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Topologies of Bihyperbolic Numbers

Ana Savić [1,†], Merve Bilgin [2,†], Soley Ersoy [2,†] and Marija Paunović [3,*,†]

1 School of Electrical and Computer Engineering, Academy of Technical and Art Applied Studies, 11000 Belgrade, Serbia
2 Department of Mathematics, Faculty of Sciences, University of Sakarya, 54050 Sakarya, Turkey
3 Faculty of Hotel Management and Tourism, University of Kragujevac, 36210 Vrnjacka Banja, Serbia
* Correspondence: majap@rcub.bg.ac.rs
† These authors contributed equally to this work.

Abstract: In this paper, we establish a correlation between the bihyperbolic numbers set and the semi-Euclidean space. There are three different norms on the semi-Euclidean space that allow us to define three different hypersurfaces on semi-Euclidean space. Hence, we construct some topological structures on these hypersurfaces called norm e, s, and t topologies. On the other hand, we introduce hyperbolic e, s, and t topologies on the bihyperbolic numbers set. Moreover, by using the idempotent and spectral representations of the bihyperbolic numbers, we introduce new topologies on the bihyperbolic numbers set.

Keywords: bihyperbolic numbers; topology; semi-Euclidean space

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1. Introduction

J. Cockle introduced Tessarine numbers as $a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$, such that $a, b, c, d \in \mathbb{R}$, $\mathrm{ij} = \mathrm{ji} = \mathrm{k}$, $\mathrm{i}^2 = -1$, $\mathrm{j}^2 = 1$, in 1848 [1–3]. Correspondingly, the discovery of the Tessarine numbers brought out the appearance of new numbers in the form of $a + c\mathrm{j}$, $a, c \in \mathbb{R}$, $\mathrm{j}^2 = 1$, $\mathrm{j} \notin \mathbb{R}$. The system of such numbers is a subalgebra of Tessarine numbers. For this reason, formerly, these numbers were called "real Tessarine" numbers. Real Tessarine numbers are also known as hyperbolic numbers because a hyperbolic number moves along a hyperbolic trajectory if this number is multiplied by an imaginary component of hyperbolic numbers, just as a complex number rotates along a circular trajectory if it is multiplied by an imaginary component. Besides, P. Fjelstad called hyperbolic numbers perplex numbers and introduced their algebraic properties and hyperbolic trigonometric functions in 1986 [4]. In addition, B. Rosenfeld named hyperbolic numbers as split-complex numbers in 1997 [5] since the algebra of these numbers includes non-real roots of 1 and also contains idempotents and zero divisors.

G. Sobczyk presented the basic properties of hyperbolic numbers and their relationship with special relativity and space–time geometry in [6]. For a long while, the hyperbolic numbers and their strict relation to the space–time geometry of two-dimensional special relativity have been an actual subject area of research [6–10]. This relation has been extended to multiple dimensions as well. For instance, the space–time or spherical hyperbolic complex numbers in dimensions three and four have been studied in a recent paper on the hyperbolic numbers together with their multidimensional generalizations [10]. W. D. Richter also introduced the hyperbolic vector product, hyperbolic vector powers, and hyperbolic vector exponential function in this paper.

In fact, the idea of working in higher dimensions dates back to old times. In 1892, Segre modified the quaternions by virtue of the commutative property in multiplication and introduced "bicomplex numbers" based on the works of Hamilton and Clifford on quaternions [11]. G. B. Price published a comprehensive book on bicomplex and multicomplex numbers in 1991 [12]. Furthermore, D. Rochon and M. Shapiro studied the algebraic properties of bicomplex and hyperbolic numbers in 2004 [13]. In this study, the importance of hyperbolic numbers and bicomplex numbers in Clifford's algebra was explained. The algebra, geometry, and analysis of bicomplex numbers were explained in detail by [14]. With the progress of time, Segre's commutative quaternions have been generalized and three types of four-dimensional commutative hypercomplex numbers $q = t + \mathrm{i}x + \mathrm{j}y + \mathrm{k}z$, where $t, x, y, z \in \mathbb{R}$, $\mathrm{i}, \mathrm{j}, \mathrm{k} \notin \mathbb{R}$, $\mathrm{i}^2 = \mathrm{k}^2 = \alpha$, $\mathrm{j}^2 = 1$, $\mathrm{ij} = \mathrm{ji} = \mathrm{k}$ such as elliptic ($\alpha < 0$), parabolic ($\alpha = 0$), and hyperbolic ($\alpha > 0$) [15,16]. The well-known bicomplex numbers correspond to the special case $\alpha = -1$. In the case of $\alpha = 1$, these numbers are called hyperbolic four complex numbers [16] or bihyperbolic numbers [17]. These numbers can be represented by a pair of hyperbolic numbers. Furthermore, the spectral representation of the bihyperbolic numbers was given in [18], and this representation allowed the definition of a partial order of bihyperbolic numbers. Furthermore, the combinatorial properties of bihyperbolic numbers of the Fibonacci and Pell types are given in the recent papers [19–21].

On the other hand, the idea of constructing topologies on bicomplex numbers was first presented by R. K. Srivastava. The norm topology, complex topology, and idempotent topology were defined on bicomplex space in 2008 [22]. R. K. Srivastava and S. Singh established the dictionary order topology in the set of bicomplex numbers in 2010 [23]. Bicomplex nets were studied by R. K. Srivastava and S. Singh in 2011 [24]. R. K. Srivastava and S. Singh studied the compactness of some sub-spaces of bicomplex spaces in 2013 [25]. A. Prakash and P. Kumar briefly introduced the topologies of bicomplex numbers and compared these topologies in 2016 [26]. S. Singh and S. Kumar studied the dictionary order topology of bicomplex numbers in 2017 [27].

Even though there are some studies constructing topological structures on bicomplex numbers sets, there is no study about topological structures on bihyperbolic numbers set. The bihyperbolic numbers are related to four-dimensional semi-Euclidean space, and defining topologies for non-Euclidean spaces is quite difficult. There have been some remarkable attempts to introduce topologies on Minkowski–Lorentz space, including [28–40]. In 1964, E. C. Zeeman stated that it is wrong to consider the usual local homogeneous Euclidean topology on Minkowski space [28,29], because the group of homeomorphisms of Euclidean space contains elements that transform space-like and time-like directions into each other. However, this is not physically possible. S. Nanda introduced t-topology and s-topology in Minkowski space [30,31]. G. Agrawal and S. Shirivastava investigated the topological properties of Minkowski space given by the t-topology and s-topology [33,34].

In light of recent research related to topologies on non-Euclidean spaces and detailed information on the bihyperbolic numbers set, the present paper aims to fill the gap in defining topologies on the bihyperbolic numbers set.

2. Preliminaries

The set of the hyperbolic numbers is

$$H = \left\{ z : z = x + \mathrm{j}y,\ \ \mathrm{j}^2 = 1,\ \ x, y \in \mathbb{R},\ \mathrm{j} \notin \mathbb{R} \right\}$$

and the hyperbolic conjugate of a $z \in H$ is $\overline{z} = x - \mathrm{j}y$ [7]. The modulus of a hyperbolic number $z \in H$ is $|z|_H = \sqrt{|z\overline{z}|} = \sqrt{|x^2 - y^2|}$ [7]. The hyperbolic numbers can be also defined as ordered pairs of reals where

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 + y_1y_2, x_1y_2 + y_1x_2)$$

that any pair (x, y) corresponds to $x + \mathrm{j}y$. Note that such numbers do not form a field, but an irregular commutative ring. Furthermore, the hyperbolic numbers form is an algebra over the field of the real numbers. For every real x, it holds

$$(x + \mathrm{j}x)(x - \mathrm{j}x) = 0,$$

which means that numbers of this form are irregular. The number $x + \mathrm{j}y$ is regular iff it is invertible iff $|x| \neq |y|$. Furthermore, if we define $\langle x_1 + \mathrm{j}y_1, x_2 + \mathrm{j}y_2 \rangle$ as $x_1x_2 - y_1y_2$, then for any numbers z_1, z_2, it holds

$$\langle z_1z_2, z_1z_2 \rangle = \langle z_1, z_1 \rangle \langle z_2, z_2 \rangle.$$

A hyperbolic number can be also considered as a point in two-dimensional Minkowski space $\mathbb{R}^2_1$. Thus, if we choose a number $z \in H$ corresponding to a point in $\mathbb{R}^2_1$, then for the position vector $z = (x, y)$ of this point, the Lorentzian inner product is given by $\langle z, z \rangle|_{\mathbb{R}^2_1} = x^2 - y^2$. Furthermore, $z = (x, y)$ is a space-like, lightlike (null), or time-like vector if $\langle z, z \rangle|_{\mathbb{R}^2_1} > 0$, $\langle z, z \rangle|_{\mathbb{R}^2_1} = 0$, or $\langle z, z \rangle|_{\mathbb{R}^2_1} < 0$, respectively. Therefore, the geometric structure of Minkowski space can be associated with the hyperbolic numbers.

We define the space cone, null cone, and time cone of $z_0 \in H$ as follows:

$$SH(z_0) = \left\{ z \in H \middle| \, (z - z_0)\overline{(z - z_0)} > 0 \text{ or } z = z_0 \right\},$$

$$NH(z_0) = \left\{ z \in H \middle| \, (z - z_0)\overline{(z - z_0)} = 0 \right\},$$

and

$$TH(z_0) = \left\{ z \in H \middle| \, (z - z_0)\overline{(z - z_0)} < 0 \text{ or } z = z_0 \right\},$$

respectively.

Therefore, the well-known hyperbolic numbers $e_1 = \frac{1+\mathrm{j}}{2}$ and $e_2 = \frac{1-\mathrm{j}}{2}$ stay in the null cone of the origin $NH(O)$ since $e_1\overline{e_1} = 0$ and $e_2\overline{e_2} = 0$.

Furthermore, e_1 and e_2 are called the idempotent elements based on $(e_1)^2 = e_1$, $(e_2)^2 = e_2$ [14]. Any hyperbolic number $z \in H$ can be written as the linear combination:

$$z = x + \mathrm{j}y = \alpha_1 e_1 + \alpha_2 e_2 \quad (1)$$

where $\alpha_1 = x + y$, $\alpha_2 = x - y$ are real numbers. This representation is called the idempotent representation of a hyperbolic number [14].

Relatively new numbers can be obtained by changing the real coefficients of a hyperbolic number by hyperbolic numbers. In this way, these numbers have the form $\zeta = z_1 + \mathrm{j}_2 z_2$, where $z_1 = x_0 + \mathrm{j}_1 x_1$, $z_2 = x_2 + \mathrm{j}_1 x_3$ $(x_0, x_1, x_2, x_3 \in \mathbb{R})$ are hyperbolic numbers and $\mathrm{j}_1, \mathrm{j}_2, \mathrm{j}_3$ are hyperbolic units such that $\mathrm{j}_1^2 = \mathrm{j}_2^2 = \mathrm{j}_3^2 = 1$, $\mathrm{j}_1\mathrm{j}_2 = \mathrm{j}_2\mathrm{j}_1 = \mathrm{j}_3$. These numbers are called bihyperbolic numbers [17]. Moreover, the term hyperbolic four complex is used for bihyperbolic numbers [16]. Especially, the algebraic properties of bihyperbolic numbers were studied in detail by [18]. The set of bihyperbolic numbers is denoted by

$$H_2 = \{ \zeta | \, \zeta = z_1 + \mathrm{j}_2 z_2, \; z_1, z_2 \in H(\mathrm{j}_1) \}$$

where $H(\mathrm{j}_1)$ is the set of hyperbolic numbers denoted by j_1. The symbol H will be used for the set $H(\mathrm{j}_1)$ in the rest of the article. There are three pairs of idempotent elements relative to the hyperbolic unit for the bihyperbolic number. These are

$$e_{1,\mathrm{j}_s} = \frac{1 + \mathrm{j}_s}{2}, \;\; e_{2,\mathrm{j}_s} = \frac{1 - \mathrm{j}_s}{2}, \;\; (s = 1, 2, 3) \quad (2)$$

and the properties $e_{1,\mathrm{j}_s} + e_{2,\mathrm{j}_s} = 1$, $e_{1,\mathrm{j}_s} e_{2,\mathrm{j}_s} = 0$, $\left(e_{1,\mathrm{j}_s}\right)^2 = e_{1,\mathrm{j}_s}$, and $\left(e_{2,\mathrm{j}_s}\right)^2 = e_{2,\mathrm{j}_s}$ are satisfied. Thus, a bihyperbolic number ζ can be written in three different forms as $\zeta = \zeta_{1,\mathrm{j}_1} e_{1,\mathrm{j}_1} + \zeta_{2,\mathrm{j}_1} e_{2,\mathrm{j}_1}$, $\zeta = \zeta_{1,\mathrm{j}_2} e_{1,\mathrm{j}_2} + \zeta_{2,\mathrm{j}_2} e_{2,\mathrm{j}_2}$, or $\zeta = \zeta_{1,\mathrm{j}_3} e_{1,\mathrm{j}_3} + \zeta_{2,\mathrm{j}_3} e_{2,\mathrm{j}_3}$ [18]. The coefficients of the idempotent representations are

$$\begin{aligned} z_1 + \mathrm{j}_3 z_2 &= \zeta_{1,\mathrm{j}_1}, \quad z_1 - \mathrm{j}_3 z_2 = \zeta_{2,\mathrm{j}_1}, \\ z_1 + z_2 &= \zeta_{1,\mathrm{j}_2}, \quad z_1 - z_2 = \zeta_{2,\mathrm{j}_2}, \\ z_1 + \mathrm{j}_1 z_2 &= \zeta_{1,\mathrm{j}_3}, \quad z_1 - \mathrm{j}_1 z_2 = \zeta_{2,\mathrm{j}_3}. \end{aligned} \tag{3}$$

It is seen that $\zeta_{1,\mathrm{j}_1}, \zeta_{2,\mathrm{j}_1} \in H_2$ and $\zeta_{1,\mathrm{j}_2}, \zeta_{2,\mathrm{j}_2}, \zeta_{1,\mathrm{j}_3}, \zeta_{2,\mathrm{j}_3} \in H(\mathrm{j}_1)$. The first and second forms of these idempotent representations were presented by [18], and the third one was given by [9].

Furthermore, the spectral representation of a bihyperbolic number $\zeta = z_1 + \mathrm{j}_2 z_2 = x_0 + \mathrm{j}_1 x_1 + \mathrm{j}_2 x_2 + \mathrm{j}_3 x_3 \in H_2$ is

$$\zeta = w_1 \mathrm{i}_1 + w_2 \mathrm{i}_2 + w_3 \mathrm{i}_3 + w_4 \mathrm{i}_4 \tag{4}$$

where $\mathrm{i}_1,\ \mathrm{i}_2,\ \mathrm{i}_3,\ \mathrm{i}_4$ are the idempotent elements such as

$$\begin{aligned} \mathrm{i}_1 &= \frac{1 + \mathrm{j}_1 + \mathrm{j}_2 + \mathrm{j}_3}{4},\ \mathrm{i}_2 = \frac{1 - \mathrm{j}_1 + \mathrm{j}_2 - \mathrm{j}_3}{4}, \\ \mathrm{i}_3 &= \frac{1 + \mathrm{j}_1 - \mathrm{j}_2 - \mathrm{j}_3}{4},\ \mathrm{i}_4 = \frac{1 - \mathrm{j}_1 - \mathrm{j}_2 + \mathrm{j}_3}{4}. \end{aligned} \tag{5}$$

These elements satisfy ${\mathrm{i}_s}^2 = \mathrm{i}_s$ for $s = 1,2,3,4$. Furthermore, $\mathrm{i}_s \mathrm{i}_k = 0$ for $s,k = 1,2,3,4$ and $s \neq k$ [16]. The coefficients in the spectral representation of ζ are as follows:

$$\begin{aligned} w_1 &= x_0 + x_1 + x_2 + x_3, \quad w_2 = x_0 - x_1 + x_2 - x_3, \\ w_3 &= x_0 + x_1 - x_2 - x_3, \quad w_4 = x_0 - x_1 - x_2 + x_3. \end{aligned} \tag{6}$$

w_k and i_k $(k = 1,\ 2,\ 3,\ 4)$ are, respectively, the eigenvalues and orthonormal eigenvectors of the associated matrix of ζ [9]. For every $k = 1,2,3,4$, the eigenvalue function λ_k defined by $\lambda_k(\zeta) = w_k$ is a surjective algebra homomorphism from H_2 to $\mathbb{R}$ with $\ker(\lambda_k) = \mathrm{Vect}\{i_s :\ s = 1,2,3,4 \text{ and } s \neq k\}$ [18].

Lastly, $\overline{\zeta}^{\mathrm{j}_1} = z_1 - \mathrm{j}_2 z_2$, $\overline{\zeta}^{\mathrm{j}_2} = \bar{z}_1 + \mathrm{j}_2 \bar{z}_2$ and $\overline{\zeta}^{\mathrm{j}_3} = \bar{z}_1 - \mathrm{j}_2 \bar{z}_2$ are called the principal conjugates of a bihyperbolic number ζ [9,18].

Example 1. *Let us consider the bihyperbolic number $\zeta = z_1 + \mathrm{j}_2 z_2$ formed by two hyperbolic numbers $z_1 = 1 - 2\mathrm{j}_1$ and $z_2 = 2 + 3\mathrm{j}_1$. Then, it can be represented as*

$$\zeta = 1 - 2\mathrm{j}_1 + 2\mathrm{j}_2 + 3\mathrm{j}_3.$$

Furthermore, three different idempotent representations of this number are

$$\zeta = (1 - 2\mathrm{j}_1 + 3\mathrm{j}_2 + 2\mathrm{j}_3) e_{1,\mathrm{j}_1} + (1 - 2\mathrm{j}_1 - 3\mathrm{j}_2 - 2\mathrm{j}_3) e_{2,\mathrm{j}_1},$$

$$\zeta = (3 + \mathrm{j}_1) e_{1,\mathrm{j}_2} - (1 + 5\mathrm{j}_1) e_{2,\mathrm{j}_2},$$

$$\zeta = 4 e_{1,\mathrm{j}_3} - (2 + 4\mathrm{j}_1) e_{2,\mathrm{j}_3}.$$

In addition to these, the spectral representation of this number is

$$\zeta = 4\mathrm{i}_1 + 2\mathrm{i}_2 - 4\mathrm{i}_3 + 4\mathrm{i}_4.$$

The principal conjugates of ζ are determined as

$$\overline{\zeta}^{\mathrm{j}_1} = z_1 - \mathrm{j}_2 z_2 = 1 - 2\mathrm{j}_1 - 2\mathrm{j}_2 - 3\mathrm{j}_3,$$

$$\overline{\zeta}^{\mathrm{j}_2} = \bar{z}_1 + \mathrm{j}_2 \bar{z}_2 = 1 + 2\mathrm{j}_1 + 2\mathrm{j}_2 - 3\mathrm{j}_3,$$

$$\overline{\zeta}^{\mathrm{j}_3} = \bar{z}_1 - \mathrm{j}_2 \bar{z}_2 = 1 + 2\mathrm{j}_1 - 2\mathrm{j}_2 + 3\mathrm{j}_3.$$

3. Bihyperbolic Numbers and Semi-Euclidean Space

Let us consider four-dimensional real affine space $\mathbb{R}^4$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$ such that $\mathbf{x} = (x_0, x_1, x_2, x_3)$ and $\mathbf{y} = (y_0, y_1, y_2, y_3)$. If the scalar product of $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle|_{\mathbb{R}^4_2} = \varepsilon_0 x_0 y_0 + \varepsilon_1 x_1 y_1 + \varepsilon_2 x_2 y_2 + \varepsilon_3 x_3 y_3$$

where arbitrary two elements of $\{\, \varepsilon_i \mid i = 0, 1, 2, 3\}$ are -1 and the others are $+1$, then the real affine four-space equipped with this scalar product is called the semi-Euclidean space with index 2 and represented by $\mathbb{R}^4_2$ [41].

Just as the geometry of the Minkowski plane can be described with hyperbolic numbers, the geometry of four-dimensional semi-Euclidean space can be described with bihyperbolic numbers. This interrelation between the points (x_0, x_1, x_2, x_3) in $\mathbb{R}^4_2$ and the bihyperbolic numbers $\zeta = z_1 + \mathrm{j}_2 z_2 = x_0 + \mathrm{j}_1 x_1 + \mathrm{j}_2 x_2 + \mathrm{j}_3 x_3$ in H_2 can be constructed by associating the semi-Euclidean norm on four-dimensional semi-Euclidean space and the real-valued norm on the bihyperbolic numbers set.

In this regard, let us explain how and where the semi-Euclidean norm with the metric signature determined relative to a suitably chosen basis such as $(+,+,-,-)$, $(+,-,+,-)$ or $(+,-,-,+)$ corresponds to the real-valued norm of bihyperbolic numbers.

First, recall the real-valued norm of a bihyperbolic number ζ given by [15,18]

$$\begin{aligned} |\zeta|_{H_2} &= \sqrt[4]{\left|\zeta \overline{\zeta}^{\mathrm{j}_1} \overline{\zeta}^{\mathrm{j}_2} \overline{\zeta}^{\mathrm{j}_3}\right|} \\ &= \sqrt[4]{\left| \sum_{i=0}^{3} x_i^4 + 8 \prod_{i=0}^{3} x_i - 2\left(x_0^2 x_2^2 + x_1^2 x_3^2 + x_1^2 x_2^2 + x_0^2 x_1^2 + x_2^2 x_3^2 + x_0^2 x_3^2\right) \right|}. \end{aligned}$$

This can be expressed in three different ways:

$$|\zeta|_{H_2} = \sqrt[4]{\left| \left(x_0^2 + x_1^2 - x_2^2 - x_3^2\right)^2 - 4(x_0 x_1 - x_2 x_3)^2 \right|},$$

$$|\zeta|_{H_2} = \sqrt[4]{\left| \left(x_0^2 - x_1^2 + x_2^2 - x_3^2\right)^2 - 4(x_0 x_2 - x_1 x_3)^2 \right|},$$

or

$$|\zeta|_{H_2} = \sqrt[4]{\left| \left(x_0^2 - x_1^2 - x_2^2 + x_3^2\right)^2 - 4(x_0 x_3 - x_1 x_2)^2 \right|}.$$

Furthermore, it is known that the products of a bihyperbolic number and its conjugates are

$$\zeta \overline{\zeta}^{\mathrm{j}_1} = x_0^2 + x_1^2 - x_2^2 - x_3^2 + 2\mathrm{j}_1(x_0 x_1 - x_2 x_3), \tag{7}$$

$$\zeta \overline{\zeta}^{\mathrm{j}_2} = x_0^2 - x_1^2 + x_2^2 - x_3^2 + 2\mathrm{j}_2(x_0 x_2 - x_1 x_3), \tag{8}$$

$$\zeta \overline{\zeta}^{\mathrm{j}_3} = x_0^2 - x_1^2 - x_2^2 + x_3^2 + 2\mathrm{j}_3(x_0 x_3 - x_1 x_2). \tag{9}$$

Example 2. *If we consider the bihyperbolic number $\zeta = 1 - 2\mathrm{j}_1 + 2\mathrm{j}_2 + 3\mathrm{j}_3$ given in Example 1, then we find the product of ζ with each of its conjugates as*

$$\zeta \overline{\zeta}^{\mathrm{j}_1} = -8 - 16\mathrm{j}_1, \quad \zeta \overline{\zeta}^{\mathrm{j}_2} = -8 + 16\mathrm{j}_2, \quad \zeta \overline{\zeta}^{\mathrm{j}_3} = 2 + 14\mathrm{j}_3,$$

respectively. It is a fact that there are three ways of computing the norms $\sqrt{\left|\zeta \overline{\zeta}^{\mathrm{j}_k}\right|_H} = \sqrt[4]{|u^2 - v^2|}$, where $\zeta \overline{\zeta}^{\mathrm{j}_k} = u + v\mathrm{j}_k$ for $k = 1, 2, 3$ gives the same real-valued norm of ζ; this also can be seen from

$$\sqrt{\left|\zeta \overline{\zeta}^{\mathrm{j}_1}\right|_H} = \sqrt{\left|\zeta \overline{\zeta}^{\mathrm{j}_2}\right|_H} = \sqrt{\left|\zeta \overline{\zeta}^{\mathrm{j}_3}\right|_H} = |\zeta|_{H_2} = \sqrt[4]{192} = 2\sqrt[4]{12}.$$

The relations (7)–(9) give rise to thought about the cases $x_0x_1 - x_2x_3 = 0$, $x_0x_2 - x_1x_3 = 0$ or $x_0x_3 - x_1x_2 = 0$. In these cases, three hypersurfaces occur in H_2 such that

$$M_1 = \{ x_0 + \mathrm{j}_1x_1 + \mathrm{j}_2x_2 + \mathrm{j}_3x_3 | x_0x_1 - x_2x_3 = 0\},$$

$$M_2 = \{ x_0 + \mathrm{j}_1x_1 + \mathrm{j}_2x_2 + \mathrm{j}_3x_3 | x_0x_2 - x_1x_3 = 0\},$$

and

$$M_3 = \{ x_0 + \mathrm{j}_1x_1 + \mathrm{j}_2x_2 + \mathrm{j}_3x_3 | x_0x_3 - x_1x_2 = 0\}.$$

It was indicated by [18] that the real-valued norm of ζ on the hypersurfaces M_k for $k = 1,2,3$ coincides with $|\zeta|_{\mathrm{j}_k} = \sqrt{\left|\zeta\overline{\zeta}^{\mathrm{j}_k}\right|_H}$ for $k = 1,2,3$ as

$$\begin{aligned} |\zeta|_{H_2} = |\zeta|_{\mathrm{j}_1} &= \sqrt{|x_0^2 + x_1^2 - x_2^2 - x_3^2|} \text{ on } M_1, \\ |\zeta|_{H_2} = |\zeta|_{\mathrm{j}_2} &= \sqrt{|x_0^2 - x_1^2 + x_2^2 - x_3^2|} \text{ on } M_2, \\ |\zeta|_{H_2} = |\zeta|_{\mathrm{j}_3} &= \sqrt{|x_0^2 - x_1^2 - x_2^2 + x_3^2|} \text{ on } M_3. \end{aligned}$$

Finally, we associate the set of bihyperbolic numbers with semi-Euclidean space $\mathbb{R}^4_2$ by the fact that the norms $|\zeta|_{\mathrm{j}_k}$ defined on the hypersurfaces M_k correspond to the semi-Euclidean norms:

$$\| \zeta \|_{\mathbb{R}^4_2} = \sqrt{\left| \langle \zeta, \zeta \rangle_{\mathbb{R}^4_2} \right|}$$

with metric signatures $(+,+,-,-)$, $(+,-,+,-)$, and $(+,-,-,+)$, respectively.

In this regard, we can introduce the cones of a bihyperbolic number $\zeta_0 \in M_k \subset H_2$ as follows:

$$SM_k(\zeta_0) = \left\{ \zeta \in M_k \middle| \ (\zeta - \zeta_0)\overline{(\zeta - \zeta_0)}^{\mathrm{j}_k} > 0 \text{ or } \zeta = \zeta_0 \right\},$$

$$NM_k(\zeta_0) = \left\{ \zeta \in M_k \middle| \ (\zeta - \zeta_0)\overline{(\zeta - \zeta_0)}^{\mathrm{j}_k} = 0 \right\},$$

and

$$TM_k(\zeta_0) = \left\{ \zeta \in M_k \middle| \ (\zeta - \zeta_0)\overline{(\zeta - \zeta_0)}^{\mathrm{j}_k} < 0 \text{ or } \zeta = \zeta_0 \right\}$$

respectively, being the space cone, null cone, and time cone of ζ_0 in hypersurfaces $M_k \subset H_2$ for $k = 1,2,3$.

In addition, let us consider the bihyperbolic numbers $\zeta = \zeta_{1,\mathrm{j}_s}e_{1,\mathrm{j}_s} + \zeta_{2,\mathrm{j}_s}e_{2,\mathrm{j}_s}$ for $s = 1,2,3$, then the products of e_{1,j_s} and e_{2,j_s} with ζ in H_2 are

$$e_{1,\mathrm{j}_1}H_2 = \left\{ e_{1,\mathrm{j}_1}\zeta = \zeta_{1,\mathrm{j}_1}e_{1,\mathrm{j}_1} : \zeta_{1,\mathrm{j}_1} \in H_2 \right\}, \quad e_{2,\mathrm{j}_1}H_2 = \left\{ e_{2,\mathrm{j}_1}\zeta = \zeta_{2,\mathrm{j}_1}e_{2,\mathrm{j}_1} : \zeta_{2,\mathrm{j}_1} \in H_2 \right\}$$

$$e_{1,\mathrm{j}_2}H_2 = \left\{ e_{1,\mathrm{j}_2}\zeta = \zeta_{1,\mathrm{j}_2}e_{1,\mathrm{j}_2} : \zeta_{1,\mathrm{j}_2} \in H \right\}, \quad e_{2,\mathrm{j}_2}H_2 = \left\{ e_{2,\mathrm{j}_2}\zeta = \zeta_{2,\mathrm{j}_2}e_{2,\mathrm{j}_2} : \zeta_{2,\mathrm{j}_2} \in H \right\},$$

$$e_{1,\mathrm{j}_3}H_2 = \left\{ e_{1,\mathrm{j}_3}\zeta = \zeta_{1,\mathrm{j}_3}e_{1,\mathrm{j}_3} : \zeta_{1,\mathrm{j}_3} \in H \right\}, \quad e_{2,\mathrm{j}_3}H_2 = \left\{ e_{2,\mathrm{j}_3}\zeta = \zeta_{2,\mathrm{j}_3}e_{2,\mathrm{j}_3} : \zeta_{2,\mathrm{j}_3} \in H \right\}.$$

Therefore, the elements of the sets $e_{1,\mathrm{j}_s}H_2$ and $e_{2,\mathrm{j}_s}H_2$ that are obtained with the idempotent representations of bihyperbolic numbers are bihyperbolic numbers for $s = 1,2,3$, and the following propositions can be given.

Theorem 1. *Let $\zeta \in SM_k(\zeta_0)$ such that $\zeta = \zeta_{1,\mathrm{j}_s}e_{1,\mathrm{j}_s} + \zeta_{2,\mathrm{j}_s}e_{2,\mathrm{j}_s}$ and $\zeta_0 = \zeta^1_{0,\mathrm{j}_s}e_{1,\mathrm{j}_s} + \zeta^2_{0,\mathrm{j}_s}e_{2,\mathrm{j}_s}$ are idempotent representations for $s = 1,2,3$:*

i. *If $s = k$ for $s,k = 1,2,3$, then $\zeta_{1,\mathrm{j}_s}e_{1,\mathrm{j}_s} \in SM_k\left(\zeta^1_{0,\mathrm{j}_s}e_{1,\mathrm{j}_s}\right)$ and $\zeta_{2,\mathrm{j}_s}e_{2,\mathrm{j}_s} \in SM_k\left(\zeta^2_{0,\mathrm{j}_s}e_{2,\mathrm{j}_s}\right)$.*

ii. *If $s \neq k$ for $s,k = 1,2,3$, then $\zeta_{1,\mathrm{j}_s}e_{1,\mathrm{j}_s} \in NM_k\left(\zeta^1_{0,\mathrm{j}_s}e_{1,\mathrm{j}_s}\right)$ and $\zeta_{2,\mathrm{j}_s}e_{2,\mathrm{j}_s} \in NM_k\left(\zeta^2_{0,\mathrm{j}_s}e_{2,\mathrm{j}_s}\right)$.*

Proof. Let $\zeta, \zeta_0 \in M_k$ and $\zeta \in SM_k(\zeta_0)$. $\zeta = z_1 + \mathrm{j}_2 z_2 = (x_0 + \mathrm{j}_1 x_1) + \mathrm{j}_2(x_2 + \mathrm{j}_1 x_3)$ and $\zeta_0 = \omega_1 + \mathrm{j}_2\omega_2 = (y_0 + \mathrm{j}_1 y_1) + \mathrm{j}_2(y_2 + \mathrm{j}_1 y_3)$, then the products of $\zeta - \zeta_0$ with their j_k-conjugates are

$$(\zeta - \zeta_0)\overline{(\zeta - \zeta_0)}^{\mathrm{j}_1} = (x_0 - y_0)^2 + (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2 > 0,$$

$$(\zeta - \zeta_0)\overline{(\zeta - \zeta_0)}^{\mathrm{j}_2} = (x_0 - y_0)^2 - (x_1 - y_1)^2 + (x_2 - y_2)^2 - (x_3 - y_3)^2 > 0,$$

$$(\zeta - \zeta_0)\overline{(\zeta - \zeta_0)}^{\mathrm{j}_3} = (x_0 - y_0)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 + (x_3 - y_3)^2 > 0$$

for $k = 1, 2, 3$, respectively. On the other hand, if we consider the multiplications of $\left(\zeta_{1,\mathrm{j}_s} e_{1,\mathrm{j}_s} - \zeta^1_{0,\mathrm{j}_s} e_{1,\mathrm{j}_s}\right)$ and also $\left(\zeta_{2,\mathrm{j}_s} e_{2,\mathrm{j}_s} - \zeta^2_{0,\mathrm{j}_s} e_{2,\mathrm{j}_s}\right)$ with their j_k-conjugates, then:

i. $\zeta_{1,\mathrm{j}_s} e_{1,\mathrm{j}_s} \in SM_k\left(\zeta^1_{0,\mathrm{j}_s} e_{1,\mathrm{j}_s}\right)$, $\zeta_{2,\mathrm{j}_s} e_{2,\mathrm{j}_s} \in SM_k\left(\zeta^2_{0,\mathrm{j}_s} e_{2,\mathrm{j}_s}\right)$ if $s = k$.
ii. $\zeta_{1,\mathrm{j}_s} e_{1,\mathrm{j}_s} \in NM_k\left(\zeta^1_{0,\mathrm{j}_s} e_{1,\mathrm{j}_s}\right)$, $\zeta_{2,\mathrm{j}_s} e_{2,\mathrm{j}_s} \in NM_k\left(\zeta^2_{0,\mathrm{j}_s} e_{2,\mathrm{j}_s}\right)$ if $s \neq k$.

   These are obtained for $s, k = 1, 2, 3$ since $\zeta, \zeta_0 \in M_k$ and $\zeta - \zeta_0 \in M_k$. □

The following two theorems can be proven by using a similar method.

Theorem 2. *Let $\zeta \in NM_k(\zeta_0)$. If $s = k$ or $s \neq k$ for $s, k = 1, 2, 3$, then $\zeta_{1,\mathrm{j}_s} e_{1,\mathrm{j}_s} \in NM_k\left(\zeta^1_{0,\mathrm{j}_s} e_{1,\mathrm{j}_s}\right)$ and $\zeta_{2,\mathrm{j}_s} e_{2,\mathrm{j}_s} \in NM_k\left(\zeta^2_{0,\mathrm{j}_s} e_{2,\mathrm{j}_s}\right)$.*

Theorem 3. *Let $\zeta \in TM_k(\zeta_0)$.*

i. If $s = k$ for $s, k = 1, 2, 3$, then $\zeta_{1,\mathrm{j}_s} e_{1,\mathrm{j}_s} \in TM_k\left(\zeta^1_{0,\mathrm{j}_s} e_{1,\mathrm{j}_s}\right)$ and $\zeta_{2,\mathrm{j}_s} e_{2,\mathrm{j}_s} \in TM_k\left(\zeta^2_{0,\mathrm{j}_s} e_{2,\mathrm{j}_s}\right)$.
ii. If $s \neq k$ for $s, k = 1, 2, 3$, then $\zeta_{1,\mathrm{j}_s} e_{1,\mathrm{j}_s} \in NM_k\left(\zeta^1_{0,\mathrm{j}_s} e_{1,\mathrm{j}_s}\right)$ and $\zeta_{2,\mathrm{j}_s} e_{2,\mathrm{j}_s} \in NM_k\left(\zeta^2_{0,\mathrm{j}_s} e_{2,\mathrm{j}_s}\right)$.

Theorem 4. *Let $\zeta \in H_2$ and the idempotent representation of ζ be $\zeta = \zeta_{1,\mathrm{j}_s} e_{1,\mathrm{j}_s} + \zeta_{2,\mathrm{j}_s} e_{2,\mathrm{j}_s}$ for $s = 1, 2, 3$. $SM_k(O)$, $NM_k(O)$, and $TM_k(O)$ are the space, null, and time cone of the origin for $k = 1, 2, 3$, respectively.*

i. If $\zeta \in SM_1(O)$, $\zeta \in NM_1(O)$ or $\zeta \in TM_1(O)$, then $\zeta_{1,\mathrm{j}_1}, \zeta_{2,\mathrm{j}_1} \in SM_1(O)$ $\zeta_{1,\mathrm{j}_1}, \zeta_{2,\mathrm{j}_1} \in NM_1(O)$ or $\zeta_{1,\mathrm{j}_1}, \zeta_{2,\mathrm{j}_1} \in TM_1(O)$, respectively, for $s = k = 1$.
ii. If $\zeta \in SM_k(O)$, $\zeta \in NM_k(O)$ or $\zeta \in TM_k(O)$, then $\zeta_{1,\mathrm{j}_s}, \zeta_{2,\mathrm{j}_s} \in SH(O)$, $\zeta_{1,\mathrm{j}_s}, \zeta_{2,\mathrm{j}_s} \in NH(O)$ or $\zeta_{1,\mathrm{j}_s}, \zeta_{2,\mathrm{j}_s} \in TH(O)$, respectively, for $s, k = 2, 3$ where $s = k$:

Proof.

i. Let $\zeta = x_0 + \mathrm{j}_1 x_1 + \mathrm{j}_2 x_2 + \mathrm{j}_3 x_3 \in SM_1(O)$ for $k = 1$. If $\zeta = 0$, the proof is obvious. Let $\zeta \neq 0$. Then, the product of ζ and its j_1-conjugate is $\zeta\overline{\zeta}^{\mathrm{j}_1} = x_0^2 + x_1^2 - x_2^2 - x_3^2 > 0$. On the other hand, the coefficients of the idempotent representation of ζ are $\zeta_{1,\mathrm{j}_1} = x_0 + \mathrm{j}_1 x_1 + \mathrm{j}_2 x_3 + \mathrm{j}_3 x_2$ and $\zeta_{2,\mathrm{j}_1} = x_0 + \mathrm{j}_1 x_1 - \mathrm{j}_2 x_3 - \mathrm{j}_3 x_2$ for $s = 1$. Hence, $\zeta_{1,\mathrm{j}_1}\overline{\zeta_{1,\mathrm{j}_1}}^{\mathrm{j}_1} = x_0^2 + x_1^2 - x_2^2 - x_3^2 > 0$ and $\zeta_{2,\mathrm{j}_1}\overline{\zeta_{2,\mathrm{j}_1}}^{\mathrm{j}_1} = x_0^2 + x_1^2 - x_2^2 - x_3^2 > 0$ are found. Therefore, $\zeta_{1,\mathrm{j}_1}, \zeta_{2,\mathrm{j}_1} \in SM_1(O)$. Similarly, if we choose $\zeta \in NM_1(O)$, then $\zeta_{1,\mathrm{j}_1}, \zeta_{2,\mathrm{j}_1} \in NM_1(O)$, or if we choose $\zeta \in TM_1(O)$, then $\zeta_{1,\mathrm{j}_1}, \zeta_{2,\mathrm{j}_1} \in TM_1(O)$ is obtained.
ii. Let $\zeta \in SM_2(O)$ for $s = k = 2$. If $\zeta = 0$, the proof is obvious. Let $\zeta \neq 0$. Then, the product of ζ and its j_2-conjugate is $\zeta\overline{\zeta}^{\mathrm{j}_2} = x_0^2 - x_1^2 + x_2^2 - x_3^2 > 0$, and the coefficients of the idempotent representation of ζ are

$$\zeta_{1,\mathrm{j}_2} = (x_0 + x_2) + \mathrm{j}_1(x_1 + x_3), \;\; \zeta_{2,\mathrm{j}_2} = (x_0 - x_2) + \mathrm{j}_1(x_1 - x_3)$$

   for $s = 2$. Moreover,

$$\zeta_{1,\mathrm{j}_2}\overline{\zeta_{1,\mathrm{j}_2}} = (x_0 + x_2)^2 - (x_1 + x_3)^2 = x_0^2 - x_1^2 + x_2^2 - x_3^2 > 0$$

and

$$\zeta_{2,j_2}\overline{\zeta_{2,j_2}} = (x_0 - x_2)^2 - (x_1 - x_3)^2 = x_0^2 - x_1^2 + x_2^2 - x_3^2 > 0$$

are obtained considering that $x_0x_2 - x_1x_3 = 0$ on the hypersurface M_2 for $k = 2$. Hence, $\zeta_{1,j_2}, \zeta_{2,j_2} \in SH(O)$. Similarly, when $\zeta \in NM_2(O)$, it is easily seen that $\zeta_{1,j_2}, \zeta_{2,j_2} \in NH(O)$ and also when $\zeta \in TM_2(O)$, $\zeta_{1,j_2}, \zeta_{2,j_2} \in TH(O)$. Furthermore, let $s = k = 3$. If we choose $\zeta \in SM_3(O)$, then $\zeta_{1,j_3}, \zeta_{2,j_3} \in SH(O)$. If $\zeta \in NM_3(O)$, then $\zeta_{1,j_3}, \zeta_{2,j_3} \in NH(O)$, and if $\zeta \in TM_3(O)$, then $\zeta_{1,j_3}, \zeta_{2,j_3} \in TH(O)$.

□

Remark 1. *Theorem 4 is not valid in the case of $s \neq k$, as can be seen by the following example.*

Example 3. *If we consider the bihyperbolic number $\zeta = z_1 + j_2z_2$, where $z_1 = 2 + 3j_1$ and $z_2 = 1 + 6j_1$, then we see that $\zeta = 2 + 3j_1 + j_2 + 6j_3$ is an element of M_1 since $2.3 - 1.6 = 0$. Furthermore, $\zeta = z_1 + j_2z_2$ belongs to $TM_1(O)$ by the fact that $4^2 + 3^2 - 1^2 - 6^2 < 0$. However, for $s = 2$, the coefficients $\zeta_{1,j_2}, \zeta_{2,j_2}$ of its idempotent representation belong to $SM_1(O)$, since this representation is given in the form $\zeta = \zeta_{1,j_2}e_{1,j_2} + \zeta_{2,j_2}e_{2,j_2} = (z_1 + z_2)e_{1,j_2} + (z_1 - z_2)e_{2,j_2} = (3 + 9j_1)e_{1,j_2} + (1 - 3j_1)e_{2,j_2}$.*

It is understood from the last four theorems that the relationship between the idempotent representations of ζ, $\zeta_{1,j_s}e_{1,j_s} + \zeta_{2,j_s}e_{2,j_s}$ for $s = 1,2,3$ with the space cone $SM_k(\zeta_0)$, the null cone $NM_k(\zeta_0)$, and the time cone $TM_k(\zeta_0)$ for $k = 1,2,3$ is meaningful when $s = k$.

Theorem 5. *Let the idempotent elements e_{1,j_s} and e_{2,j_s} for $s = 1,2,3$ and $SM_k(O)$, $NM_k(O)$, and $TM_k(O)$ denote the space, null, and time cone of the origin for $k = 1,2,3$, respectively. Then:*

i. $e_{1,j_1}, e_{2,j_1} \notin M_1$, $e_{1,j_1}, e_{2,j_1} \in NM_2(O)$ *and* $e_{1,j_1}, e_{2,j_1} \in NM_3(O)$ *for* $s = 1$;

ii. $e_{1,j_2}, e_{2,j_2} \in NM_1(O)$, $e_{1,j_2}, e_{2,j_2} \notin M_2$ *and* $e_{1,j_2}, e_{2,j_2} \in NM_3(O)$ *for* $s = 2$;

iii. $e_{1,j_3}, e_{2,j_3} \in NM_1(O)$, $e_{1,j_3}, e_{2,j_3} \in NM_2(O)$ *and* $e_{1,j_3}, e_{2,j_3} \notin M_3$ *for* $s = 3$.

Proof.

i. Let $s = 1$ and $\zeta = x_0 + j_1x_1 + j_2x_2 + j_3x_3 \in M_1$ for $k = 1$. Hence, $x_0x_1 - x_2x_3 = 0$. The idempotent elements $e_{1,j_1}, e_{2,j_1} \in H \subseteq H_2$ are

$$e_{1,j_1} = \frac{1 + j_1}{2} = \frac{1}{2} + \frac{j_1}{2} + j_2 0 + j_3 0,$$

$$e_{2,j_1} = \frac{1 - j_1}{2} = \frac{1}{2} - \frac{j_1}{2} + j_2 0 + j_3 0.$$

Since $x_0x_1 - x_2x_3 \neq 0$ for the idempotent elements, $e_{1,j_1}, e_{2,j_1} \notin M_1$. On the other hand, $x_0x_2 - x_1x_3 = 0$ and $\zeta\overline{\zeta}^{j_2} = x_0^2 - x_1^2 + x_2^2 - x_3^2 = 0$ for the bihyperbolic number $\zeta \in NM_2(O)$ for $k = 2$. If this is considered, then $e_{1,j_1}, e_{2,j_1} \in NM_2$. If the bihyperbolic number is chosen such as $\zeta \in NM_3(O)$ for $k = 3$, then $x_0x_3 - x_1x_2 = 0$ and $\zeta\overline{\zeta}^{j_3} = x_0^2 - x_1^2 - x_2^2 + x_3^2 = 0$ are obtained. Therefore, $e_{1,j_1}, e_{2,j_1} \in NM_3$.
Cases (ii.) and (iii.) can be proven similarly.

□

4. Topologies of Bihyperbolic Numbers

In this section, we establish topologies on the hypersurfaces $M_k \subset H_2$ for $k = 1,2,3$ from a similar point of view as that of Zeeman and Nanda [28–31]. They introduced and developed e-, s-, and t-topologies on Minkowski space. For further details, the readers are referred to [32–34] and the references therein. The basic difference in our approach is modeling the semi-Euclidean space with bihyperbolic numbers and defining open balls via bihyperbolic numbers.

4.1. Norm Topologies of Bihyperbolic Numbers

Let us restrict ourselves to one of the hypersurface $M_k \subset H_2$ for $k = 1, 2, 3$ based on the definitions of cones of a bihyperbolic number ζ_0 in H_2. Assume that $\zeta = x_0 + \mathrm{j}_1 x_1 + \mathrm{j}_2 x_2 + \mathrm{j}_3 x_3 \in M_k$, then the Euclidean norm of the vector $\zeta = (x_0, x_1, x_2, x_3)$ is

$$\|\zeta\| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

Now, we define the norm e-, s-, and t-topologies by considering the Euclidean norm on $M_k \subset H_2$.

4.1.1. Norm e-Topology

The Euclidean open balls of radius δ and center $\zeta_0 \in M_k$ are

$$D(\zeta_0, \delta) = \{\zeta \in M_k,\ \|\zeta - \zeta_0\| < \delta\}.$$

For all $\delta > 0$ and $\zeta_0 \in M_k$, the family of the Euclidean open balls is denoted by

$$B_N^E = \{D(\zeta_0, \delta) :\ \delta > 0, \zeta_0 \in M_k\}.$$

The topology on M_k generated by the basis B_N^E is called the norm e-topology and indicated by τ_N^E.

Nevertheless, the semi-Euclidean space is not locally homogeneous, and the Euclidean topology on it is inadequate. Zeeman suggested in [28,29] a new topology instead of the Euclidean topology. Subsequently, the space topology (s-topology) and time topology (t-topology), which are weaker versions of Zeeman's fine topology, were introduced [30–34]. With similar thoughts, we define norm s- and t-topologies on the hypersurfaces of H_2 as follows.

4.1.2. Norm s-Topology

Let $D(\zeta_0, \delta)$ be any Euclidean open ball with center ζ_0 and radius δ in B_N^E and $SM_k(\zeta_0)$ be the space cone, then

$$D^S(\zeta_0, \delta) = D(\zeta_0, \delta) \cap SM_k(\zeta_0)$$

is called the s-ball on hypersurface M_k. The family of all s-balls is denoted by B_N^S. Hence, B_N^S becomes a basis for a topology on M_k. The topology generated by the basis B_N^S is called the norm s-topology on M_k and denoted by τ_N^S.

Similarly, we can define the norm t-topology by changing $SM_k(\zeta_0)$ with the time cone $TM_k(\zeta_0)$.

4.1.3. Norm t-Topology

Any t-ball on hypersurface M_k is defined as

$$D^T(\zeta_0, \delta) = D(\zeta_0, \delta) \cap TM_k(\zeta_0).$$

The family of all t-balls is denoted by B_N^T and composes a basis for a topology on M_k. The topology generated by the basis B_N^T is called the norm t-topology on M_k and represented with τ_N^T.

There are three types of Cartesian products on H_2. These products are given by the following definitions.

Definition 1. *Let $X \subseteq H_2$ and images of X under transformations h_1 and h_2 be represented by*

$$h_1(X) = X_1 = \{z_1 | z_1 = h_1(\zeta),\ \zeta = z_1 + \mathrm{j}_2 z_2 \textit{ and } \zeta \in X\},$$
$$h_2(X) = X_2 = \{z_2 | z_2 = h_2(\zeta),\ \zeta = z_1 + \mathrm{j}_2 z_2 \textit{ and } \zeta \in X\}.$$

Then, the hyperbolic Cartesian product of X_1 and X_2 according to the basis $\{1, \mathrm{j}_2\}$ is given by

$$X_1 \times_H X_2 = \{z_1 + \mathrm{j}_2 z_2 | z_1 \in X_1,\ z_2 \in X_2\}$$

and it is called the Cartesian hyperbolic representation of X.

Definition 2. *Let $X \subseteq H_2$ and images of X under transformations h_{1,j_s} and h_{2,j_s} for $s = 1,2,3$ be represented by*

$$h_{1,\mathrm{j}_s}(X) = X_{1,\mathrm{j}_s} = \left\{ \zeta_{1,\mathrm{j}_s} \middle| \zeta_{1,\mathrm{j}_s} = h_{1,\mathrm{j}_s}(\zeta),\ \zeta = \zeta_{1,\mathrm{j}_s} e_{1,\mathrm{j}_s} + \zeta_{2,\mathrm{j}_s} e_{2,\mathrm{j}_s} \text{ and } \zeta \in X \right\},$$

$$h_{2,\mathrm{j}_s}(X) = X_{2,\mathrm{j}_s} = \left\{ \zeta_{2,\mathrm{j}_s} \middle| \zeta_{2,\mathrm{j}_s} = h_{2,\mathrm{j}_s}(\zeta),\ \zeta = \zeta_{1,\mathrm{j}_s} e_{1,\mathrm{j}_s} + \zeta_{2,\mathrm{j}_s} e_{2,\mathrm{j}_s} \text{ and } \zeta \in X \right\}.$$

Then, the idempotent Cartesian product of X_{1,j_s} and X_{2,j_s} according to the basis $\{e_{1,\mathrm{j}_s}, e_{2,\mathrm{j}_s}\}$ given by

$$X_{1,\mathrm{j}_s} \times_{I_{\mathrm{j}_s}} X_{2,\mathrm{j}_s} = \left\{ \zeta_{1,\mathrm{j}_s} e_{1,\mathrm{j}_s} + \zeta_{2,\mathrm{j}_s} e_{2,\mathrm{j}_s} \middle| \zeta_{1,\mathrm{j}_s} \in X_{1,\mathrm{j}_s},\ \zeta_{2,\mathrm{j}_s} \in X_{2,\mathrm{j}_s} \right\}$$

is called the Cartesian idempotent representation of X.

The Cartesian idempotent representation for $s = 3$ was given in [9].

Definition 3. *Let $X \subseteq H_2$ and images of X under transformations λ_1, λ_2, λ_3 and λ_4 be represented by*

$$\lambda_1(X) = X_1 = \{w_1 | w_1 = \lambda_1(\zeta),\ \zeta = w_1\mathrm{i}_1 + w_2\mathrm{i}_2 + w_3\mathrm{i}_3 + w_4\mathrm{i}_4 \text{ and } \zeta \in X\},$$

$$\lambda_2(X) = X_2 = \{w_2 | w_2 = \lambda_2(\zeta),\ \zeta = w_1\mathrm{i}_1 + w_2\mathrm{i}_2 + w_3\mathrm{i}_3 + w_4\mathrm{i}_4 \text{ and } \zeta \in X\},$$

$$\lambda_3(X) = X_3 = \{w_3 | w_3 = \lambda_3(\zeta),\ \zeta = w_1\mathrm{i}_1 + w_2\mathrm{i}_2 + w_3\mathrm{i}_3 + w_4\mathrm{i}_4 \text{ and } \zeta \in X\},$$

$$\lambda_4(X) = X_4 = \{w_4 | w_4 = \lambda_4(\zeta),\ \zeta = w_1\mathrm{i}_1 + w_2\mathrm{i}_2 + w_3\mathrm{i}_3 + w_4\mathrm{i}_4 \text{ and } \zeta \in X\}.$$

Then, the spectral Cartesian product of X_1, X_2, X_3, and X_4 according to the basis $\{\mathrm{i}_1, \mathrm{i}_2, \mathrm{i}_3, \mathrm{i}_4\}$ is

$$X_1 \times_U X_2 \times_U X_3 \times_U X_4 = \{w_1 i_1 + w_2 i_2 + w_3 i_3 + w_4 i_4 | w_s \in X_s,\ s = 1,2,3,4\}$$

and $X = X_1 \times_U X_2 \times_U X_3 \times_U X_4$ is called the Cartesian spectral representation of X.

The symbol $\times_U$ is defined, since $w_s \in X_s$ are the real numbers for $s = 1,2,3,4$, and the usual topology will be taken on the real numbers set $\mathbb{R}$ in the next subsections. Now, we can define the hyperbolic, idempotent, and spectral topologies of bihyperbolic numbers by using the new Cartesian representations.

4.2. Hyperbolic Topologies of Bihyperbolic Numbers

$\|z\| = \sqrt{x_1{}^2 + x_2{}^2}$ for $z = x_1 + \mathrm{j}_1 x_2 \in H$ is the Euclidean norm on H. Hence, we can easily define the Euclidean topology on the set of hyperbolic numbers H. Namely, the Euclidean open ball is $D(z_0, \delta) = \{z \in H,\ \|z - z_0\| < \delta\}$ for $z_0 \in H$ and $\delta > 0$. Clearly, the family of these Euclidean open balls is a basis for the usual Euclidean topology on H and is denoted by B^E.

4.2.1. Hyperbolic e-Topology

Let D_1 and D_2 be any Euclidean open balls on H, then

$$B^E \times B^E = \left\{ D_1 \times D_2 \middle| D_1, D_2 \in B^E \right\}$$

is a basis for a product topology on $H \times H$. Since $H \times H \cong H_2$,

$$B_H^E = \left\{ D_1 \times_H D_2 \middle| D_1, D_2 \in B^E \right\}$$

is a basis for a topology on H_2 such that

$$\begin{aligned} D_1 \times_H D_2 &= D_1\left(z_0^1, \delta_1\right) \times_H D_2\left(z_0^2, \delta_2\right) \\ &:= \left\{ \zeta = z_1 + \mathrm{j}_2 z_2 \middle| \left\| z_1 - z_0^1 \right\| < \delta_1, \ \left\| z_2 - z_0^2 \right\| < \delta_2 \right\}. \end{aligned}$$

The topology generated by this basis is called the hyperbolic e-topology and denoted by τ_H^E.

4.2.2. Hyperbolic s-Topology

Let us consider $D(z_0, \delta) \in B^E$ for all $z_0 \in H$ and all $\delta > 0$, then

$$D(z_0, \delta) \cap SH(z_0) = D^S(z_0, \delta)$$

is an s-ball. In this case, the family of all s-balls is B^S, and B^S is a basis for a topology on H. Moreover,

$$B^S \times B^S = \left\{ D_1^S \times D_2^S \middle| D_1^S, D_2^S \in B^S \right\}$$

is a basis for a product topology on $H \times H$. The family on $H \times H \cong H_2$

$$B_H^S = \left\{ D_1^S \times_H D_2^S \middle| D_1^S, D_2^S \in B^S \right\}$$

is a basis for a topology on H_2 such that

$$D_1^S \times_H D_2^S = D_1^S\left(z_0^1, \delta_1\right) \times_H D_2^S\left(z_0^2, \delta_2\right) := \left\{ \zeta = z_1 + \mathrm{j}_2 z_2 \middle| z_1 \in D_1^S, \ z_2 \in D_2^S \right\}.$$

The topology generated by B_H^S on H_2 is named the hyperbolic s-topology and denoted by τ_H^S.

4.2.3. Hyperbolic t-Topology

Let $D(z_0, \delta) \in B^E$ for $z_0 \in H$ and $\delta > 0$, then

$$D(z_0, \delta) \cap TH(z_0) = D^T(z_0, \delta)$$

is called the t-ball. The family of all t-balls is shown as B^T, and it is a basis for a topology on H. Hence, the family

$$B^T \times B^T = \left\{ D_1^T \times D_2^T \middle| D_1^T, D_2^T \in B^T \right\}$$

is a basis for a product topology on $H \times H$, and moreover, the family

$$B_H^T = \left\{ D_1^T \times_H D_2^T \middle| D_1^T, D_2^T \in B^T \right\}$$

is a basis for a topology on H_2 where

$$D_1^T \times_H D_2^T = D_1^T\left(z_0^1, \delta_1\right) \times_H D_2^T\left(z_0^2, \delta_2\right) := \left\{ \zeta = z_1 + \mathrm{j}_2 z_2 \middle| z_1 \in D_1^T, \ z_2 \in D_2^T \right\}.$$

The topology generated by this basis is called the hyperbolic t-topology on H_2 and denoted by τ_H^T.

4.3. Idempotent Topologies of Bihyperbolic Numbers

The idempotent e-, s-, and t-topology can be defined by making use of the idempotent representations of the bihyperbolic numbers. These idempotent representations are given by the equalities of (3). We only use the second and third idempotent representations since the coefficients of the first representation are not hyperbolic numbers. They are bihyperbolic numbers. This means that we consider the representations of a bihyperbolic numbers according to e_{1,j_l} and e_{2,j_l} $(l = 2, 3)$. Moreover, we construct another topology by using the spectral representation given by Equation (4).

4.3.1. Idempotent e-Topology

The family of the open balls $D(\zeta_{0,\mathrm{j}_l}, \delta)$ is B^E for all $\zeta_{0,\mathrm{j}_l} \in H$ $(l = 2, 3)$ and $\delta > 0$, then B^E is a basis for the usual Euclidean topology on H. Furthermore, the family

$$B^E_{I_{\mathrm{j}_l}} = \left\{ D_1 \times_{I_{\mathrm{j}_l}} D_2 \middle| \, D_1, D_2 \in B^E, \, l = 2, 3 \right\}$$

is a basis for a topology on H_2. Here, the Cartesian product of D_1 and D_2 is defined by

$$\begin{aligned} D_1 \times_{I_{\mathrm{j}_l}} D_2 &= D_1\left(\zeta^1_{0,\mathrm{j}_l}, \delta_1\right) \times_{I_{\mathrm{j}_l}} D_2\left(\zeta^2_{0,\mathrm{j}_l}, \delta_2\right) \\ &:= \left\{ \zeta_{1,\mathrm{j}_l} e_{1,\mathrm{j}_l} + \zeta_{2,\mathrm{j}_l} e_{2,\mathrm{j}_l} \middle| \, \left\| \zeta_{1,\mathrm{j}_l} - \zeta^1_{0,\mathrm{j}_l} \right\| < \delta_1, \, \left\| \zeta_{2,\mathrm{j}_l} - \zeta^2_{0,\mathrm{j}_l} \right\| < \delta_2, \, l = 2, 3 \right\}. \end{aligned}$$

The topology generated by this basis on H_2 is called the idempotent e-topology and indicated by $\tau^E_{I_{\mathrm{j}_l}}$.

4.3.2. Idempotent s-Topology

Let $D(\zeta_{0,\mathrm{j}_l}, \delta) \in B^E$ for $\zeta_{0,\mathrm{j}_l} \in H$ $(l = 2, 3)$ and $\delta > 0$, then the s-ball is

$$D(\zeta_{0,\mathrm{j}_l}, \delta) \cap SH(\zeta_{0,\mathrm{j}_l}) = D^S(\zeta_{0,\mathrm{j}_l}, \delta)$$

and the family of all s-balls is denoted by B^S. We know that B^S is the basis for a topology on H. Hence,

$$B^S \times B^S = \left\{ D^S_1 \times D^S_2 \middle| \, D^S_1, D^S_2 \in B^S \right\}$$

is a basis for product topology on $H \times H \cong H_2$ and

$$B^S_{I_{\mathrm{j}_l}} = \left\{ D^S_1 \times_{I_{\mathrm{j}_l}} D^S_2 \middle| \, D^S_1, D^S_2 \in B^S, \, l = 2, 3 \right\}$$

is a basis for a topology on the H_2, where

$$\begin{aligned} D^S_1 \times_{I_{\mathrm{j}_l}} D^S_2 &= D^S_1\left(\zeta^1_{0,\mathrm{j}_l}, \delta_1\right) \times_{I_{\mathrm{j}_l}} D^S_2\left(\zeta^2_{0,\mathrm{j}_l}, \delta_2\right) \\ &:= \left\{ \zeta = \zeta_{1,\mathrm{j}_l} e_{1,\mathrm{j}_l} + \zeta_{2,\mathrm{j}_l} e_{2,\mathrm{j}_l} \middle| \, \zeta_{1,\mathrm{j}_l} \in D^S_1, \, \zeta_{2,\mathrm{j}_l} \in D^S_2, \, l = 2, 3 \right\}. \end{aligned}$$

The topology generated by this basis on H_2 is called the idempotent s-topology and denoted by $\tau^S_{I_{\mathrm{j}_l}}$ for $l = 2, 3$.

4.3.3. Idempotent t-Topology

The family of the open balls $D(\zeta_{0,\mathrm{j}_l}, \delta)$ for $\zeta_{0,\mathrm{j}_l} \in H$ and $\delta > 0$ is B^E, then the t-ball is denoted by

$$D(\zeta_{0,\mathrm{j}_l}, \delta) \cap TH(\zeta_{0,\mathrm{j}_l}) = D^T(\zeta_{0,\mathrm{j}_l}, \delta).$$

Let the family of the t-balls be B^T, then B^T becomes a basis for a topology on H. Hence,

$$B^T \times B^T = \left\{ D^T_1 \times D^T_2 \middle| \, D^T_1, D^T_2 \in B^T \right\}$$

is a basis for a product topology on $H \times H$. On the other hand, since $H \times H \cong H_2$,

$$B^T_{I_{j_l}} = \left\{ D^T_1 \times_{I_{j_l}} D^T_2 \middle| \, D^T_1, D^T_2 \in B^T, \; l = 2,3 \right\}$$

is a basis for a topology on H_2, where

$$\begin{aligned} D^T_1 \times_{I_{j_l}} D^T_2 &= D^T_1\left(\zeta^1_{0,j_l}, \delta_1\right) \times_{I_{j_l}} D^T_2\left(\zeta^2_{0,j_l}, \delta_2\right) \\ &:= \left\{ \zeta = \zeta_{1,j_l} e_{1,j_l} + \zeta_{2,j_l} e_{2,j_l} \middle| \, \zeta_{1,j_l} \in D^T_1, \; \zeta_{2,j_l} \in D^T_2, \; l = 2,3 \right\}. \end{aligned}$$

This topology is called the idempotent t-topology and denoted by $\tau^T_{I_{j_l}}$ for $l = 2,3$.

4.3.4. Spectral Topology of Bihyperbolic Numbers

Let $\zeta \in H_2$; the spectral representation of ζ is $\zeta = w_1 i_1 + w_2 i_2 + w_3 i_3 + w_4 i_4$, where $w_1, w_2, w_3, w_4 \in \mathbb{R}$. It is well known that the family of all open intervals of $\mathbb{R}$ generates the usual topology on $\mathbb{R}$. If any open interval in $\mathbb{R}$ is represented by D, then the family

$$B_U = \{ D_1 \times_U D_2 \times_U D_3 \times_U D_4 | D_m = (x_m - \delta_m, x_m + \delta_m), \; m = 1,2,3,4 \}$$

becomes a basis for a topology on H_2, where

$$D_1 \times_U D_2 \times_U D_3 \times_U D_4 := \{ \, w_1 i_1 + w_2 i_2 + w_3 i_3 + w_4 i_4 | \, w_m \in D_m, \; m = 1,2,3,4 \}.$$

This topology is called the spectral topology on H_2.

5. Comparative Evaluation

There are a few reasons why we might consider different topologies on the set of bihyperbolic numbers. First, we have discovered the roles of bihyperbolic numbers in the special theory of relativity by associating the real-valued norm of a bihyperbolic numbers with the structures of hypersurfaces in the semi-Euclidean space. It is known that the semi-Euclidean space is endowed with a bilinear structure that is symmetric and non-degenerate, but not positive-definite, which do not, in general, induce a basis of topology via the collection of the usual open balls with different radii. This means that there is no nice generated topology to coincide locally with the Euclidean topology. Zeeman [28,29] explained the roots of the problem obviously as follows: the semi-Euclidean space is not locally homogeneous such as there is an associated light cone that separates space-like vectors from time-like vectors at each point in this space. Furthermore, the group of all homeomorphisms of four-dimensional Euclidean space is inadequate physically. In this regard, Zeeman introduced alternative topologies, now known as the Zeeman topology or the finest topology and, alternatively, the t-topology, which is the finest topology such that the Euclidean topology is induced on the time axes only, and the s-topology, which is the finest topology such that the Euclidean topology is induced only on the space-like hyperplanes. Zeeman's perspective attracted a great deal of attention on the research based on topologies of non-Euclidean spaces and followed by studies such as [30–40].

As is known, to study the space–time geometry of special relativity, taking the hyperbolic numbers is a useful approach. While this relationship has been generally used in the investigation of the two-dimensional case, there is a more recent study [10] on the hyperbolic numbers together with their multidimensional generalizations. In this paper, W. D. Richter introduced the space–time or spherical hyperbolic complex numbers in 3 and 4 dimensions. Alternative topologies on 3- and 4-dimensional Minkowski space–time may be defined based on [10,28–40].

Topological structures on the bihyperbolic numbers set have not been clarified yet. Just as the hyperbolic numbers are related to the Minkowski plane, the bihyperbolic numbers are related to four-dimensional semi-Euclidean space, and defining topologies for non-

Euclidean spaces is quite difficult as mentioned above. This gap has been closed in the present study. By constructing these topologies, we presented a new mathematical tool to analyze, explain, elaborate, and exemplify a variety of subjects related to differential geometry and physics.

6. Conclusions

In light of the approach of Zeeman, we constructed new convenient topologies on the set of bihyperbolic numbers. For this purpose, we defined the cones at any point of the hypersurfaces in the semi-Euclidean space through bihyperbolic numbers and examined their structures. Finally, all deductions and the alternative representations of bihyperbolic numbers allowed us to define these topologies.

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