.15}$$

may be applied only for components of $[D]^{\text{ISO}}$ different from zero, i.e.,

$$D^{0}_{\Gamma\Lambda} \neq 0, \quad (\Gamma, \Lambda) \in \{(1,1), (1,2), \ldots, (3,3), (4,4), (5,5), (6,6)\}$$

In order to show the elastic material symmetry in the simplest way we calculated the Young modulus as a function of m-direction by means of $E^{\text{eff}}(m) = m \diamond \mathbb{D}^{\text{eff}}$ (cf. Fig. 2 and Fig. 3).

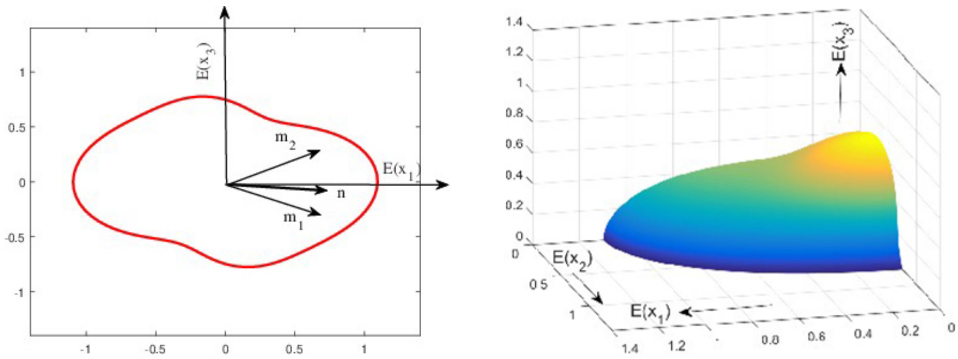

FIGURE 2. Spatial distribution of the Young modulus $(E(m)/E_0)$ for two subgroups of parallel voids (normal m_1, concentration $c_{c1} = 0.2$) and (normal m_2, concentration $c_{c2} = 0.1$) mutually rotated by $\pi/3$.

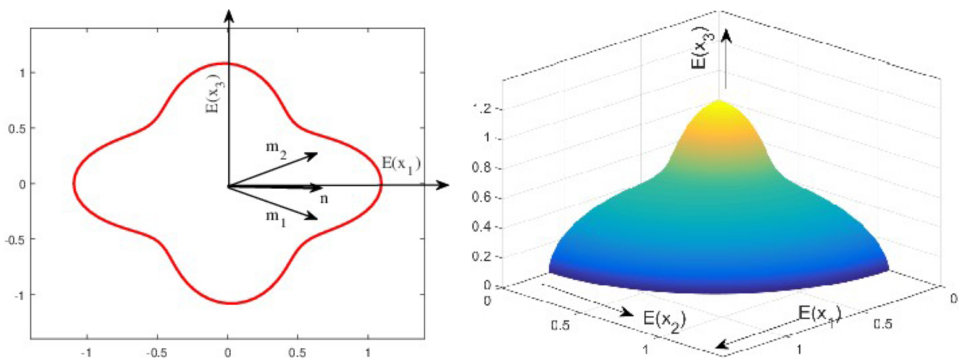

FIGURE 3. Spatial distribution of the Young modulus $(E(m)/E_0)$ for two subgroups of parallel voids (normal m_1, concentration $c_{c1} = 0.2$) and (normal m_2, concentration $c_{c2} = 0.1$) mutually rotated by $\pi/18$.

In these figures, the ratio of the effective Young modulus and Young modulus of the matrix i.e. $E^{\text{eff}}(m)/E_0$ is shown. The figures for $\theta = \pi/108$ and $\theta = \pi/648$ are omitted since they are practically the same as for $\theta = \pi/18$. This ratio is depicted as radius vector for all m-directions in the $E(x_1)$, $E(x_2)$, $E(x_3)$ space.

Damage matrices, calculated in this way for subgroup disorientation angles, $\{\pi/3, \pi/18, \pi/108, \pi/648\}$, are given below

$$[\mathbf{\Omega}]_{\theta=\pi/3} = \left\{ \begin{array}{cccccc} 0.0969 & -0.0018 & -0.0107 & 0 & 0 & 0 \\ & 0.2301 & -0.2598 & 0 & 0 & 0 \\ & & 0.2868 & 0 & 0 & 0 \\ & & & 0.4025 & 0 & 0 \\ & & & & 0.6190 & 0 \\ & & & & & 0.3180 \end{array} \right\},$$

$$[\mathbf{\Omega}]_{\theta=\pi/18} = \left\{ \begin{array}{cccccc} 0.0969 & 0.0026 & -0.0151 & 0 & 0 & 0 \\ & 0.1020 & -0.0250 & 0 & 0 & 0 \\ & & 0.2136 & 0 & 0 & 0 \\ & & & 0.7547 & 0 & 0 \\ & & & & 0.76490 & \\ & & & & & 0.1721 \end{array} \right\},$$

Inspecting the above elasticity as well as damage matrices, we can see that in all the cases except $\theta = \pi/3$ they are very close to transverse isotropy. The nondiagonal elements of damage matrices are then negligible and elements connecting shear strains and shear stresses, namely, Ω_{44}, Ω_{55}, Ω_{66} are too large and describe shear strains caused by shear stresses on the assumption that voids preserve original orientations and aspect ratios. Thus, by now, the reliable damage components are Ω_{11}, $\Omega_{22} \approx \Omega_{11}$ and Ω_{33}. Taking this into account, we will restrict our further consideration to elastic transverse isotropy for $\theta \leqslant \pi/18$.

3. Evolution equations by endochronic thermodynamics

Let us now try to characterize damage evolution making use of the thermodynamics of irreversible processes following [16] and [10]. In this section we will restrict our attention either to a single family of parallel inclusions or to the sequential approach to finding effective constants. Staying in the range of small strains, for thermoinelastic strain of the considered composite we have:

$$\boldsymbol{\varepsilon}_e = \mathbb{M}^{\text{eff}} \boldsymbol{\sigma} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\text{in}} - \boldsymbol{\alpha}^{\text{eff}}(T - T_0), \tag{3.1}$$

where $\boldsymbol{\varepsilon}^{\text{in}}$ is *damage-plastic strain*, $\mathbb{M}^{\text{eff}}$ is inverse of $\mathbb{D}^{\text{eff}}$, $T - T_0$ the temperature increment and $\boldsymbol{\alpha}^{\text{eff}}$ the effective thermal expansion tensor (determined for two families of fibres and voids in [12]). Here and in the sequel, for the sake of simpler writing, we drop brackets in $\langle \boldsymbol{\sigma} \rangle$ and $\langle \boldsymbol{\varepsilon} \rangle$. Strictly speaking, if the damage-plastic strain changes with time, then the process is thermo-inelastic while $\boldsymbol{\varepsilon}^{\text{in}} = \text{const}$ corresponds to an elastic region. It is assumed (cf. [12]) that the internal energy

U has a purely elastic part U^e and an inelastic part U^Π associated with defects. In our case the defects are spheroidal voids. Then

$$U = U^e + U^\Pi = \frac{1}{2}\boldsymbol{\sigma}\mathbb{M}^{\text{eff}}(\boldsymbol{\rho})\boldsymbol{\sigma} + U^\Pi, \tag{3.2}$$

where $\boldsymbol{\rho}$ is the *defect density* 2-tensor of Mark Kachanov [9]. Now the first law of thermodynamics reads:

$$dU = \boldsymbol{\sigma}\, d\boldsymbol{\varepsilon} + dQ. \tag{3.3}$$

Introducing the *dissipation coefficient* Π (where $0 \leqslant \Pi < 1$) by means of:

$$dQ = -(1-\Pi)\boldsymbol{\sigma}\, d\boldsymbol{\varepsilon}^{\text{in}}, \tag{3.4}$$

we have:

$$dU^\Pi = \Pi\boldsymbol{\sigma}\, d\boldsymbol{\varepsilon}^{\text{in}} + \frac{1}{2}\boldsymbol{\sigma}\, d\mathbb{M}^{\text{eff}}(\boldsymbol{\rho})\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\alpha}^{\text{eff}} dT. \tag{3.5}$$

On the other hand, the second law of thermodynamics in our case has the form (with the absolute temperature T):

$$dF = \boldsymbol{\sigma}\, d\boldsymbol{\varepsilon} - S\, dT - \Big(\aleph + \frac{1}{T^2}\boldsymbol{q}\,\text{grad}\, T\Big) T\, dt, \tag{3.6}$$

where $\aleph > 0$ is total dissipation, F is free energy, S - entropy, t - time, while the term in brackets is inelastic dissipation denoted by $\aleph^i = \aleph + (\frac{1}{T^2})\boldsymbol{q}\,\text{grad}\, T$.

Following [16] and [10], we introduce *thermodynamic time* by the next hereditary function

$$\zeta(t) := \int_0^t (\aleph^i(t'))^a dt'. \tag{3.7}$$

It is shown in [10] that the exponent a is of great importance since it shows the speed of ageing. For example, $a < 1$ may be named *decelerated ageing*, whereas $a > 1$ would be defined as *accelerated ageing*. By such a classification Vakulenko's value $a = 1$ might be termed as *steady ageing*. The function $\zeta(t)$ is piecewise continuous and nondecreasing in the way that $d\zeta(t)/dt = 0$ within elastic ranges and $d\zeta(t)/dt > 0$ when inelastic deformation takes place. Splitting the whole time history into a sequence of infinitesimal segments, Vakulenko represented the inelastic strain tensor as a functional of stress and stress rate history:

$$\boldsymbol{\varepsilon}^{\text{in}}(\zeta) \equiv 0\zeta\boldsymbol{\Delta}(\zeta - \xi, \boldsymbol{\sigma}(\boldsymbol{\xi}))d\xi. \tag{3.8}$$

For simplicity, suppose that $\partial\boldsymbol{\Delta}/\partial\zeta = \mathbf{0}$. Then

$$\frac{d\boldsymbol{\varepsilon}^{\text{in}}}{dt} = \boldsymbol{\Delta}(0, \boldsymbol{\sigma})\ (\aleph^i)^a. \tag{3.9}$$

Suppose, moreover, that an associate flow rule holds i.e. that $\boldsymbol{\Delta}(0, \boldsymbol{\sigma}) = \partial\Omega/\partial\boldsymbol{\sigma}$. Then

$$\frac{d\boldsymbol{\varepsilon}^{\text{in}}}{dt} = (\aleph^i)^a \partial\Omega/\partial\boldsymbol{\sigma} \tag{3.10}$$

Let $dU^\Pi \equiv \boldsymbol{\sigma} d\boldsymbol{\varepsilon}^{\text{in}}$. Usually free energy is assumed to have the form: $F = F^e(\boldsymbol{\varepsilon}_e, \theta) + F^\Pi(U^\Pi, \theta)$. Then, by means of the notation $\bar{\Pi} \equiv \Pi + \partial F/\partial U^\Pi$, taking into account that $\boldsymbol{\rho} = \rho m \otimes m$ and inserting all these expressions into the second law (3.6) as well as the first law of thermodynamics (3.3), we arrive at the expression for inelastic dissipation

$$\aleph^i = (1-\bar{\Pi})\boldsymbol{\sigma}\frac{d\boldsymbol{\varepsilon}^{\text{in}}}{dt} + \Big(\frac{dU^{\Pi}}{d\rho} + \frac{1}{2}\boldsymbol{\sigma}\frac{d\mathbb{M}^{\text{eff}}}{d\rho}\boldsymbol{\sigma}\Big)\frac{d\rho}{dt}. \tag{3.11}$$

Time rate of $\boldsymbol{\rho}$ is determined for the case of transverse isotropy in [14]. The orthotropic symmetry will be considered below.

3.1. Orthotropic QRI materials. When the material body possesses three preferred anisotropy directions, then the arguments of the evolution equation have to include the diadics $\mathbf{M}_k = \mathbf{m}_k \otimes \mathbf{m}_k$, $k = 1, 2, 3$. If $\mathbf{m}_k$ are unit vectors then $tr\mathbf{M}_k = 1$. Thus,

$$\mathbf{D}_P = \Lambda \partial_{\boldsymbol{\sigma}} \Omega(\sigma, \mathbf{e}_P, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3) \tag{3.12}$$

with $\Omega = \Omega(\boldsymbol{\sigma}, \mathbf{e}_P, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3)$. We restrict our consideration here to a reduced set of invariants to be used as the source of tensor generators (notation $\boldsymbol{\sigma}_d$ stands for the stress deviator):

$$s_k = tr\boldsymbol{\sigma}\mathbf{M}_k, \quad s_{k+3} = tr\boldsymbol{\sigma}^{\mathbf{2}}\mathbf{M}_k, \qquad k = 1, 2, 3, \tag{3.13}$$

omitting eigen and mixed invariants of the inelastic strain tensor. Suppose now that Ω is a polynomial of third order in $\boldsymbol{\sigma}$. Then the loading function has the following simple form (material constants $a_1, \ldots, a_9, b_1, \ldots, b_{13}$ could depend on inelastic strain):

$$\begin{aligned} 2\Omega = {} & a_1 s_1^2 + a_2 s_2^2 + a_3 s_3^2 + a_4 s_4 + a_5 s_5 + a_6 s_6 + a_7 s_1 s_2 + a_8 s_2 s_3 + a_9 s_3 s_1 \\ & + b_1 s_1^3 + b_3 s_3^3 + b_2 s_2^3 + b_4 s_1 s_4 + b_5 s_1 s_5 + b_6 s_1 s_6 + b_7 s_2 s_4 \\ & + b_8 s_2 s_5 + b_9 s_2 s_6 + b_{10} s_3 s_4 + b_{11} s_3 s_5 + b_{12} s_3 s_6 + b_{13} s_7 \end{aligned} \tag{3.14}$$

and the evolution equation reads:

$$\begin{aligned} \frac{1}{\Lambda}\frac{d\boldsymbol{\varepsilon}^{\text{in}}}{dt} = {} & \mathbf{M}_{1d}(2a_1 s_1 + a_7 s_2 + a_9 s_3 + 3b_1 s_1^2 + b_4 s_4 + b_5 s_5 + b_6 s_6) \\ & + \mathbf{M}_{2d}(2a_2 s_2 + a_7 s_1 + a_8 s_3 + 3b_2 s_2^2 + b_7 s_4 + b_8 s_5 + b_9 s_6) \\ & + \mathbf{M}_{3d}(2a_3 s_3 + a_8 s_2 + a_9 s_1 + 3b_3 s_3^2 + b_{10} s_4 + b_{11} s_5 + b_{12} s_6) \\ & + (\mathbf{M}_1\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{M}_1)_d(a_4 + b_4 s_1 + b_7 s_2 + b_{10} s_3) \\ & + (\mathbf{M}_2\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{M}_2)_d(a_5 + b_5 s_1 + b_8 s_2 + b_{11} s_3) \\ & + (\mathbf{M}_3\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{M}_3)_d(a_6 + b_6 s_1 + b_9 s_2 + b_{12} s_3) + 3b_{13}(\boldsymbol{\sigma}^2)_d. \end{aligned} \tag{3.15}$$

In [13, Chapter 5] it was reported that, based on dynamic tests on stainless steels AISI 316H performed in JRC, Ispra, the function Λ was calibrated to have the form

$$\Lambda = \eta(\sigma_{eq} - Y)\Big(\frac{\sigma_{eq}}{Y_0} - 1\Big)^{\lambda}\frac{d\sigma_{eq}}{dt}\exp(-M). \tag{3.16}$$

Here Y is the dynamic initial equivalent yield stress, Y_0 is its static counterpart, $\eta(x)$ is Heaviside's function, λ is a material constant and M is the material constant. It is worthy of note that inserting (3.16) into (3.15) leads to an evolution equation of incremental form seemingly characteristic for rate-independent materials. At first sight the evolution equation for plastic stretching looks rate-independent since it can be transformed into an incremental equation if it is multiplied by an infinitesimal time increment. However, the rate dependence appears in the stress-rate dependent value of the initial yield stress Y which has a triggering role for inelasticity onset. This phenomenon has been detected during dynamic straining experiments where yield stress is higher at higher stress rates. The model could be termed *quasi-rate-independent*. It is remarkable that the constant M covers inelastic behavior of AISI 316 stainless steel at multiaxial stress histories and strain rates from $10^{-3}s^{-1}$ to $10^{3}s^{-1}$ [1].

On the other hand, a comparison of (3.15) with (3.10) reveals that this function Λ is proportional to the inelastic dissipation $\aleph^{i}$.

3.2. Classical J2 theory of orthotropic materials. In the classical theory of plasticity of orthotropic materials the evolution equation is based on *Hill's yield function*:

$$\frac{3h}{2}(f+1) = F(\sigma_2-\sigma_3)^2 + G(\sigma_3-\sigma_1)^2 + H(\sigma_1-\sigma_2)^2 \tag{3.17}$$

and *equivalent inelastic strain* $\varepsilon^{\text{in}}_{eq} := \int_0^t \|d\boldsymbol{\varepsilon}^{\text{in}}(\tau)/d\tau\| d\tau$ such that the corresponding evolution equation reads:

$$\frac{d\boldsymbol{\varepsilon}^{\text{in}}}{dt} = \partial_{\boldsymbol{\sigma}} f = \frac{1}{2h(\varepsilon^{\text{in}}_{eq})}\partial_{\boldsymbol{\sigma}}\sigma^2_{eq}. \tag{3.18}$$

A comparison of QRI with J2 approach is possible if all the b-constants in (3.15) are negligible. However, a linearized evolution equation of QRI has a larger number of constants being more capable of description of multiaxial stress histories. It has to be underlined here that Hill's yield function (3.17) is incorrect for nonproportional stress paths (cf. [13] for details).

4. Concluding remarks

Following the approach in the paper [10] total thermo-magnetostrictive-elasto-plastic strain of ferromagnetic polycrystals was here also assumed to obey linear decomposition. Indeed, such an assumption is correct only for small strains. Applicability of the results by the effective field method for large strains is obtained here since we combine Kröner's incompatibility with the Eshelbian implanting approach. Results of this paper are briefly summarized as follows:

- By making use of Kanaun–Jeulin stochastic analysis of the self-consistent method (the effective field approach) the effective stiffness 4-tensor is formulated and found numerically. The simultaneous embedding and Kanaun–Jeulin theory are employed to a composite with voids and fibres as inclusions.
- Damage deterioration described by a temporal change of effective stiffness and effective thermal expansion is then related to the inelastic history by

means of modified Vakulenko's approach to endochronic thermodynamics. The explicit results open up a possibility for further application to thermoinelasticity of damaged steels deteriorated by voids.
- Development of damage induces elastic dissymmetry which deserves attention in attempts to develop a multiphase self-consistent theory. An extension of the present results to some other and more complex distribution of voids deserves special attention.

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О НЕЕЛАСТИЧНОСТИ ОШТЕЋЕНИХ ОРТОТРОПНИХ МАТЕРИЈАЛА НАИЗГЛЕД НЕЗАВИСНИХ ОД БРЗИНЕ

Резиме. Рад се бави телом са насумичним тродимензионим распоредом две фазе укључака: сфероидних међусобно паралелних шупљина као и различито орнјентисанних ојачавајућих паралелних крутих сфероидних кратких влакана. Приступом ефективног поља формулисан је и нумерички пронађен тензор ефектнвне крутости четвртог реда. Упоређена су симултана и узастопна уграђивања укључака. Еволуција оштеђења је описана модификованим Вакуленковим приступом ендохроној термодинамици. Дат је кратак приказ проблема ефектнвне еластичне симетрије. Резултати теорије примењенн су на еластовископластично деформисање са оштећењем реакторског нерђајућег челика AISI 316H.

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