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# ON DIFFUSE INSTABILITY OF ORTHOTROPIC VISCOPLASTIC PLATES

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**Abstract**

Elastic strain is covered by the effective medium homogenization method inside a representative volume element (RVE). It has an incremental quasi rate-independent (QRI) form obtained by the endochronic concept of thermodynamic time. The rate dependence takes place by means of stress rate dependent value of the initial yield stress. Free meso rotations and constrained micro rotations within a representative volume element (RVE) are assumed. A comparison between QRI and J2 diffuse instability equations is presented for orthotropic materials. A new QRI nonlinear evolution equation for orthotropic materials is derived by tensor function representation with Spencer-Boehler structural tensors.

**Keywords:** Orthotropic, metal forming, endochronic, directionality

## 1. Introduction

The motivation for this paper lies in the fact that Hill's orthotropic yield function, based on a simple generalization of Mises isotropic yield function, is not able to cover diverse stress directions in metal forming. This especially holds for non-proportional stress-strain histories (cf. Micunovic, 2009b) even for initially isotropic materials. The second cause is that overwhelming majority of so-called J2 theories of viscoplasticity is built using only uniaxial tension tests. Trials to apply such theories to multiaxial case are useless since they lead to introducing a lot of material constants. However, such calibrated constants predict very badly material behavior at shear and other multiaxial strain histories. The third is to extend region of diffuse instability analysis to higher strain rates which may appear in metal forming technology. Finally, an appropriate approach to inelastic strain induced anisotropy, based on micro-meso analysis, is certainly welcome.

For all these reasons, another approach is needed. First, a geometry of deformation based on Kroener's incompatibility approach as well on Eshelbian implanting eigenstrains is given briefly. By using the self-consistent effective medium approach, an effective Hooke's law for anisotropic media (cf. Micunovic 2005) is used to model an orthotropic medium by stiff prolate spheroidal

inclusions. This method, applied to inelastic polycrystals, is based on constrained micro-rotations and free meso-rotations.

Then, the evolution equation for inelastic stretching is based on quasi rate independence (QRI) and Rabotnov's inelastic delay with tensor representation. A Vakulenko's type of endochronic thermodynamics with the concept of thermodynamic time (Vakulenko 1970) lies behind such a QRI approach. Experiments in JRC, Ispra, Italy (Micunovic et al. 2007) confirmed that for isotropic reactor AISI reactor steels even a small number of material constants in the QRI-approach leads to very high correlation coefficient. These tests gave rise to a universal material constant constituting the most important part of the hereditary Rabotnov's kernel. The constant holds for diverse multiaxial stress histories and a wide range of strain rates - as it has been found in Micunovic et al (1997). It must be mentioned here that tensor function representation has been extensively used by Sawczuk, Murakami, Boehler and the others (cf. Micunovic et al 1997; Micunovic 1981; 2005; 2006; 2009a; 2009b; Micunovic and Kudrjavceva 2014; 2019a; Boehler 1979) in the field of inelasticity. Such an approach makes the question of induced anisotropy logical and easy. The controversial issue of inelastic spin has been solved in the paper by Micunovic (2009b) by the concept of fixing of orientations of intermediate reference configurations.

In the next sections is given a brief review of QRI approach as well as the J2 approach to viscoplasticity of orthotropic materials.

## 2. Evolution and constitutive equations

### *2.1 Geometric preliminaries*

As a prerequisite, it is necessary to build a correct geometric description of analyzed inelastic deformation process. Consider a polycrystalline body in a real configuration $(k(t))\ (\mathrm{k(t)})$ with dislocations and an inhomogeneous temperature field $T(X,t)\ \mathrm{T(X,t)}$ (where $t\ t$ stands for time and $X\ \mathrm{X}$ for the place of a generic material element), a body subject to surface tractions. Corresponding to $(k(t))\ (\mathrm{k(t)})$ there exists a reference configuration $(k(t_0))\ (\mathrm{k(t_0)})$ with (differently distributed) dislocations at a homogeneous temperature $T_0\ \mathrm{T_0}$ without surface tractions. Due to these defects such a configuration is not stress-free but contains an equilibrated residual stress (often named as "back-stress"). It is generally accepted that linear mapping function $\mathrm{F}(.,t):(k(t_0))\rightarrow(k(t))\ \mathrm{F(.,t)}\colon(\mathrm{k(t_0)})\rightarrow(\mathrm{k(t)})$ is a compatible second rank *total deformation gradient*. Here time $t$ as the scalar parameter allows for a family of deformed configurations $(k)\ (\mathrm{k})$. In the papers dealing with continuum representations of dislocation distributions configuration $(k(t))\ (k(t))$ is imagined to be cut into small elements denoted by $(n(t))\ (\mathrm{n(t)})$. The deformation tensor $\Phi(.,t):(n(t))\rightarrow(k(t))\ \Phi(.,\mathrm{t})\colon(\mathrm{n(t)})\rightarrow(\mathrm{k(t)})$ obtained in such a way is incompatible and should be called the (purely) *elastic distortion tensor* whereas $(n(t))\ (\mathrm{n(t)})$-elements are commonly named as *natural state local reference configurations*. Of course, the corresponding *distortion tensor* $\Pi(.,t):=\Phi(t)(.,t)^{-1}\mathrm{F}(.,t)$ $\Pi(.,\mathrm{t}):=\Phi(\mathrm{t})(.,\mathrm{t})^{-1}\mathrm{F}(.,\mathrm{t})$ is not compatible, whereas $\mathrm{F}\ \mathrm{F}$ is found by comparison of material fibres in $(k(t_0))\ (\mathrm{k(t_0)})$ and $(k(t))\ (\mathrm{k(t)})$ while $\Phi(t)\ \Phi(\mathrm{t})$ is determined by crystallographic vectors in $(n(t))\ (\mathrm{n(t)})$ and $(k(t))\ (\mathrm{k(t)})$. Multiplying above formula from the left hand side by $\Phi(.,t)\ \Phi(.,\mathrm{t})$ we reach the original Kröner's decomposition rule which is often wrongly named as Lee's decomposition formula.

Following [9] let us imagine that a typical (n)-element (called in the sequel representative volume element and denoted by RVE) is composed of $N$ monocrystal grains, such that each $\Lambda$-th grain has $N_s$ slip systems $\mathbf{A}_{\alpha\Lambda} \equiv \mathbf{s}_{\alpha\Lambda} \otimes \mathbf{n}_{\alpha\Lambda},\ \alpha \in \{1, N_s\}$. For instance, for FCC crystals $N_s = 12$. Here $\mathbf{s}_{\alpha\Lambda}$ is the unit slip vector and $\mathbf{n}_{\alpha\Lambda}$ is the unit vector normal to the slip plane.

If the time rate of residual microelastic strains is negligible, the volumes of all the grains inside the considered RVE are the same, and the microspins are very small (for details see [7]) then the inelastic stretching tensor for RVE reads (with time rate $\mathrm{D}\gamma_{\alpha\Lambda}$ of the $\gamma_{\alpha\Lambda}$ slip):

$$\mathrm{D}_P \approx \frac{1}{N} \sum_{\Lambda} \sum_{\alpha} \left( \mathrm{A}_{\alpha\Lambda}^T + \mathrm{A}_{\alpha\Lambda} \right) D\gamma_{\alpha\Lambda}. \tag{1}$$

*2.2 Hooke's Law by Homogenization Approach*

Let the microelastic strain of a $\Lambda$-grain inside a RVE be denoted by $\mathbf{E}_{\Lambda E}$. Its volume average i.e., $\boldsymbol{E}_E = \langle \boldsymbol{E}_{\Lambda E} \rangle$ is called the macroelastic strain, with $\mathbf{E}_E = (\boldsymbol{\Phi}^T \boldsymbol{\Phi} - \mathbf{1})/2$. It must be noted, however, that microelastic strain of a $\Lambda$-grain escorting mapping $(n(t)) \to (k(t))$ of RVE is different from residual microelastic strain $(\mathbf{U}_{\Lambda E}^2 - \mathbf{1})/2$ whose source is inhomogeneity of grains inside a RVE (appearing at mapping $(n(t_0)) \to (k(t_0))$). It is natural to assume that $\|\boldsymbol{E}_{\Lambda E}\| >> \|\boldsymbol{E}_{\Lambda E}^{res}\|$. Then residual microstrains are negligible for monocrystals. If the microelastic strain is provoked by the corresponding microstress $\mathbf{S}_\Lambda$, then its volume average reads $\mathbf{S} = \langle \mathbf{S}_\Lambda \rangle$. Here the second Piola-Kirchhoff stress tensor $\mathrm{S} = \boldsymbol{\Phi}^{-1} \mathbf{T} \boldsymbol{\Phi}^{-T}$ is calculated with respect to the local reference $(n(t))$-configuration. Hooke's law for the $\Lambda$-grain reads

$$\mathrm{S}_\Lambda = \mathfrak{D}_\Lambda : \mathrm{E}_{\Lambda E}, \tag{2}$$

where $\mathfrak{D}_\Lambda$ is elasticity tensor of $\Lambda$-grain. It is assumed here that elastic strain is small and inelastic finite. Then, the volume averaging of the above relation throughout the RVE gives the familiar equation of homogenization approach:

$$\langle \mathrm{S}_\Lambda \rangle = \mathfrak{D}_{eff} : \langle \mathrm{E}_{\Lambda E} \rangle,\ \textit{i.e.},\ \mathrm{S} = \mathfrak{D}_{eff} : \mathrm{E}_E. \tag{3}$$

In homogenization theories for composites with particulate inclusions there are two distinct self-consistent approaches:

*a) effective medium approach* where it is assumed that each inclusion behaves as isolated and immersed into a medium having effective constants $\mathfrak{D}_{eff}$ and

*b)effective field approach* with an assumption that again each inclusion behaves approximately as isolated and situated into the matrix with elasticity constants $D_M$ while influence of neighbouring inclusions is taken into account by means of the effective strain field $\mathbf{E}_*$ acting on the considered inclusion (Kanaun and Levin 1993).

In the paper (Levin 1982) the author proposed the approach that for polycrystals the considered grain is understood as an inclusion in the matrix composed by all the other grains applying in such a way the effective field approach. If instead of an infinite medium we employ this reasoning to the considered *RVE* then a direct application of the Levin's expression for the effective elastic moduli fourth rank tensor may be written as follows (index ``$M''$ stands for matrix while the notation $\langle \bullet \rangle_\omega$ means averaging by orientation only):

$$\mathfrak{D}_{eff} = \mathfrak{D}_M + [\mathfrak{D}] \left( \mathfrak{I} - \langle \mathfrak{A}\ \mathfrak{P} \rangle_\omega \right)^{-1} \langle \mathfrak{A} \rangle_\omega. \tag{4}$$

Here $\mathfrak{D}_M = \langle \mathfrak{D} \rangle_\omega$ and $\mathfrak{I}_{abcd} = \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}$ is the unit fourth rank tensor. By means of $\mathfrak{K}_{abcd} = (\partial_a \partial_d G_{bc})_{(ab)}$, we have

$$\mathfrak{P}_\Lambda = \mathfrak{S}_\Lambda \ \mathfrak{D}_M^{-1} \equiv -\int_{\Delta V_\Lambda} \mathfrak{K}(x - x')\, dV',$$

with $\mathfrak{A}_\Lambda = (I + P_\Lambda[D])^{-1}$ and $[\mathfrak{D}] = \mathfrak{D}_\Lambda - \mathfrak{D}_M$. In these relations $\mathfrak{S}_\Lambda$ is Eshelby's fourth rank tensor and $\mathbf{G}$ is the Green's function (second rank tensor) for the considered anisotropic crystal. The above expressions may be used for an analytical determination of the effective elastic constants.

Let us assume that a matrix is strengthened by some identical prolate parallel spheroidal strong fibres with symmetry axes aligned with a Cartesian axis $x_3$. Suppose now that one half of fibres population are rotated by some angle $\theta$ around axis $x_1$ whereas the remaining ones are rotated by $-\theta$ around the same axis. Let concentrations of both subgroups be the same $c_1 = c_2$. Aspect ratios are supposed to be the same. In this way we obtain a composite with planar symmetry with mirror axis $x_1$ and one family of fibres with two subfamilies of parallel identical voids. Otherwise fibres inside each subgroup are randomly distributed. If elasticity modulus of each fibre is much larger than the elasticity modulus of the underlying matrix, then by using (4) effective compliance matrix is calculated. Then, the obtained effective elastic symmetry is approximately orthotropic [20]. Details of the calculation are given in Micunovic and Kudrjavceva (2019a) where are analyzed more complex distributions of spheroidal prolate thin fibres and oblate thin ellipsoidal voids.

*2.3 Evolution equation – micro to meso transition*

A special attention is paid to the associativity of flow rule based on the loading function $\Omega$ and derived by Rice (Rice 1971). His evolution equation for inelastic stretching is based on the *loading function* and *PIR* - pattern of internal rearrangements:

$$\mathrm{D}_P = \partial_{\mathbf{S}} \Omega(\mathrm{S}, PIR). \tag{5}$$

The experimental evidence (cf. Micunovic et al. 1997) has shown that real time in the evolution equation for inelastic stretching may be replaced by some nondecreasing scalar function $\zeta$ of inelastic strain history responsible for aging whose ultimate value leads finally to rupture of the body. Vakulenko called this function - thermodynamic time (cf. Vakulenko 1970; Micunovic 2005). This is the main idea in this so-called endochronic thermodynamics and in such a concept purely elastic strain does not contribute to the thermodynamic time. Rather lengthy details are given in Micunovic (2006). Replacement of real time by $\zeta$ gives us a single evolution equation for inelastic stretching from very low to very high strain rates (cf. also Micunovic 2009b, p.74).

## 3. Orthotropic QRI materials

If the thermodynamic time is the same for all the grains, then normality of the inelastic stretching onto a yield (or a loading) surface could hold. In such a case, the meso evolution equation for RVE reads:

$$\mathrm{D}_P = \ \Lambda \ \partial_{\mathbf{S}} \Omega\left(\mathrm{S}, \mathrm{e}_P, \mathrm{M}_\Sigma\right) \ \text{with} \ \ \Lambda = \eta(\sigma_{eq}/\mathrm{Y}\text{-}1)^{\wedge} \lambda \ \ D_t \sigma_{eq} \quad \exp(\text{-M}).) \tag{6}$$

Here S is the the second Piola-Kirchhoff stress, $\mathbf{e}_P = (\mathbf{1} - \mathbf{\Pi}^{-T}\mathbf{\Pi}^{-1})/2$ – inelastic strain, $\sigma_{eq} \equiv \| \boldsymbol{S} \|$, $M_\Sigma$ – Spencer-Boehler's structural tensors describing anisotropy type, $\eta$ – Heaviside function, $Y$ – dynamic yield stress. Since the initial yield stress depends on value of

stress rate the evolution equation is rate dependent. Incrementality is based on almost exact experimentally observed proportionality of $\mathbf{D}_P$ and $D_t\sigma_{eq}$ (cf. Micunovic et al. 1997). Constant $M$ in the $\Lambda$-function covers this proportionality very well for reactor and some other steels.

Let the characteristic structural tensors be given by orthonormal vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Then, structural Spencer-Boehler tensors (cf. Micunovic and Kudrjavceva 2019b) are $\mathbf{M}_k = \mathbf{a}_k \otimes \mathbf{a}_k, k = 1,2,3$. For orthotropic materials $\mathbf{M}_{\Sigma 1} = \mathbf{M}_{11}$ and $\mathbf{M}_{\Sigma 2} = \mathbf{M}_{22}$ are sufficient since $\mathbf{M}_3 = \mathbf{1} - \mathbf{M}_1 - \mathbf{M}_2$. Thus, according to isotropicization theorem for anisotropic tensor functions [20], $\Omega = \Omega(\mathbf{S}, \mathbf{e}_P, \mathbf{M}_\Sigma)$ may be taken as an isotropic function of its arguments.

When thermodynamic time is a nonlinear function of inelastic power (i.e., non-steady aging happens) the equation (6) covers non-steady aging as well. For this it is necessary to replace time rates by thermodynamic time rates in (6).

When the material body possesses three preferred anisotropy directions, then the arguments of the evolution equation have to include the diadics $\mathbf{M}_k = \mathbf{a}_k \otimes \mathbf{a}_k, k = 1,2,3$. If $\mathbf{a}_k$ are unit vectors then $tr\,\mathbf{M}_k = 1$. Thus, (6) becomes

$$\mathrm{D}_P = \Lambda\ \partial_{\mathbf{S}}\Omega\left(\mathrm{S}, \mathrm{e}_P, \mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3\right). \tag{7}$$

with $\Omega = \Omega(\mathbf{S}, \mathbf{e}_P, \mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3)$. We restrict our consideration here to a reduced set of invariants to be used as the source of tensor generators (notation $\mathbf{S}_d$ stands for stress deviator):

$$s_k = tr\left\{\mathrm{S}\,\mathrm{M}_k\right\},\ \ s_{k+3} = tr\left\{\mathrm{S}^2\,\mathrm{M}_k\right\},\ \ \ s_7 = tr\left\{\mathrm{S}^3\right\}\ \ \ k=1,2,3, \tag{8}$$

omitting eigen and mixed invariants of inelastic strain tensor (cf. [12] for the complete set of invariants). Suppose now that $\Omega$ is a polynomial of third order in S. Then the loading function has the following simple form (material constants $a_1, \ldots, a_9, b_1, \ldots, b_{13}$ could depend on inelastic strain):

$$\begin{aligned} 2\Omega = {} & a_1 s_1^2 + a_2 s_2^2 + a_3 s_3^2 + a_4 s_4 + a_5 s_5 + a_6 s_6 + a_7 s_1 s_2 + a_8 s_2 s_3 + a_9 s_3 s_1 \\ & + b_1 s_1^3 + b_3 s_3^3 + b_2 s_2^3 + b_4 s_1 s_4 + b_5 s_1 s_5 + b_6 s_1 s_6 + b_7 s_2 s_4 \\ & + b_8 s_2 s_5 + b_9 s_2 s_6 + b_{10} s_3 s_4 + b_{11} s_3 s_5 + b_{12} s_3 s_6 + b_{13} s_7 \end{aligned} \tag{9}$$

and the evolution equation reads:

$$\begin{aligned} \frac{1}{\Lambda}\mathrm{D}_P = {} & \mathrm{M}_{1d}\left(2a_1 s_1 + a_7 s_2 + a_9 s_3 + 3b_1 s_1^2 + b_4 s_4 + b_5 s_5 + b_6 s_6\right) \\ & + \mathrm{M}_{2d}\left(2a_2 s_2 + a_7 s_1 + a_8 s_3 + 3b_2 s_2^2 + b_7 s_4 + b_8 s_5 + b_9 s_6\right) \\ & + \mathrm{M}_{3d}\left(2a_3 s_3 + a_8 s_2 + a_9 s_1 + 3b_3 s_3^2 + b_{10} s_4 + b_{11} s_5 + b_{12} s_6\right) \\ & + (\mathrm{M}_1\mathrm{S} + \mathrm{S}\mathrm{M}_1)_d\,(a_4 + b_4 s_1 + b_7 s_2 + b_{10} s_3) \\ & + (\mathbf{M}_2\mathbf{S} + \mathbf{S}\mathbf{M}_2)_d\,(a_5 + b_5 s_1 + b_8 s_2 + b_{11} s_3) \\ & + (\mathrm{M}_3\mathrm{S} + \mathrm{S}\mathrm{M}_3)_d\,(a_6 + b_6 s_1 + b_9 s_2 + b_{12} s_3) + 3b_{13}(\mathrm{S}^2)_d \end{aligned} \tag{10}$$

## 4. Classical J2 theory of orthotropic materials

Hill in his book (Hill 1950), as well as Logan and Hosford (1980), gave the definition of equivalent stress for orthotropic materials (with different properties in all the three directions) by means of principal stresses:

$$2f=\frac{2}{3}\frac{\sigma_{eq}^2}{h}-1=\frac{2}{3h}[F(\sigma_2-\sigma_3)^2+G(\sigma_3-\sigma_1)^2+H(\sigma_1-\sigma_2)^2]-1, \tag{11}$$

$$2f_H=\frac{2}{3h}[F|\sigma_2-\sigma_3|^{m_H}+G|\sigma_3-\sigma_1|^{m_H}+H|\sigma_1-\sigma_2|^{m_H}]-1. \tag{12}$$

In classical theory of plasticity of orthotropic materials the evolution equation is based on the above J2 yield function and equivalent inelastic strain $\varepsilon_{eq}^P:=\int_0^t||d\boldsymbol{\varepsilon}^P(\tau)/d\tau||d\tau$ such that the corresponding evolution equation reads:

$$\frac{d\varepsilon^P}{dt}=\partial_{\mathbf{S}}f=\frac{1}{2h(\varepsilon_{eq}^{in})}\partial_{\mathbf{S}}\sigma_{eq}^2. \tag{13}$$

All the models belonging to *J2* class are based on the notion of universal flow curve. The family of functions

$$f\left(\sigma_{eq},\varepsilon_{eq}^P,\dot{\varepsilon}_{eq}^P\right)=\sigma_{eq}-\Phi\left(\sigma_{eq},\varepsilon_{eq}^P,\dot{\varepsilon}_{eq}^P\right) \tag{14}$$

defines for $f=0$ viscoinelastic (i.e., rate dependent) deformation. On the other hand

$$\sigma_{eq}-\Phi\left(\sigma_{eq},\varepsilon_{eq}^P,0\right)<0$$

corresponds to elastic deformation. In a more traditional understanding constant values of $\dot{\varepsilon}_{eq}^P=c_1,c_2,\dots,c_n$ define family of universal flow curves. The word "universal" means that such curves should hold for all multiaxial strain states. It is allowed for straining history to be accounted for but strictly limited to scalar functionals of the form

$$\sigma_{eq}-\Phi\left(\sigma_{eq},\varepsilon_{eq}^P,0\right)<0 \tag{15}$$

where presence of $\dot{\varepsilon}_{eq}^P(\tau)(0<\tau<t)$ takes into account inelastic strain rate history while $\varepsilon_{eq}^P(\tau)(0<\tau<t)$ is responsible for inelastic strain history.

Now, associate flow rule formulated by Prager and Hochenemser (and then applied by Drucker, Perzyna and others) is based on normality of inelastic stretching tensor $\mathbf{D}_P$ on the yield surface $f=const$. Therefore such postulate gives

$$\mathrm{D}_P=\Lambda\,\partial_{\mathbf{S}}f. \tag{16}$$

It must be noted that, although seemingly tensorial, (16) is completely specified by scalars listed in (14). Traditionally, all these scalars have been found experimentally by tension experiments and then applied to arbitrary multiaxial strain states. In the sequel we will see how such an assumption is wrong. Experiments do not support the notion of universal flow curves (Micunovic et al. 1997). Therefore, yield function must include not only $\sigma_{eq}$ and $\varepsilon_{eq}^P$ but must be be of the include tensors $\mathbf{S}_d,\mathbf{e}_P,\mathbf{D}_P$ where tensors (i.e., their selected proper and mixed invariants) instead of scalars for multiaxial stress states are to be taken into account. Despite this we proceed here with such an approach based on universal flow curves. For orthotropic materials

whose anisotropy directions are given by the orthogonal unit vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, the corresponding stress invariants aligned to such directions are $A_{\sigma_k} = \boldsymbol{S}: \boldsymbol{M}_k \equiv \mathbf{a}_k \cdot \boldsymbol{S} \cdot \mathbf{a}_k, k = 1,2,3$, with $\boldsymbol{M}_1 \equiv \mathbf{a}_1 \otimes \mathbf{a}_1$, $\boldsymbol{M}_2 \equiv \mathbf{a}_2 \otimes \mathbf{a}_2$ and $\mathbf{M}_3 = \mathbf{1} - \mathbf{M}_1 - \mathbf{M}_2$. To take into account such an anisotropy for a fixed inelastic strain rate the yield function:

$$2f = \frac{2\sigma_{eq}^2}{3h(\varepsilon_{eq}^P)} - 1 \qquad \text{with} \qquad \sigma_{eq}^2 = \frac{3}{2}\left\{ tr\mathrm{S}_d^2 + 3\frac{(2-b)A_{\sigma_1}^2 + (2-c)A_{\sigma_2}^2}{2b+2c-5} \right\}. \tag{17}$$

Here, an extended equivalent stress, necessary for orthotropic materials, is introduced. Moreover the following notation is here convenient

$$\mathrm{D}_P \, dt \equiv d\mathrm{e}_P, \tag{18}$$

with inelastic strain increments along anisotropy directions $\mathbf{M}_1, \mathbf{M}_2$:

$$A_{d\varepsilon_k} = \mathrm{a}_k \cdot d\mathrm{e}_P \cdot \mathrm{a}_k \equiv \mathrm{M}_k : d\mathrm{e}^P, \quad (k=1,2).$$

Here $d\mathbf{e}_P$ is increment of inelastic strain tensor induced by the inelastic stretching $\mathbf{D}_P$ and $A_{d\varepsilon_1}$, $A_{d\varepsilon_2}$ are its invariants aligned with the anisotropy directions $\mathbf{M}_{11}, \mathbf{M}_{22}$. Let us remark that $\mathrm{D}_P$ is a deviator when inelastic incompressibility holds. According to [19] equivalent inelastic strain rate is defined by means of equality of inelastic powers (cf. (17)) $\sigma_{eq}\, d\varepsilon_{eq}^P = \mathbf{S}: d\,\mathbf{e}^P$. The commonly accepted associate flow rule was constructed by Prager and Hochenemser in the form (derived from (18) by means of (16)):

$$d\mathrm{e}_P = d\lambda \;\; \partial_{\mathbf{S}} f, \qquad \text{with} \quad d\lambda = \sigma_{eq}\, d\varepsilon_{eq}^P. \tag{19}$$

In this way we arrive at evolution equations which in the case of isotropy are called Levi-Mises equations. Dyadic deviators $\mathbf{M}_{1d}$ and $\mathbf{M}_{2d}$ are formed from $\mathbf{M}_1$ and $\mathbf{M}_2$ shown above. Making use of Levi-Mises equations leads to the definition of the equivalent inelastic strain increment:

$$d\ \mathrm{e}_P = \frac{3}{2}\frac{d\varepsilon_{eq}^P}{\sigma_{eq}}\left\{ \mathrm{S}_d + \frac{3}{2b+2c-5}\left[ (2-b)A_{\sigma_1}\mathrm{M}_{1d} + (2-c)A_{\sigma_2}\ \mathrm{M}_{2d} \right] \right\}, \tag{20}$$

$$\left(d\varepsilon_{eq}^P\right)^2 = \frac{2}{3}tr\{d\ \mathrm{e}^P\}^2 + \frac{2}{3}\frac{1}{bc-1}\ \ [(b-2)(2b-1)A_{d\varepsilon_1}^2 + (c-2)(2c-1)A_{d\varepsilon_2}^2$$
$$-2(b-2)(c-2)A_{d\varepsilon_1}A_{d\varepsilon_2}].$$

The meaning of material constants is found from Levi-Mises equations when either only $S_{11}$ or $S_{22}$ are different from zero. Writing explicitly (20) for these two special cases of loading we get

$$b \equiv 1 + r_2 = 1 + \left.\frac{d\varepsilon_{33}^P}{d\varepsilon_{11}^P}\right|_{S_{11}=0, S_{12}=0}, \quad c \equiv 1 + r_1 = 1 + \left.\frac{d\varepsilon_{33}^P}{d\varepsilon_{22}^P}\right|_{S_{22}=0, S_{12}=0}. \tag{21}$$

In the special case of isotropy $r_1 = r_2 = 1$ would lead to $b = c = 2$. For an orthotropic sheet with in-plane preferred directions $\mathbf{a}_1, \mathbf{a}_2$ comparison of (21) with (11) gives rise to

$$F = \frac{b-1}{2b+2c-5}, \quad G = \frac{c-1}{2b+2c-5}, H = \frac{1}{2b+2c-5}.$$

Here we have made the simplest check of the validity of Hill's yield function. In Micunovic (2009b) it has been demonstrated that such yield function is much worse for nonproportional stress-strain histories. A simple remedy would be to take some functions of inelastic strain tensor instead of constants $F, G, H$ in (11). For the case of transverse isotropy this has been done in Micunovic and Kudrjavceva (2019b).

Note 1

A comparison of QRI with J2 approach is possible if all the b – constants in (10) are negligible. However, a linearized evolution equation of QRI (derived and discussed in Micunovic and Kudrjavceva 2019b) has larger number of constants being more capable for description of multiaxial stress histories. It has to be underlined here that either Hill's (11) or Logan-Hosford yield function (12) are incorrect for nonproportional stress paths (cf. Micunovic and Kudrjavceva 2019a for details).

## 5. Diffuse instability

According to McClintock, (cf. McClintock 1971), a nonuniform strain field may develop twofold: (a) thinning, caused by tension loads, occurs very gradually in dimensions comparable with specimen dimensions and (b) it occurs in a region comparable with sheet (or specimen) thickness. The first is called *diffuse instability*, whereas for the second phenomenon is named *localized instability*. Due to above distinction the diffuse instability could appear mainly when cylindrical or some other 3D specimen are used.

At this point Hill's stability postulate (cf. Hill 1957) is invoked stating in its local form that at a bifurcation point the following equality holds

$$tr\{D_t\mathrm{S}_d\ \mathrm{D}_P\}=tr\{\mathrm{D}_P\mathrm{S}_d\mathrm{D}_P\}, \tag{22}$$

for "rigid" inelastic materials (i.e., materials where elastic strain is so many times smaller than the inelastic strain that it may be neglected). In above equality $D_t\mathrm{S}_d$ is the time rate of the stress deviator whereas $\mathbf{D}_P$ is the inelastic stretching tensor. If the associativity of flow rule holds i.e., $\mathbf{D}_P = \Lambda\, \partial_{\mathbf{S}}\Omega$ holds, then the above equality has the form:

$$\Lambda\ tr\{D_t\mathrm{S}_d\ \partial_{\mathbf{S}}\Omega\} - \Lambda^2\ tr\{\partial_{\mathbf{S}}\Omega\ \mathrm{S}_d\ \partial_{\mathbf{S}}\Omega\}=0. \tag{23}$$

Initiation of diffuse instability begins when (19) is satisfied. Explicitly, for longitudinal uniaxial stress $\sigma_1$, uniaxial transverse stress $\sigma_2$ as well as shear stress $\tau$ the following equations hold if diffuse instability commences

$$\begin{aligned} &D_t\sigma_1[1-\sigma_1^2 exp(-M)(\frac{\sigma_1}{Y0}-1)^{\lambda}\{(8a_1+2a_2+2a_3+12a_4-4a_7+2a_8-4a_9)/9 \\ &+\sigma_1(24b_1-3b_2-3b_3+14b_4+2b_5+2b_6-7b_7 \\ &-b_8-b_9-7b_{10}-b_{11}-b_{12}+18b_{13})/27\}]=0, \end{aligned} \tag{24}$$

$$\begin{aligned} &D_t\sigma_2[1-\sigma_2^2 exp(-M)(\frac{\sigma_2}{Y0}-1)^{\lambda}[(12a_5+2a_1+8a_2+2a_3-4a_7-4a_8+2a_9)/9 \\ &+\sigma_2(-3b_1+24b_2-3b_3-b_4-16b_5-b_6+2b_7 \\ &+32b_8+2b_9-b_{10}-16b_{11}-b_{12}+18b_{13}/3)/27]=0, \end{aligned} \tag{25}$$

$$D_t\tau[1-\sqrt{3}(b_4+b_5+b_7+b_8-2b_{10}-2b_{11}+6b_{13})\tau^2(\frac{\sqrt{3}\tau}{Y0}-1)^{\lambda}\,exp(-M)/3]. \tag{26}$$

All the equations are algebraic. From the last equation we see that a diffuse instability in the case of shear is predicted solely by nonlinear quadratic terms.

Let us see how the diffuse instability is predicted by J2 approach. Replacing (24) into (18) we arrive at the following differential equations

$$-D_t\sigma_1+\frac{c\sigma_1^2}{2h\left(2b+2c-5\right)}=0, \tag{27}$$

$$-D_t\sigma_2+\frac{b\sigma_2^2}{2h\left(2b+2c-5\right)}=0, \tag{28}$$

$$\frac{\tau}{h}D_t\tau=0, \tag{29}$$

for longitudinal uniaxial stress $\sigma_1$, for transverse uniaxial stress $\sigma_2$ and for shear $\tau$, respectively. If the material functions $h, \mathrm{b}, \mathrm{c}$ are known, then the above three differential equations could be solved. It should be underlined here that the $\mathrm{J}_2$ approach allows diffuse instability only for small strain rates and that a shear diffuse instability is impossible by J2 approach. It is essential to note that if J2 approach is applied, then this is always a differential equation which gives few hundreds larger inelastic stretching for relatively small increase of stress. Thus, results of J2 approach are correct only for small inelastic strain rates. On the contrary, the factor $\Lambda$ in the QRI approach is proportional to stress rate (cf. Micunovic 2009a). Thus, we solve an algebraic equation which holds for small to large strain rates.

## 6. Some conclusions

Some of the results of this paper may be summarized as follows.

Although the majority of the features of the QRI modelling are listed in the introduction, we could underline here that the above consideration of diffuse instability has shown its advantage in comparison with the classical J2 modeling. Since the J2 theory is based on uniaxial experiments (mainly tension) it is blind for directionality when we try to apply tension data to shear (cf. Micunovic 2009b for details).

It is shown that J2 approach with Hill's orthotropic yield function cannot cover all tension stress directions. This holds especially for shear.

Explicit diffuse instability equations are written for orthotropic materials applying $\mathrm{J}_2$ as well as QRI approach.

The basic novelties of this paper are the following: 1. application of tensor function approach to orthotropic inelastic materials by using endochronic evolution equation with extended Vakulenko's thermodynamic time, 2. formulation of viscoplasticity with loading function dependent on stress, inelastic strain and structural tensors and 3. a new QRI nonlinear evolution equation for orthotropic materials is derived by tensor function representation with Spencer-Boehler structural tensors.

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