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Research Article

Solution of a Fractional Integral Equation Using the Darbo Fixed Point Theorem

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The concept of measure of noncompactness in a Banach space is used in this paper to extend some tripled fixed point theorems. We prove the existence of fractional integral equation solutions using a generalized Darbo fixed point theorem. To demonstrate the validity of the main result, an example is provided.

1. Introduction

Noncompactness measure has ushered in a new branch of nonlinear analysis. It covers a wide range of applications in operator theory. Noncompactness measures have a wide range of applications in FP theory and are particularly useful in differential and integral equations, as well as fractional calculus. Kuratowski [1] investigated the first definition of a noncompactness measure. In 1955, Darbo [2] ensured the existence of fixed points for some mappings using the notion of noncompactness measures, which were obtained by generalizing the Schauder FP theorem [3] and the Banach contraction principle. Many authors use the term "noncompactness measure" to make Darbo FP theorem more general.

The goal of this paper is to extend Darbo's FP theorem and to apply our findings to determine the existence of solutions of fractional integral equations.

We begin with preliminaries, notations, concepts, and definitions that will be used throughout the paper.

Let us have a real Banach space $(\mathbb{B}, \|\,.\,\|)$, and $B(g,r) = \{v \in \mathbb{B}\colon \|v-g\| \le r\}$. Let $\mathbb{S}(\neq\phi)\subseteq\mathbb{B}$. Also, let

(a) $\mathbb{R} = (-\infty, \infty)$.
(b) $\mathbb{R}_+ = [0, \infty)$.
(c) $\overline{\mathbb{S}}$ = the closure of $\mathbb{S}$.
(d) $\operatorname{Conv}\mathbb{S}$ = the convex closure of $\mathbb{S}$.
(e) $\mathbb{G}_{\mathbb{B}}$ = the set of all nonempty and bounded subsets of $\mathbb{B}$.
(f) $\mathbb{H}_{\mathbb{B}}$ = the set of all relatively compact sets.

We provide the below definition of MNC, which is referenced in [4].

Definition 1. A mapping $\Lambda\colon \mathbb{G}_{\mathbb{B}} \longrightarrow \mathbb{R}_+$ is said to be a MNC in $\mathbb{B}$ if it fulfills the following axioms:

(i) The family $\ker\Lambda = \{\mathbb{S}\in\mathbb{G}_{\mathbb{B}}\colon \Lambda(\mathbb{S}) = 0\} \neq \phi$ and $\ker\Lambda \subset \mathbb{H}_{\mathbb{B}}$.
(ii) $\mathbb{S}\subseteq P \Rightarrow \Lambda(\mathbb{S}) \le \Lambda(P)$.
(iii) $\Lambda(\overline{\mathbb{S}}) = \Lambda(\mathbb{S})$.
(iv) $\Lambda(\operatorname{Conv}\mathbb{S}) = \Lambda(\mathbb{S})$.
(v) $\Lambda(L\mathbb{S} + (1-L)P) \le L\Lambda(\mathbb{S}) + (1-L)\Lambda(P)$ for any $L \in [0,1]$.
(vi) If $\mathbb{S}_q \in \mathbb{G}_{\mathbb{B}}$, $\mathbb{S}_q = \overline{\mathbb{S}}_q$, $\mathbb{S}_{q+1} \subset \mathbb{S}_q$ for $q = 1, 2, \ldots$ and $\lim_{q\longrightarrow\infty}\Lambda(\mathbb{S}_q) = 0$, then $\mathbb{S}_\infty = \cap_{q=1}^{\infty}\mathbb{S}_q \neq \phi$.

Since $\Lambda(\mathbb{S}_\infty) \le \Lambda(\mathbb{S}_q)$ for all q, $\Lambda(\mathbb{S}_\infty) = 0$, and so $\mathbb{S}_\infty = \cap_{q=1}^{\infty} \mathbb{S}_q \in \ker\Lambda$.

In the theory of fixed points, the Schauder FP principle and Darbo theorem are crucial.

Theorem 1 (see [3]) (Schauder). *For a nonempty, bounded, closed, and convex subset (NBCCS) $\mathbb{D}$ of a Banach space $\mathbb{B}$, if $\Upsilon: \mathbb{D} \longrightarrow \mathbb{D}$ is a continuous and compact mapping, it must have at least one FP.*

Theorem 2 (see [2]) (Darbo). *For a NBCCS $\mathbb{W}$ of a Banach space $\mathbb{B}$, if $\mathbb{T}: \mathbb{W} \longrightarrow \mathbb{W}$ is a continuous self-mapping with*

$$\Lambda(\mathbb{T}\Omega) \le p\Lambda(\Omega), \quad \Omega \subseteq \mathbb{W}, \tag{1}$$

where $p \in [0,1)$ and Λ is an arbitrary MNC on $\mathbb{B}$, then $\mathbb{T}$ has a FP.

In fractional calculus, fixed point theorems have numerous applications. Let us have a look at some of the work that has been done in this area.

In [5], Sahoo et al. developed numerous new inequalities for twice differentiable convex functions that are coupled with the Hermite–Hadamard integral inequality by using an integral equality related to the k-Riemann–Liouville fractional operator. In addition, for various types of convex functions, certain fresh examples of the established conclusions are derived. This fractional integral adds the symmetric properties of Riemann–Liouville and Hermite–Hadamard inequalities. The authors in [6] explored the existence and uniqueness of solutions to two-dimensional Volterra integral equations, Riemann–Liouville integrals, and Atangana–Baleanu integral operators.

Deng et al. [7] examined the existence of mild solutions for a class of impulsive neutral stochastic functional differential equations in Hilbert spaces with noncompact semigroup. The Hausdorff measure of noncompactness and the Mönch fixed point theorem are used to find sufficient conditions for the existence of mild solutions. The presence of an almost periodic solution to a fractional differential equation with impulse and fractional Brownian motion under nonlocal conditions was the subject of the essay [8].

2. Main Result

We now recall some important definitions that are helpful to our work.

Definition 2. Let $\mathbb{V}$ be the set of all maps $v: \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$\sum_{i=1}^{\infty} v(s_n) = \infty, \tag{2}$$

for all $\{s_n\} \subseteq \mathbb{R}$.

Definition 3 (see [9]). Let $\mathbb{Q}$ be the set of all functions $Q: \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ that fulfills the axioms:

(1) $\max\{\ell_1, \ell_2\} \le Q(\ell_1, \ell_2)$ for $\ell_1, \ell_2 \ge 0$.

(2) Q is continuous.

(3) $Q(\ell_1 + \ell_2, n_1 + n_2) \le Q(\ell_1, n_1) + Q(\ell_2, n_2)$.

Example 1. $Q(\ell_1, \ell_2) = \ell_1 + \ell_2$ is an example of the class $\mathbb{Q}$.

Using the above two classes of control functions, we prove the following results.

Theorem 3. *Let $\mathbb{W}$ be a NBCCS of a Banach space $\mathbb{B}$. Also, let $\Gamma: \mathbb{W} \longrightarrow \mathbb{W}$ be a continuous mapping with*

$$\begin{aligned} Q[\Lambda(\Gamma G), \varpi(\Lambda(\Gamma G))] \le {} & Q\{\Lambda(G), \varpi(\Lambda(G))\} \\ & - v[Q\{\Lambda(G), \varpi(\Lambda(G))\}], \end{aligned} \tag{3}$$

for all $G \subseteq \mathbb{W}$, where Λ is an arbitrary MNC, $v \in \mathbb{V}$, and $Q \in \mathbb{Q}$. Also, let $\varpi: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a nondecreasing continuous mapping. So, Γ has at least one FP in $\mathbb{W}$.

Proof. We define the sequence $(\mathbb{W}_n)_n$ as follows:

$$\begin{cases} \mathbb{W}_0 = \mathbb{W}, \\ \mathbb{W}_n = \operatorname{Conv}(\Gamma \mathbb{W}_n), \quad n = 1, 2, 3, \ldots. \end{cases} \tag{4}$$

We can easily see through induction that

$$\mathbb{W}_{n+1} \subseteq \mathbb{W}_n, \quad n = 0, 1, 2, \ldots. \tag{5}$$

If $N \in \mathbb{N}$ so that $Q\{\Lambda(\mathbb{W}_N), \varpi(\Lambda(\mathbb{W}_N))\} = 0$, then $\Lambda(\mathbb{W}_N) = 0$, that is, $\mathbb{W}_N$ is a relatively compact set. So, by Theorem 1, Γ admits a FP in $\mathbb{W}$.

Now, we may assume that $Q\{\Lambda(\mathbb{W}_N), \varpi(\Lambda(\mathbb{W}_N))\} > 0$ for each $N \in \mathbb{N}$.

On the contrary, we have

$$\begin{aligned} Q\{\Lambda(\mathbb{W}_{n+1}), \varpi(\Lambda(\mathbb{W}_{n+1}))\} &= Q\{\Lambda(\Gamma\mathbb{W}_n), \varpi(\Lambda(\Gamma\mathbb{W}_n))\} \\ &\leqslant Q\{\Lambda(\mathbb{W}_n), \varpi(\Lambda(\mathbb{W}_n))\} - v[Q\{\Lambda(\mathbb{W}_n), \varpi(\Lambda(\mathbb{W}_n))\}] \\ &\leqslant Q\{\Lambda(\mathbb{W}_{n-1}), \varpi(\Lambda(\mathbb{W}_{n-1}))\} - v[Q\{\Lambda(\mathbb{W}_{n-1}), \varpi(\Lambda(\mathbb{W}_{n-1}))\}], \\ -v[Q\{\Lambda(\mathbb{W}_n), \varpi(\Lambda(\mathbb{W}_n))\}] &\leqslant \cdots \\ &\leqslant Q\{\Lambda(\mathbb{W}_0), \varpi(\Lambda(\mathbb{W}_0))\} - \sum_{i=0}^{n} v[Q\{\Lambda(\mathbb{W}_i), \varpi(\Lambda(\mathbb{W}_i))\}]. \end{aligned} \tag{6}$$

Since $\sum_{i=0}^{n} v[Q\{\Lambda(\mathbb{W}_i), \varpi(\Lambda(\mathbb{W}_i))\}] \longrightarrow \infty$, then $Q\{\Lambda(\mathbb{W}_{n+1}), \varpi(\Lambda(\mathbb{W}_{n+1}))\} \longrightarrow 0$, as $n \longrightarrow +\infty$.

This implies that

$$\Lambda(\mathbb{W}_n) \longrightarrow 0, \text{ as } n \longrightarrow +\infty. \tag{7}$$

Since $\mathbb{W}_n \supseteq \mathbb{W}_{n+1}$, by Definition 1, we obtain that $\mathbb{W}_\infty := \cap_{n=1}^{\infty} \mathbb{W}_n$ is a nonempty, closed, and convex subset of $\mathbb{W}$ and $\mathbb{W}_\infty$ is Γ invariant.

So, Theorem 1 concludes that Γ has a FP in $\mathbb{W}$.

Hence, we have the completed proof. □

The following is a crucial consequence of Theorem 3.

Corollary 1. *Let $\mathbb{W}$ be a NBCCS of a Banach space $\mathbb{B}$. Also, let $\Gamma: \mathbb{W} \longrightarrow \mathbb{W}$ be a continuous mapping with*

$$\Lambda(\Gamma G) + \varpi(\Lambda(\Gamma G)) \leq \Lambda(G) + \varpi(\Lambda(G)) - v[\Lambda(G) + \varpi(\Lambda(G))], \tag{8}$$

for all $G \subseteq \mathbb{W}$, where Λ is an arbitrary MNC and $v \in \mathbb{V}$. Also, let $\varpi: \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a nondecreasing continuous mapping. So, Γ has at least one FP in $\mathbb{W}$.

Proof. Putting $Q(\ell_1, \ell_2) = \ell_1 + \ell_2$ in Theorem 3, we get the above corollary. □

Corollary 2. *Let $\mathbb{W}$ be a NBCCS of a Banach space $\mathbb{B}$. Also, let $\Gamma: \mathbb{W} \longrightarrow \mathbb{W}$ be a continuous mapping with*

$$\Lambda(\Gamma G) \leq \Lambda(G) - v[\Lambda(G)], \tag{9}$$

for all $G \subseteq \mathbb{W}$, where Λ is an arbitrary MNC and $v \in \mathbb{V}$. So, Γ has at least one FP in $\mathbb{W}$.

Proof. Setting $\varpi(\ell) = 0$ in Corollary 1, we obtain the above corollary. □

Corollary 3. *Let $\mathbb{W}$ be a NBCCS of a Banach space $\mathbb{B}$. Also, let $\Gamma: \mathbb{W} \longrightarrow \mathbb{W}$ be a continuous mapping with*

$$\Lambda(\Gamma G) \leq \sigma \Lambda(G), \tag{10}$$

for all $G \subseteq \mathbb{W}$, where Λ is an arbitrary MNC and $\sigma = (k/k+1) \in (0,1]$. So, Γ has at least one FP in $\mathbb{W}$.

Proof. Setting $v(s) = (1/k+1)s$ in Corollary 2, we obtain the above corollary. □

Definition 4. (see [10]). A mapping $\mathcal{J}: \mathbb{W} \times \mathbb{W} \times \mathbb{W} \longrightarrow \mathbb{W}$ is called to have a tripled fixed point $(a, x, \hbar) \in \mathbb{W}^3$ if $\mathcal{J}(a, x, \hbar) = a$, $\mathcal{J}(a, x, \hbar) = x$ and $\mathcal{J}(a, x, \hbar) = \hbar$.

Theorem 4 (see [4]). *Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ be an MNC in $\mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_n$, respectively. Additionally, suppose that the mapping $\mathfrak{P}: \mathbb{R}_+^{\rho} \longrightarrow \mathbb{R}_+$ is convex with $\mathfrak{P}(y_1, y_2, \ldots, y_\rho) = 0 \Leftrightarrow y_\sigma = 0$ for $\sigma = 1, 2, 3, \ldots, \rho$. Then, $\check{\Lambda}(\Omega) = \mathfrak{P}(\Lambda_1(\Omega_1), \Lambda_2(\Omega_2), \ldots, \Lambda_n(\Omega_n))$ will be an MNC in $\mathbb{B}_1 \times \mathbb{B}_2 \times \cdots \times \mathbb{B}_n$.*

Example 2 (see [11]). Let $\mathfrak{P}(a, x, \hbar) = a + x + \hbar$, for $(a, x, \hbar) \in \mathbb{R}_+^3$. Now, $\mathfrak{P}(a, x, \hbar) = a + x + \hbar = 0 \Leftrightarrow a = x = \hbar = 0$. As $\mathfrak{P}$ is convex which fulfills all conditions of Theorem 4, $\check{\Lambda}(\Omega) = \mathfrak{P}(\Lambda_1(\Omega_1), \Lambda_2(\Omega_2), \Lambda_3(\Omega_3))$ is an MNC on $\mathbb{B}_1 \times \mathbb{B}_2, \times \mathbb{B}_3$, where Ω_σ is the natural projection of Ω into $\mathbb{B}_\sigma$ for $\sigma = 1, 2, 3$.

Example 3 (see [12]). Let $\mathfrak{P}(a, x, \hbar) = \max\{a, x, \hbar\}$, for $(a, x, \hbar) \in \mathbb{R}_+^3$. Now, $\mathfrak{P}(a, x, \hbar) = \max\{a, x, \hbar\} = 0 \Leftrightarrow a = x = \hbar = 0$. As $\mathfrak{P}$ is convex which fulfills all conditions of Theorem 4, $\check{\Lambda}(\Omega) = \mathfrak{P}(\Lambda_1(\Omega_1), \Lambda_2(\Omega_2), \Lambda_3(\Omega_3))$ is an MNC on $\mathbb{B}_1 \times \mathbb{B}_2, \times \mathbb{B}_3$, where Ω_σ is the natural projection of Ω into $\mathbb{B}_\sigma$ for $\sigma = 1, 2, 3$.

Theorem 5. *Let $\mathbb{W}$ be a NBCCS of a Banach space $\mathbb{B}$. Also, let $\mathcal{J}: \mathbb{W} \times \mathbb{W} \times \mathbb{W} \longrightarrow \mathbb{W}$ be a continuous mapping with*

$$Q\{\Lambda(\mathcal{J}(\varpi_1 \times \varpi_2 \times \varpi_3)), \varpi(\Lambda(\mathcal{J}(\varpi_1 \times \varpi_2 \times \varpi_3)))\} \leq \frac{\mu}{3}\{\Lambda(\varpi_1 \times \varpi_2 \times \varpi_3) + \varpi(\Lambda(\varpi_1 \times \varpi_2 \times \varpi_3))\}, \tag{11}$$

for all $\varpi_1 \times \varpi_2 \times \varpi_3 \subseteq \mathbb{W}$, where Λ is an arbitrary MNC and ϖ and Q are as in Theorem 1. Also, let $\mu(a + x + \hbar) \leq \mu(a) + \mu(x) + \mu(\hbar); a, x, \hbar \geq 0$ and $\varpi(a + x + \hbar) \leq \varpi(a) + \varpi(x) + \varpi(\hbar); a, x, \hbar \geq 0$. So, $\mathcal{J}$ has at least a tripled fixed point in $\mathbb{W}$.

Proof. We consider a function $\check{\mathcal{J}}: \mathbb{W}_3 \longrightarrow \mathbb{W}_3$ by

$$\check{\mathcal{J}}(\varpi_1, \varpi_2, \varpi_3) = (\mathcal{J}(\varpi_1, \varpi_2, \varpi_3), \mathcal{J}(\varpi_2, \varpi_3, \varpi_1), \mathcal{J}(\varpi_3, \varpi_1, \varpi_2)), \tag{12}$$

for all $(\varpi_1, \varpi_2, \varpi_3) \in \mathbb{W}$. It is trivial that $\check{\mathcal{J}}$ is continuous.

Since $\mathcal{J}$ is continuous, assume that $\varpi \subset \mathbb{W}^3$ is nonempty. We have

$$\check{\Lambda}(\varpi) = \Lambda(\varpi_1) + \Lambda(\varpi_2) + \Lambda(\varpi_3), \tag{13}$$

where $\varpi_1, \varpi_2, \varpi_3$ represent $\mathbb{W}$'s natural projections.

Now, we get

$$
\begin{aligned}
&Q\{\check{\Lambda}(\check{\mathcal{J}}(\varpi)),\varpi(\check{\Lambda}(\check{\mathcal{J}}(\varpi)))\}\\
&\leq Q\left[\begin{array}{c}\check{\Lambda}(\mathcal{J}(\varpi_1\times\varpi_2\times\varpi_3)\times\mathcal{J}(\varpi_2\times\varpi_3\times\varpi_1)\times\mathcal{J}(\varpi_3\times\varpi_1\times\varpi_2)),\\ \varpi(\check{\Lambda}(\mathcal{J}(\varpi_1\times\varpi_2\times\varpi_3)\times\mathcal{J}(\varpi_2\times\varpi_3\times\varpi_1)\times\mathcal{J}(\varpi_3\times\varpi_1\times\varpi_2)))\end{array}\right]\\
&=Q\left[\begin{array}{c}\Lambda(\mathcal{J}(\varpi_1\times\varpi_2\times\varpi_3))+\Lambda(\mathcal{J}(\varpi_2\times\varpi_3\times\varpi_1))+\Lambda(\mathcal{J}(\varpi_3\times\varpi_1\times\varpi_2)),\\ \varpi(\Lambda(\mathcal{J}(\varpi_1\times\varpi_2\times\varpi_3)))+\Lambda(\mathcal{J}(\varpi_2\times\varpi_3\times\varpi_1))+\Lambda(\mathcal{J}(\varpi_3\times\varpi_1\times\varpi_2))\end{array}\right]\\
&\leq Q\left[\begin{array}{c}\Lambda(\mathcal{J}(\varpi_1\times\varpi_2\times\varpi_3))+\Lambda(\mathcal{J}(\varpi_2\times\varpi_3\times\varpi_1))+\Lambda(\mathcal{J}(\varpi_3\times\varpi_1\times\varpi_2)),\\ \varpi(\Lambda(\mathcal{J}(\varpi_1\times\varpi_2\times\varpi_3)))+\Lambda(\mathcal{J}(\varpi_2\times\varpi_3\times\varpi_1))+\Lambda(\mathcal{J}(\varpi_3\times\varpi_1\times\varpi_2))\end{array}\right]\\
&\leq Q[\Lambda(\mathcal{J}(\varpi_1\times\varpi_2\times\varpi_3)),\varpi(\Lambda(\mathcal{J}(\varpi_1\times\varpi_2\times\varpi_3)))]\\
&\quad+Q[\Lambda(\mathcal{J}(\varpi_2\times\varpi_3\times\varpi_1))],\varpi(\Lambda(\mathcal{J}(\varpi_2\times\varpi_3\times\varpi_1)))\\
&\quad+Q[\Lambda(\mathcal{J}(\varpi_3\times\varpi_1\times\varpi_2)),\varpi(\Lambda(\mathcal{J}(\varpi_3\times\varpi_1\times\varpi_2)))]\\
&\leq\mu\{\Lambda(\varpi_1)+\Lambda(\varpi_2)+\Lambda(\varpi_3)+\varpi(\Lambda(\varpi_1)+\Lambda(\varpi_2)+\Lambda(\varpi_3))\}\\
&=\mu\{\check{\Lambda}(\varpi)+\varpi(\check{\Lambda}(\varpi))\}=\mu\{Q(\check{\Lambda}(\varpi),\varpi(\check{\Lambda}(\varpi)))\}.
\end{aligned} \tag{14}
$$

We can conclude from Theorem 1 that $\check{\mathcal{J}}$ has a minimum of one FP in $\mathbb{W}^3$.

Now, from Theorem 1, $\mathcal{J}$ admits a tripled fixed point. □

3. Measure of Noncompactness on $C([0,T])$

Let $\mathbb{B}=C(\mathcal{J})$ be the space of real continuous functions on $\mathcal{J}$, where $\mathcal{J}=[0,T]$, which is equipped with

$$\|\mu\|=\sup\{|\mu(t)|:\ t\in\mathcal{J}\},\quad \mu\in\mathbb{B}. \tag{15}$$

Let $J(\neq\phi)\subseteq\mathbb{B}$ be bounded. For $\mu\in J$ and $\delta>0$, denote by $\Lambda(\mu,\delta)$ the modulus of the continuity of μ, i.e.,

$$\Lambda(\mu,\delta)=\sup\{|\mu(h_1)-\mu(h_2)|:\ h_1,h_2\in\mathcal{J},|h_1-h_2|\leq\delta\}. \tag{16}$$

Moreover, we set

$$\Lambda(J,\delta)=\sup\{\Lambda(\mu,\delta):\ \mu\in J\};\ \Lambda_0(J)=\lim_{\delta\longrightarrow 0}\Lambda(J,\delta). \tag{17}$$

It is generally known that the mapping Λ_0 is a MNC in $\mathbb{B}$, and $\Gamma(J)=(1/2)\Lambda_0(J)$ will be the Hausdorff MNC (see [4]).

4. Solvability of Fractional Integral Equations

In this part, we show how our conclusions concerning the existence of a solution to a fractional integral equation in a Banach space can be applied.

Consider the following fractional integral equation [13]:

$$\psi(h)=\psi_0+f(h,\psi(h))+\frac{1}{\Gamma(\chi)}\int_0^h(h-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))d\ell, \tag{18}$$

where $0\leq\chi<1,\psi(0)=\psi_0\geq 0,h\in\mathcal{J}=[0,T]$.

Let

$$Q_{r_0}=\{\psi\in\mathbb{B}:\ \|\psi\|\leq r_0\}. \tag{19}$$

Assume that

(A) $f\colon\mathcal{J}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function and there exists a constant $\beta_1\geq 0$ satisfying

$$|f(h,\psi)-f(h,\psi_1)|\leq\beta_1|\psi-\psi_1|,\quad h\in\mathcal{J};\psi,\psi_1\in\mathbb{R}. \tag{20}$$

Also,

$$\widehat{F}=\sup\{|f(h,0)|:\ h\in\mathcal{J}\}. \tag{21}$$

(B) $\sigma\colon\mathcal{J}\times\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is a continuous function and there exists a nondecreasing function $\varpi\colon\mathbb{R}_+\longrightarrow\mathbb{R}_+$ satisfying

$$|\sigma(h,\psi)|\leq\varpi(|\psi|);\ (h,\psi)\in\mathcal{J}\times\mathbb{R}. \tag{22}$$

(C) There exists a positive solution r_0 for the following inequality:

$$\psi_0+\beta_1 r_0+\widehat{F}+\frac{\varpi(r_0)}{\Gamma(\chi+1)}.T^{\chi}\leq r_0. \tag{23}$$

Theorem 6. *If constraints (A)–(C) hold, equation (18) has at least one solution in* $\mathbb{B}$.

Proof. Consider the following operator $P\colon\mathbb{B}\longrightarrow\mathbb{B}$ such that

$$(P\psi)(h)=\psi_0+f(h,\psi(h))+\frac{1}{\Gamma(\chi)}\int_0^h(h-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))d\ell. \tag{24}$$

□

Step 1. We show that P maps Q_{r_0} into Q_{r_0}. Let $\psi\in Q_{r_0}$, and we now have

$$\begin{aligned}|(P\psi)(h)| &\leq \left|\psi_0\right| + |f(h,\psi(h))| + \left|\frac{1}{\Gamma(\chi)}\int_0^h (h-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))\mathrm{d}\ell\right| \\ &\leq \psi_0 + |f(h,\psi(h)) - f(h,0)| + |f(h,0)| + \left|\frac{1}{\Gamma(\chi)}\int_0^h (h-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))\mathrm{d}\ell\right| \\ &\leq \psi_0 + \beta_1|\psi(h)| + \widehat{F} + \left|\frac{1}{\Gamma(\chi)}\int_0^h (h-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))\mathrm{d}\ell\right|.\end{aligned} \tag{25}$$

Also,

$$\begin{aligned}\left|\frac{1}{\Gamma(\chi)}\int_0^h (h-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))\mathrm{d}\ell\right| &\leq \frac{1}{\Gamma(\chi)}\int_0^h (h-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))\mathrm{d}\ell \\ &\leq \frac{\varpi(\|\psi\|)}{\Gamma(\chi)}\int_0^h (h-\ell)^{\chi-1}\mathrm{d}\ell \leq \frac{\varpi(\|\psi\|)}{\Gamma(\chi+1)}T^{\chi}.\end{aligned} \tag{26}$$

Hence, $\| \psi \| < r_0$ gives

$$\|P\psi\| \leq \psi_0 + \beta_1 r_0 + \widehat{F} + \frac{\varpi(r_0)}{\Gamma(\chi+1)}.T^{\chi} \leq r_0. \tag{27}$$

Due to assumption (C), P maps Q_{r_0} into Q_{r_0}.

Step 2. We show that P is continuous on Q_{r_0}. Let $\delta > 0$ and $\psi, \psi_1 \in Q_{r_0}$ such that $\|\psi - \psi_1\| < \delta$. For all $h \in \mathcal{I}$, we have

$$\begin{aligned}\left|(P\psi)(h) - (P\psi_1)(h)\right| &\leq \left|\psi_0 - \psi_0\right| + \left|f(h,\psi(h)) - f(h,\psi_1(h))\right| \\ &\quad + \left|\frac{1}{\Gamma(\chi)}\int_0^h (h-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))\mathrm{d}\ell - \frac{1}{\Gamma(\chi)}\int_0^h (h-\ell)^{\chi-1}\sigma(\ell,\psi_1(\ell))\mathrm{d}\ell\right| \\ &\leq \beta_1\left|\psi(h) - \psi_1(h)\right| + \frac{1}{\Gamma(\chi)}\int_0^h (h-\ell)^{\chi-1}\sigma(\ell,\psi(\ell)) - \sigma(\ell,\psi_1(\ell))\mathrm{d}\ell \\ &\leq \beta_1\left\|\psi - \psi_1\right\| + \frac{1}{\Gamma(\chi)}\int_0^h (h-\ell)^{\chi-1}\left|\sigma(\ell,\psi(\ell)) - \sigma(\ell,\psi_1(\ell))\right|\mathrm{d}\ell \\ &< \beta_1\left\|\psi - \psi_1\right\| + \frac{1}{\Gamma(\chi)}\Lambda_{r_0}(\delta)\int_0^h (h-\ell)^{\chi-1}\mathrm{d}\ell \\ &< \beta_1\left\|\psi - \psi_1\right\| + \frac{1}{\Gamma(\chi+1)}\Lambda_{r_0}(\delta)T^{\chi},\end{aligned} \tag{28}$$

where

$$\Lambda_{r_0}(\delta) = \sup\left\{\begin{array}{c}\left|\sigma(\ell,\psi(\ell)) - \sigma(\ell,\psi_1(\ell))\right|: \left|\psi - \psi_1\right| \leq \delta; \ell \in I; \\ \psi, \psi_1 \in [-r_0, r_0]\end{array}\right\}. \tag{29}$$

Hence, $\|\psi - \psi_1\| < \delta$ gives

$$\left|(P\psi)(h) - (P\psi_1)(h)\right| < \beta_1\delta + \frac{1}{\Gamma(\chi+1)}\Lambda_{r_0}(\delta)T^{\chi}. \tag{30}$$

As $\delta \longrightarrow 0$, we get $|(P\psi)(h) - (P\psi_1)(h)| \longrightarrow 0$. This clearly proves that P is continuous on Q_{r_0}.

Step 3. An estimation of P with respect to Λ_0: now, assume that $\Delta(\neq \phi) \subseteq Q_{r_0}$. Let $\delta > 0$ be arbitrary. Also, choose $\psi \in \Delta$ with $h_1, h_2 \in \mathcal{I}$ such that $|h_2 - h_1| \leq \delta$ and $h_2 \geq h_1$.

Now,

$$
\begin{aligned}
\left|(P\psi)(h_2)-(P\psi)(h_1)\right| &= \left|\psi_0+f(h_2,\psi(h_2))+\frac{1}{\Gamma(\chi)}\int_0^{h_2}(h_2-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))\mathrm{d}\ell\right.\\
&\quad \left.-\psi_0-f(h_1,\psi(h_1))-\frac{1}{\Gamma(\chi)}\int_0^{h_1}(h_1-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))\mathrm{d}\ell\right|\\
&\le \left|f(h_2,\psi(h_2))-f(h_1,\psi(h_1))\right|+\frac{1}{\Gamma(\chi)}\left|\int_0^{h_2}(h_2-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))\mathrm{d}\ell-\int_0^{h_1}(h_1-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))\mathrm{d}\ell\right|\\
&\le \left|f(h_2,\psi(h_2))-f(h_1,\psi(h_1))\right|\\
&\quad +\frac{1}{\Gamma(\chi)}\left|\int_{h_1}^{h_2}(h_2-\ell)^{\chi-1}\sigma(\ell,\psi(\ell))\mathrm{d}\ell+\int_0^{h_1}\left\{(h_2-\ell)^{\chi-1}-(h_1-\ell)^{\chi-1}\right\}\sigma(\ell,\psi(\ell))\mathrm{d}\ell\right|\\
&\le \left|f(h_2,\psi(h_2))-f(h_2,\psi(h_1))\right|+\left|f(h_2,\psi(h_1))-f(h_1,\psi(h_1))\right|\\
&\quad +\frac{\varpi(|\psi|)}{\Gamma(\chi)}\left(\int_{h_1}^{h_2}(h_2-\ell)^{\chi-1}\mathrm{d}\ell+\int_0^{h_1}\left\{(h_2-\ell)^{\chi-1}-(h_1-\ell)^{\chi-1}\right\}\mathrm{d}\ell\right)\\
&\le \beta_1\left|\psi(h_2)-\psi(h_1)\right|+\left|f(h_2,\psi(h_1))-f(h_1,\psi(h_1))\right|+\frac{\varpi(r_0)}{\Gamma(\chi+1)}\left[h_2^{\chi}-h_1^{\chi}\right]\\
&\le \beta_1\Lambda(\psi,\delta)+\Lambda_f(r_0,\delta)+\frac{\varpi(r_0)}{\Gamma(\chi+1)}\left[h_2^{\chi}-h_1^{\chi}\right],
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
\Lambda_f(r_0,\delta) &= \sup\left\{\begin{array}{c}\left|f(h_2,\psi(h_1))-f(h_1,\psi(h_1))\right|:\ \left|h_2-h_1\right|\le\delta;\ h_1,h_2\in\mathscr{I};\\ |\psi|\le r_0\end{array}\right\},\\
\Lambda(\psi,\delta) &= \sup\left\{\left|\psi(h_2)-\psi(h_1)\right|\le\delta:\ \left|h_2-h_1\right|\le\delta;\ h_1,h_2\in\mathscr{I}\right\}.
\end{aligned}
\tag{32}
$$

As $\delta\longrightarrow 0$, $h_2\longrightarrow h_1$, so we get

$$
\lim_{\delta\longrightarrow 0}\frac{\varpi(r_0)}{\Gamma(\chi+1)}\left[h_2^{\chi}-h_1^{\chi}\right]\longrightarrow 0. \tag{33}
$$

Hence,

$$
\left|(P\psi)(h_2)-(P\psi)(h_1)\right|\le\beta_1\Lambda(\psi,\delta)+\Lambda_f(r_0,\delta), \tag{34}
$$

i.e.,

$$
\Lambda(P\psi,\delta)\le\beta_1\Lambda(\psi,\varepsilon)+\Lambda_f(r_0,\delta). \tag{35}
$$

By the uniform continuity of f on $\mathscr{I}\times[-r_0,r_0]$, we now obtain $\lim_{\delta\longrightarrow 0}\Lambda_f(r_0,\delta)\longrightarrow 0$, as $\delta\longrightarrow 0$.

Taking $\sup_{\psi\in\Delta}$ and $\delta\longrightarrow 0$, we get

$$
\Lambda_0(P\Delta)\le\beta_1\Lambda_0(\Delta). \tag{36}
$$

Hence, by Corollary 3, P has a FP in $\Delta\subseteq Q_{r_0}$. That is, equation (18) has a solution in $\mathbb{B}$.

Example 4. Consider the following fractional integral equation:

$$
\psi(h)=\frac{|\psi|}{2}+\frac{\psi}{10+h^4}+\frac{1}{\Gamma(1/2)}\int_0^h(h-\ell)^{1/2}\sin^{-1}\left(\frac{\psi^2(\ell)}{1-\ell^2}\right)\mathrm{d}\ell, \tag{37}
$$

for $h\in[0,2]=\mathscr{I}$, which is a particular case of equation (18).

Here,

$$\psi_0 = \frac{|\psi|}{2},$$
$$f(h, \phi(h)) = \frac{\psi}{10 + h^4},$$
$$\chi = \frac{1}{2},$$
$$\sigma(\ell, \psi(\ell)) = \sin^{-1}\left(\frac{\psi^2(\ell)}{1 - \ell^2}\right). \tag{38}$$

Also, it is trivial that f is continuous and satisfies

$$\left|f(h, \psi(h)) - f(h, \psi_1(h))\right| \leq \frac{\left|\psi - \psi_1\right|}{10}. \tag{39}$$

Therefore, $\beta_1 = 1/10$.

If $\|\psi\| \leq r_0$, then

$$\psi_0 = \frac{r_0}{2},$$
$$\widehat{F} = \frac{r_0}{10}, \tag{40}$$
$$|\sigma(\ell, \psi)| \leq \left|\psi^2\right|.$$

So,

$$\varpi(r_0) = r_0^2. \tag{41}$$

Putting these values in the inequality of assumption (C), we get

$$\frac{r_0}{2} + \frac{1}{10}r_0 + \frac{r_0}{10} + \frac{r_0^2}{\Gamma(3/2)}(2)^{1/2} \leq r_0$$
$$\Rightarrow \frac{r_0^2}{\Gamma(3/2)}(2)^{1/2} \leq \frac{3}{10}r_0 \Rightarrow r_0 \leq \frac{3\Gamma(3/2)}{10(2)^{1/2}}. \tag{42}$$

However, assumption (C) is also fulfilled for $r_0 = 3\Gamma(3/2)/10(2)^{1/2}$.

We can see that all of Theorem 5's assumptions are achieved, from (A) to (C). Equation (37), according to Theorem 5, has a solution in $\mathbb{B} = C(\mathcal{I})$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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