

Miljan Leković¹
Tanja Stanišić²
Nemanja Pantić³

University of Kragujevac, Faculty of Hotel Management
and Tourism in Vrnjačka Banja

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TRADE OFF BETWEEN SIMPLICITY AND PRECISION OF INDEX MODELS

Abstract

In the basis of contemporary portfolio theory is Markowitz model of portfolio analysis which accurately defines a set of efficient portfolios for a relatively small number of securities in its composition. With the increase in the number of securities in the portfolio, the application of the Markowitz's model becomes complex, so financial theory found the solution of the problem in the single-index Sharpe's model. The later emergence of multi-index models, which better reflect reality, increased precision in determining a set of efficient portfolios, but at the cost of greater complexity of the model. The aim of the research is to analyze a kind of substitution between the simplicity and precision of the model, and to search answer to the question of what is the optimal number of explanatory factors of the model. Using qualitative economic analysis method, it was concluded that the number of factors (indexes) in the model should be increased until marginal benefits in the form of increased precision are equalized with marginal costs in the form of increased complexity, reduced applicability and associated costs of obtaining informations. In striving for greater precision of models, financial analysts must not overlook that the index models emerged from the practical necessity of simplifying the original Markowitz's model.

Key words: single-index model, market model, two-index model, two-sector index model, multi-index models

JEL classification: G00, G10, G11

TRADE OFF ИЗМЕЋУ ЈЕДНОСТАВНОСТИ И ПРЕЦИЗНОСТИ ИНДЕКСНИХ МОДЕЛА

Апстракт

У основи савремене портфолио теорије налази се Markowitz-ев модел портфолио анализе који прецизно одређује сет ефикасних портфолија за релативно мали број хартија од вредности у његовом саставу. Са повећањем броја хартија од вредности у саставу портфолија примена Markowitz-евог

¹ m.lekovic@kg.ac.rs, ORCID ID: 0000-0002-4952-3991

² tanja.stanisic@kg.ac.rs, ORCID ID: 0000-0001-5809-794X

³ nemanja.pantic@kg.ac.rs, ORCID ID: 0000-0003-0030-6950

модела постаје сложена, па је решење проблема финансијска теорија пронашла у једноиндексном Sharpe-овом моделу. Каснијом појавом вишеиндексних модела, који боље одражавају реалност, повећана је прецизност приликом одређивања сета ефикасних портфолија, али по цени веће сложености модела. Циљ истраживања је анализирање својеврсне супституције између једноставности и прецизности модела, и тражење одговора на питање који је то оптималан број објашњавајућих фактора модела. Применом метода квалитативне економске анализе закључено је да број фактора (индекса) у моделу треба повећавати све док се маргиналне користи у виду повећане прецизности не изједначе са маргиналним трошковима у виду повећане комплексности, смањене апликативности и пратећих трошкова прибављања информација. У тежњи за већом прецизношћу модела, финансијски аналитичари не смеју изгубити из вида да су се индексни модели појавили из практичне нужности поједностављења оригиналног Markowitz-евог модела.

Кључне речи: једноиндексни модел, тржишни модел, двоиндексни модел, двосекторски индексни модел, вишеиндексни модели

Introduction

Markowitz model of portfolio analysis, when determining a set of effective portfolios, requires an estimate of the expected return and variance for each security, as well as an estimate of the covariance between each pair of analyzed securities. (Markowitz, 1952; 1959). The total number of inputs required to successfully operate the Markowitz's model is $\frac{n^2+3n}{2}$, which for the case of 10, 100 and 1000 available securities is 65, 5150 and 501500 data. The latest case of estimating over half a million data represented an impossible mission for analysts, and therefore it comes to finding simpler methods that would require less input.

The solution to the problem described in the form of a single-index model was offered by William F. Sharpe in his paper "A Simplified Model for Portfolio Analysis" (Sharpe, 1963). Sharpe's single-index model and all other index, that is, factor models are based on a return generation process that describes how and from which components the securities returns are created. According to index models, one or more factors systematically affect the returns of all securities. Thus, the correlation of the returns of two securities is not determined directly, but indirectly based on their relationship with one or more factors contained in the model. In this way, the number of required covariance is equated with the number of securities analyzed. It decreases from earlier $\frac{n^2-n}{2}$ in Markowitz's model to n , while the total number of inputs decreases from $\frac{n^2+3n}{2}$ to $(k+2)n+2k$, or equivalent, $2n+2k+kn$, where: n – number of securities, and k – the number of factors used in the model. The epilogue of the above mentioned is a significant simplification of the process of determining a set of effective portfolios, that is, of drawing an efficient limit, which is truly obtained at the cost of less exactness in relation to the original Markowitz's model.

The factors contained in the index models explain the systemic variability of returns, i.e. system component of stochastic movements of securities' returns. The

remaining unexplained part of the stochastic return movements is attributed to the unexpected effects specific to the security and its issuer.

Bearing in mind the above mentioned, the subject of research are single-index model, which explains the systemic variability of returns using one factor, and multi-index models, which use two or more factors to explain the systemic variability of securities' returns. The aim of the research is to present the positive and negative aspects of these models and their variations such as the original single-index market model, Jensen's single-index market model, two-index model and two-sector index model to the investment public in the Republic of Serbia.

Starting from a defined subject and formulated research objective, the basic research question is: What number of explanatory factors of the model is optimal? The method of qualitative economic analysis will be applied in the research in order to make valid conclusions about the research problem by studying the relevant literature.

Single-index model – simplicity at the expense of precision

The single-index model represents the simplest form of the return generation process. The total number of data required for its successful functioning is $3n + 2$. The basic premise of the single-index model is that the returns of securities are sensitive to the movement of one common factor that systematically changes prices, and therefore, the returns of all securities.

The general single-index model takes the following form (Leković, 2017):

$$r_{it} = a_i + b_i F_t + \varepsilon_{it}, \quad (1)$$

where:

r_{it} – the return rate, i.e., the return in the holding period of the security i ,

a_i – the expected return of the security i for the case of zero value of factor F ,

b_i – the sensitivity of return of the security i to the changes in factor F ,

F_t – the value of a factor that systematically affects the price of security i in the period t ,

ε_{it} – the random error, i.e., the random variable with an expected zero value in the period t .

The previous equation divides the total return of the security i (r_{it}) на systemic ($a_i + b_i F_t$) and non-systemic component (ε_{it}). The systemic component of the total return is explained by a common factor F , while the non-systemic component represents the unique (specific) return of the observed security.

It is important to point out the autonomous component of the return of security i (a_i), independent of the impact of the common factor F , consisting of a_i and ε_{it} :

$$a_i = a_i + \varepsilon_{it}. \quad (2)$$

In the financial literature, market movement, i.e., market index is most often cited as the common factor explaining the systemic variability of securities returns. Other factors of systemic variability of return are also in use, such as the unexpected growth rate of gross domestic product

(Petković et al., 2020), the unexpected increase rate of inflation (Pantić & Milojević, 2019), and similar. According to Elton et al. (2011), the fact that securities' prices generally rise with market growth, that is, they fall when the market is in crisis, suggests that one of the reasons for the correlation between the returns of securities lies in their common response to market changes. Therefore, many authors propose interrelating the return of individual security and the market rate of return as a useful measure of the correlation of the observed securities' returns. Single-index model that uses the market rate of return as explanatory factor is called the market model and has the following form (Ferruz et al., 2010, p. 271):

$$r_{it} = a_i + \beta_i r_{mt} + \varepsilon_{it}, \quad (3)$$

where:

β_i – the sensitivity of return of security i to changes in the market index,

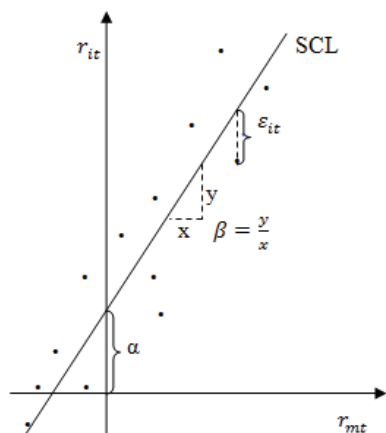
r_{mt} – the market rate of return (the rate of return on the market index) in the period t .

Using single-index market model requires estimation of beta coefficient (β_i) for each security. Beta coefficient can be obtained by subjective estimates of analysts, or by estimating a historical beta based on historical data. Historical beta coefficients provide useful information about future beta coefficients, if they are relatively stable over time. A more stable historical beta means a more reliable estimate of the future beta and also greater reliability of the entire model.

The value of the estimated beta coefficient is interpreted as follows: if β_i equals, for example, +0.5 the return of the observed security will increase (decrease) by 0.5% in the case of a rise (fall) in the market index for 1%. Exceptionally, if the beta coefficient takes a negative value, a change in the market index will result in a change in the return of the observed security in the opposite direction.

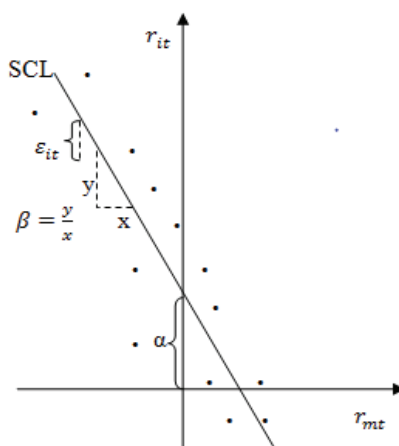
Graphical representation of single-index market model is done by using *Security Characteristic Line (SCL)*. Security Characteristic Line describes the relationship between the return of the observed security (r_{it}) and market return (r_{mt}) (Figures 1 and 2).

Figure 1: Security Characteristic Line in the case of a positive beta



Source: Authors

Figure 2: Security Characteristic Line in the case of a negative beta



Source: Authors

Each point in the diagram indicates one pair of returns for a particular security and a market return, and a regression line constructed between these points such that the sum of all square deviations from that line is minimal is called Security Characteristic Line. The degree of deviation of the return points from the Security Characteristic Line indicates the level of correlation between the return of a particular security and the market return. The greater vertical deviation of the return points from the characteristic line (marked with ε_{it}), implies the smaller correlation. The perfect correlation of observed and market return exists only in the case of zero residual ($\varepsilon_{it} = 0$), when the return points lie on the characteristic line. This situation is an idealized theoretical case, since the imperfect correlative relation corresponds more to reality. Because of the above, the return points generally do not lie on the characteristic line.

The slope of the characteristic line is determined by the beta coefficient. In other words, the beta coefficient is the slope coefficient of the characteristic line. The characteristic line has a positive slope in the case of a positive beta (Figure 1), or a negative slope in the case of a negative beta (Figure 2). In Figure 1, the return of the observed security (r_{it}) rises (falls) with rising (falling) market return (r_{mt}), while the movement of the security return and the market return in Figure 2 is inverse. The return points that lie on the characteristic line of positive slope indicate the perfect positive correlation between the return of a particular security and the market return, while the return points from the regression line of negative slope indicate a perfect negative correlation.

It is also important to interpret the alpha coefficient (α) which shows the expected return of the observed security in the case of zero market return. In Figures 1 and 2, the alpha coefficient represents the distance from the coordinate origin to the intersection point of the characteristic line and the y-axis. The alpha coefficient indicates the deviation of the actual from the expected return:

- if alpha is positive, the actual return is higher than expected and the security is undervalued;
- if alpha is negative, the actual return is lower than expected and the security is overestimated;
- the zero alpha coefficient indicates the absence of an undervaluation, that is, an overestimation of the observed security and the presence of a price equilibrium.

Due to the functioning of the market mechanism and market laws, the first two situations in the final instance result in the third. In the first case, return higher than expected and price below the equilibrium price to attract buyers who increase demand (Milašinović et al., 2019). Demand growth affects price growth, which leads to a gradual decrease in the real return down to the equilibrium level represented by the third situation. In the second case, the return lower than expected and the price higher than the equilibrium price reject customers who reduce demand. The fall in demand causes the price to fall, resulting in a gradual increase in the real return to the equilibrium level.

Important assumptions single-index market model related to random error (ε_{it}) are (Francis & Kim, 2013, p. 167):

- the expected value of the random error (residual) is zero ($E(\varepsilon_{it}) = 0$),
- the variance of random error is constant ($\sigma^2_{\varepsilon_i} = \text{const}$),
- the random error is uncorrelated with market return ($\text{Cov}(\varepsilon_{it}, r_{mt}) = 0$),
- the random errors are serially uncorrelated ($\text{Cov}(\varepsilon_{it}, \varepsilon_{is}) = 0$, $\forall t \neq s$),
- the random errors of different securities are mutually uncorrelated ($\text{Cov}(\varepsilon_{it}, \varepsilon_{jt}) = 0$, $\forall i \neq j$).

The last assumption about the mutual independence of the residuals of the analyzed securities is the most important of the above mentioned, since it implies that the only cause of the systemic variability of the return of the different securities is the chosen factor – in this case, the market movement (market rate of return). Non-correlation of residuals indicates that there are no additional factors that systematically affect securities' returns and that the presented single-factor model is valid.

Using the single-index market model, the following terms are derived (Elton et al., 2011, p. 134):

- expected return of individual security: $\bar{r}_i = \alpha_i + \beta_i \bar{r}_m$,
- variance of return of individual security: $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{\varepsilon i}^2$,
- covariance of returns between securities: $\sigma_{ij} = \beta_i \beta_j \sigma_m^2$.

In analogy to the division of the overall return of the security to the systemic and non-systemic component, the total variance of return (σ_i^2) is divided into systemic and non-systemic variance, i.e., on systemic (factor) and non-systemic (non-factor) risk. Systemic risk is represented by the product of the squared beta coefficient and the market variance ($\beta_i^2 \sigma_m^2$), and non-systemic risk by the variance of the residual of the individual security ($\sigma_{\varepsilon i}^2$). Since the market variance (σ_m^2) is the same for all securities, the beta coefficient (β_i) is considered the right measure of systemic risk:

- if $\beta_i = 1$, the security has the same systemic risk as the total market;
- if $\beta_i > 1$, securities have higher systemic risk than the total market;
- if $\beta_i < 1$, securities have less systemic risk than the total market.

According to the *Modern Portfolio Theory* (MPT), decisions should be made in the context of portfolios, not in the context of individual securities. This caused even greater practical value that have the following terms based also on the single-index market model:

- Return in the holding period of the securities' portfolio: $r_{pt} = \alpha_p + \beta_p r_{mt} + \varepsilon_{pt}$

;

- The expected return of the securities' portfolio: $\bar{r}_p = \alpha_p + \beta_p \bar{r}_m$;

where:

$$\alpha_p = \sum_{i=1}^n w_i \alpha_i, \quad (4)$$

$$\beta_p = \sum_{i=1}^n w_i \beta_i, \quad (5)$$

$$\varepsilon_{pt} = \sum_{i=1}^n w_i \varepsilon_{it}. \quad (6)$$

Alpha coefficient, beta coefficient and portfolio random error (residual) (α_p , β_p , ε_{pt}) are weighted averages of alpha coefficients, beta coefficients and residuals of component securities. Thereby, parts of the total portfolio value invested in a particular security are used as weights (w_i).

- The variance of the securities' portfolio: $\sigma_p^2 = \beta_p^2 \sigma_m^2 + \sigma_{\varepsilon p}^2$;

where:

$$\sigma_{\varepsilon p}^2 = \sum_{i=1}^n w_i^2 \sigma_{\varepsilon i}^2. \quad (7)$$

If the residuals of the return rates of different securities are not mutually correlated, the variance of the portfolio residual represents the weighted average of the variance of the residuals of the individual securities in its composition. Assuming that the same proportion of money is invested in each of the securities ($w_i = \frac{1}{n}$), the formula for

residual portfolio variance becomes:

$$\sigma_{\varepsilon p}^2 = \frac{1}{n} \sum_{i=1}^n \frac{1}{n} \sigma_{\varepsilon i}^2. \quad (8)$$

It is clear that by diversifying and increasing the number of component securities, the non-systemic variance (variance of residual) of the portfolio decreases drastically and disappears in the final instance, so that the overall risk of the portfolio becomes:

$$\sigma_p^2 = \beta_p^2 \sigma_m^2, \quad (9)$$

or equivalent,

$$\sigma_p = \beta_p \sigma_m. \quad (10)$$

In addition to the original single-index market model that uses return in the holding period, it is useful to present Jensen's single-index market model that uses the excess return (risk premium) instead of return in the holding period:

$$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it}, \quad (11)$$

where:

R_{it} – the excess return (risk premium) of the security i in period t , that is, the return rate of a particular security above the risk-free rate of return in the observed period;

R_{mt} – the market excess return (market risk premium) in period t , that is, the return rate of the market portfolio above the risk-free rate of return in the observed period.

Michael C. Jensen presented this model in his doctoral dissertation, published in 1968, citing its high utility value and its numerous advantages in the evaluation of investment performance as a reason for introducing (Jensen, 1968).

It is important to point out that, under the assumption of constant risk-free rate of return ($r_f = \text{const}$), the original single-index market model and Jensen's single-index market model are very similar. Beta coefficients (β_i) and random errors (ε_{it}) do not differ in the described versions of the single-index models, since the same constant (r_f) is subtracted from the dependent and independent variables. Only alpha coefficients differ, and the relationship between the original alpha (α_i) and Jensen's alpha (α_i') can be represented as (Francis & Kim, 2013, p. 171):

$$\alpha_i' = \alpha_i - r_f(1 - \beta_i). \quad (12)$$

These alphas have different values, meanings and different uses. The introduction of Jensen's alpha coefficient significantly facilitates the process of measuring realized investment performance.

On the other hand, under conditions of fluctuating risk-free rate of return (r_{ft}), the original and Jensen's single-index market models are moving away from each other, i.e., they lose similarities because they have different not only alpha, but also beta coefficients.

Multi-index models – precision at the expense of simplicity

If the residuals of the component securities' return rates are mutually correlated, the single-index model loses validity and usability, and it is necessary to introduce additional indices (factors) that together with the existing factor would better explain the

systemic component of the stochastic movements of the portfolio's securities. According to Lee et al. (2010), the simplest way to construct a multi-index model is to supplement a market model based on a market index with other factors such as an index that shows the movement of the industry to which the enterprise belongs. With increasing number of factors, multi-index models are generated and the covariance among residuals approaches zero. This increases the precision and complexity of the model.

The simplest variant of a multi-index model is a two-index model, with the required number of inputs $4n + 4$. For each security it is necessary to determine the alpha coefficient (α_i), beta coefficient relative to the first index (β_{i1}), beta coefficient relative to the second index (β_{i2}), as well as residual variance ($\sigma_{\varepsilon_i}^2$). It is also necessary to determine the expected rates of return and variances for both selected indices.

The basic premise of the two-index model is that securities' returns are dependent on the systemic impact of two common factors, which explain the systemic component of stochastic movements in returns of component securities. Thereby, the unexplained non-systemic component is attributed to the unanticipated effects specific to the particular security and its issuer.

According to Sharpe et al. (1995), the economy is not a monolithic entity, therefore, a number of factors can influence the return of securities: the growth rate of gross domestic product, the level of interest rates, the inflation rate, the level of the oil price.

The two-index model, which uses the Gross Domestic Product (GDP) and unexpected inflation rate (INF) as explanatory factors, takes the following form (Leković, 2017):

$$r_{it} = \alpha_i + \beta_{i1}GDP_t + \beta_{i2}INF_t + \varepsilon_{it}, \quad (13)$$

where:

α_i – the expected return of security i for the case of zero value of factors GDP and INF ,

β_{i1} – the sensitivity of the return of security i to changes in the growth rate of the Gross Domestic Product (GDP),

β_{i2} – the sensitivity of the return of security i to changes in the inflation rate (INF).

The systemic component of the total return of the observed security is represented by the sum of the first three elements of the right part of the equation ($\alpha_i + \beta_{i1}GDP_t + \beta_{i2}INF_t$), while the last fourth element (ε_{it}) indicates the non-systemic component of the total return.

The graphical representation of the two-index model described is done using a characteristic plane. Thereby, each point in a two-dimensional space indicates a combination of the return of a particular security, the growth rate of gross domestic product and the inflation rate.

Earlier assumptions of single-index model related to random error (ε_{it}) apply to both two-index and multi-index models. An additional assumption, aimed at simplifying the computational process, is the non-correlation of the indices used, that is, the selected factors ($Cov(GDP_t, INF_t) = 0$). The possible impact of one factor on another can be eliminated by an orthogonalization process that turns correlated factors into uncorrelated ones.

The present two-index model leads to the appropriate formulas for:

- the expected return of individual security: $\bar{r}_i = \alpha_i + \beta_{i1}\overline{GDP} + \beta_{i2}\overline{INF}$,
- the variance of return of individual security: $\sigma_i^2 = \beta_{i1}^2\sigma_{GDP}^2 + \beta_{i2}^2\sigma_{INF}^2 + \sigma_{\varepsilon_i}^2$,

- the covariance of returns between securities: $\sigma_{ij} = \beta_{i1}\beta_{j1}\sigma_{GDP}^2 + \beta_{i2}\beta_{j2}\sigma_{INF}^2$.

In the formula for the total variance of the return of an individual security ($\sigma_i^2 = \beta_{i1}^2\sigma_{GDP}^2 + \beta_{i2}^2\sigma_{INF}^2 + \sigma_{\varepsilon_i}^2$) systemic risk is represented by the sum of the first two elements of the right part of the equation: 1) the squared beta coefficient of the security i relative to the first index multiplied by variance growth rate of gross domestic product ($\beta_{i1}^2\sigma_{GDP}^2$) and 2) the squared beta coefficient of the security i relative to the second index multiplied by variance rate of inflation ($\beta_{i2}^2\sigma_{INF}^2$). On the other hand, non-systemic risk is represented by the residual variance of individual security ($\sigma_{\varepsilon_i}^2$).

In the case of portfolio, using the two-index model results in the following terms:

- The return in the holding period of the securities' portfolio of:
 $r_{pt} = \alpha_p + \beta_{p1}GDP_t + \beta_{p2}INF_t + \varepsilon_{pt}$;
- The expected return of the securities' portfolio: $\bar{r}_p = \alpha_p + \beta_{p1}\overline{GDP} + \beta_{p2}\overline{INF}$;

where:

$$\beta_{p1} = \sum_{i=1}^n w_i \beta_{i1}, \quad (14)$$

$$\beta_{p2} = \sum_{i=1}^n w_i \beta_{i2}. \quad (15)$$

The beta coefficient of the portfolio relative to the first index and the beta coefficient of the portfolio relative to the second index (β_{p1} и β_{p2}) are the weighted averages of the beta coefficients of individual securities relative to the first or second index, whereby parts of the total portfolio value invested in a particular security are used as weights (w_i).

- The variance of return of the securities' portfolio: $\sigma_p^2 = \beta_{p1}^2\sigma_{GDP}^2 + \beta_{p2}^2\sigma_{INF}^2 + \sigma_{\varepsilon_p}^2$.

By increasing the number of securities in the portfolio, the non-systemic risk of the portfolio, represented by the residual variance of the portfolio ($\sigma_{\varepsilon_p}^2$), drastically decreases and approaches zero, so the total portfolio risk is reduced to systemic risk:

$$\sigma_p^2 = \beta_{p1}^2\sigma_{GDP}^2 + \beta_{p2}^2\sigma_{INF}^2. \quad (16)$$

In addition to the classic two-index model, it is important to introduce *two-sector-factor model*. Prices of the securities in the same sector often show a high degree of correlation, indicating the systemic impact of a particular sector factor. The basic premise of this model is that all securities are divided into two sectors, with their returns being affected solely by the factor characteristic for the sector to which the securities belong. Thus, factor characteristic for the first sector (F_1) systematically affects the returns of securities of the first sector, while the factor related to the second sector (F_2) systematically affects the returns of securities of the second sector. The sensitivity of the return of the first sector securities to changes of factor F_2 is equal to zero, and inversely, the sensitivity of the return of the second sector securities to changes of factor F_1 is also zero. The above mentioned indicates that in the general two-index model:

$$r_{it} = \alpha_i + b_{i1}F_{1t} + b_{i2}F_{2t} + \varepsilon_{it}, \quad (17)$$

either b_{i1} or b_{i2} will be equal to zero, depending on the sector to which the security belongs. If the security belonging to the first sector is marked with i and the security belonging to the second sector is marked with j , the corresponding two-sector index models will take the following form:

$$r_{it} = \alpha_i + b_{i1}F_{1t} + \varepsilon_{it}, \quad (18)$$

or,

$$r_{jt} = \alpha_j + b_{j2}F_{2t} + \varepsilon_{jt}. \quad (19)$$

So, unlike the classic two-index model whose required number of inputs is $4n + 4$, the total number of data necessary for the successful functioning two-sector index model is smaller and amounts $3n + 4$, which is also a key advantage of this model.

Extending the two-index model with additional factors leads to more complex multi-index models that require larger number of inputs:

- three-index model ($5n + 6$ inputs),
- four-index model ($6n + 8$ inputs),
- five-index ($7n + 10$ inputs),
- k -index model ($((k + 2)n + 2k$, or equivalent, $2n + 2k + kn$ inputs, where: n – number of securities, and k – number of factors in the model).

General multi-index model with k factors of systemic variability of the securities' return has the following form:

$$r_{it} = \alpha_i + b_{i1}F_{1t} + b_{i2}F_{2t} + \dots + b_{ik}F_{kt} + \varepsilon_{it}. \quad (20)$$

The following analytical expressions are obtained using this model:

- the expected return of individual security: $\bar{r}_i = \alpha_i + b_{i1}\bar{F}_1 + b_{i2}\bar{F}_2 + \dots + b_{ik}\bar{F}_k$,
- variance of return of individual security: $\sigma_i^2 = b_{i1}^2\sigma_{F_1}^2 + b_{i2}^2\sigma_{F_2}^2 + \dots + b_{ik}^2\sigma_{F_k}^2 + \sigma_{\varepsilon_i}^2$,
- covariance of returns between securities: $\sigma_{ij} = b_{i1}b_{j1}\sigma_{F_1}^2 + b_{i2}b_{j2}\sigma_{F_2}^2 + \dots + b_{ik}b_{jk}\sigma_{F_k}^2$,
- retrun in the holding period of the securities' portfolio: $r_{pt} = \alpha_p + b_{p1}F_{1t} + b_{p2}F_{2t} + \dots + b_{pk}F_{kt} + \varepsilon_{pt}$,
- the expected return of the securities' portfolio: $\bar{r}_p = \alpha_p + b_{p1}\bar{F}_1 + b_{p2}\bar{F}_2 + \dots + b_{pk}\bar{F}_k$,
- variance of the securities' portfolio: $\sigma_p^2 = b_{p1}^2\sigma_{F_1}^2 + b_{p2}^2\sigma_{F_2}^2 + \dots + b_{pk}^2\sigma_{F_k}^2 + \sigma_{\varepsilon_p}^2$.

It is not difficult to conclude that the concept is exactly the same as in the case of the two-index model. However, the key problem with the multi-index model is the choice of index, i.e. factors that systemically influence the return generation process. According to Grinblatt and Titman (2001), the three basic ways of assessing common systemic risk factors are:

- the use of statistical techniques, such as factor analysis,
- the specification of macroeconomic factors, such as unexpected changes in interest rates, unexpected changes in the level of economic activity,
- the specification of the characteristics of the securities or companies as microeconomic factors.

Despite numerous researches (Chen et al., 1986; Idris & Bala, 2015; Jamaludin et al., 2017; Kim, 2006; Sharpe, 1982; Tudor, 2010; Zhu, 2012), the financial literature has not yet reached a consensus on the most important systemic risk factors. Over time, some models such as the Barr Rosenberg Associates (BARRA) model (Rosenberg, 1974), Fama-French three-factor model (Fama & French, 1993), Burmeister-Ibbotson-Roll-Ross (BIRR) model (Burmeister et al., 1994), Carhart' for-factor model (Carhart, 1997) have found application in practice. However, decades of research have not been sufficient for making the final judgment about the factors that systemically influence the return generation process.

The latest researches, among which study carried out by Harvey et al. (2016) stands out in particular, show that hundreds of factors are associated with returns at a

statistically significant level. This study conducted by Harvey et al. (2016) indicates that at least 316 factors are in statistically significant relation to returns. In intention to indicate the appearance and abundance of new factors, Cochrane (2011) uses a picturesque expression “*zoo of new factors*”.

It is clear that a greater number of factors implies greater model exactness, while at the same time a greater number of required inputs for the result has greater model complexity. The multi-index model, according to the precision and the number of inputs required, occupies mid position between the original Markowitz' model and the single-index model.

Conclusion

By choosing between Markowitz's, single-index and multi-index model, a kind of *trade off* is made between the simplicity and precision of the model. Striving for greater simplicity of the model, it must not overlook the simultaneous loss of precision in determining a set of efficient portfolios. Inversely, striving for greater precision, the simultaneous loss of simplicity must not be neglected.

As Markowitz model of portfolio analysis requires the calculation of a correlation for each pair of securities within a portfolio, its application to portfolio expansion becomes more complex. With this in mind, William F. Sharpe has offered a simpler solution in the form of a single-index model that involves the systemic impact of one common factor on the returns of all securities and determining the correlation between the returns of securities based on their relationship with the common factor. Compared to the Markowitz's model, the application of Sharpe's single-index model is characterized by simplicity caused by fewer required inputs, but also by less precision in determining a set of efficient portfolios.

In order to determine the efficient limits more accurately and to explain more fully the systemic variability of securities' returns, it is proposed to introduce additional factors and to generate a multi-index model. Introducing additional factors that systemically affect securities' returns increases the precision and complexity of a multi-index model that gradually approaches original Markowitz's model in terms of its characteristics. Each additional factor means greater precision while reducing the applicability of the model in real conditions. Therefore, during a search for optimality, a number of factors should be increased until the marginal benefits in terms of increased model precision are greater than marginal costs in terms of increased complexity, reduced applicability and the associated costs of obtaining information. The above mentioned conclusion is comparable to the conclusion reached by Benjelloun and Siddiqi (2006) and Statman (2004) when examining the optimal portfolio size: the portfolio size should be increased until the marginal benefits of diversification in terms of reduced investment risk are greater than the marginal costs of diversification in terms of increased portfolio management costs.

Based on the above mentioned, it is concluded that the single-index and multi-index models represent a simplification of Markowitz model of portfolio analysis, and that the multi-index model occupies a central position between the original Markowitz's model and Sharpe's single-index model, because it is characterized by medium complexity and medium level of precision in determining the set of efficient portfolios.

The qualitative, but not quantitative, analysis of the optimal number of explanatory factors of the model was performed in the paper, and, therefore, future research should be directed to a comprehensive empirical analysis to support the conclusions drawn using the qualitative methodology.

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