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An analytical approach for free vibration analysis of Euler-Bernoulli stepped beams with axial-bending coupling effect

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ABSTRACT

Free vibration of eccentrically stepped beams with one step change in cross-section is considered. It is assumed that the longitudinal symmetry axes of the beam segments are translationally shifted along the vertical direction with respect to each other. The effect of that arrangement of the segments on the coupling of axial and bending vibrations of the stepped beam is analyzed. The beam segments are modeled in the frame of the Euler-Bernoulli theory of elastic beams. Two numerical examples are presented.

KEYWORDS

Stepped beam, Step eccentricity, Axial-bending coupling, Euler-Bernoulli theory, Natural frequency, Free vibration

1. INTRODUCTION

Stepped beams often appear as an integral part of various devices and structures in mechanical and civil engineering. That is why the buckling and vibration analysis of stepped beams is extremely important for engineering practice. Despite the large number of papers published in connection with this problem (see e.g. [1-22]), this field of scientific research is still actual with intensive development and generation of new scientific problems. In the available literature, there are mainly references dealing with homogeneous stepped beams. Also, tapered beams can be modeled as stepped beams in a manner described in [23, 24]. In the recent years stepped beams made of functionally graded materials represent an actual research field due to special mechanical characteristics of this kind of materials [25-27]. Decoupled axial and bending vibrations are mainly considered for stepped beams. However, often these two types of vibrations can be coupled. The reason for this may be, for example, the shape of cross-section of beams [28, 29], rigid bodies eccentrically attached to the stepped beams [23, 30-33], angled-beam joints in the frame structures [31, 34] and the compliant mechanisms [35], varying material characteristics of beam segments through their thickness direction [36] as well as mutual eccentric positions of longitudinal symmetry axes of stepped beams segments [22].

The last-mentioned cause of coupling is considered in this paper. In this sense, the objective of our paper is to extend the approach described in [31, 33] to the case of Euler-Bernoulli eccentrically stepped beams with one step change in cross-section. In the frame of the Euler-Bernoulli theory of elastic beams, to the authors' best knowledge of the literature, the appearance of axial-bending coupling effect in the case of this type of stepped beams was not considered.

2. FORMULATION OF GOVERNING EQUATIONS

An eccentrically stepped beam of rectangular cross-section with one step discontinuity is shown in Figure 1. The width and thicknesses of cross-sections of segments (S_1) and (S_2) are denoted by $b_i\,(i=1,2)$ and $h_i\,(i=1,2)$, respectively. In the undeformed configuration of the stepped beam, local stationary inertial coordinate frames $\{x_i, y_i, z_i\}(i=1,2)$ are placed in the manner shown in Figure 1. Also, the longitudinal symmetry axes of segments (S_1) and (S_2) are translationally shifted in the vertical direction by an amount e. The quantities $u_i(z_i,t)(i=1,2)$ and $w_i(z_i,t)(i=1,2)$ represent the axial and transverse displacements, respectively, of any point of the neutral axes of the beam segments. The material and geometric characteristics of segments (S_1) and (S_2) are: E_i is the modulus of elasticity, $I_{x(i)}$ is the cross-sectional area moment of inertia about axis x_i, A_i is the cross-sectional area, ρ_i is the mass density, and L_i is the length of the i-th beam segment. The partial differential equations of bending and axial free vibrations of segments (S_1) and (S_2) are as follows [37, 38]:

$$E_i I_{x(i)} \frac{\partial^4 w_i(z_i,t)}{\partial z_i^4} + \rho_i A_i \frac{\partial^2 w_i(z_i,t)}{\partial t^2} = 0,\ \ i=1,2, \tag{1}$$

$$E_i A_i \frac{\partial^2 u_i(z_i,t)}{\partial z_i^2} - \rho_i A_i \frac{\partial^2 u_i(z_i,t)}{\partial t^2} = 0,\ \ i=1,2. \tag{2}$$

Based on the method of separation of variables [37, 38], the displacements $u_i(z_i,t)(i=1,2)$ and $w_i(z_i,t)(i=1,2)$ can be written as:

$$w_i(z_i,t) = W_i(z_i)T(t), \qquad u_i(z_i,t) = U_i(z_i)T(t), \tag{3}$$

where $U_i(z_i)(i=1,2)$ and $W_i(z_i)(i=1,2)$ are the mode shapes in free axial and bending vibrations, respectively, and $T(t)=e^{i\omega t}$, $i=\sqrt{-1}$, and ω is the natural angular frequency of vibration of the stepped beam.

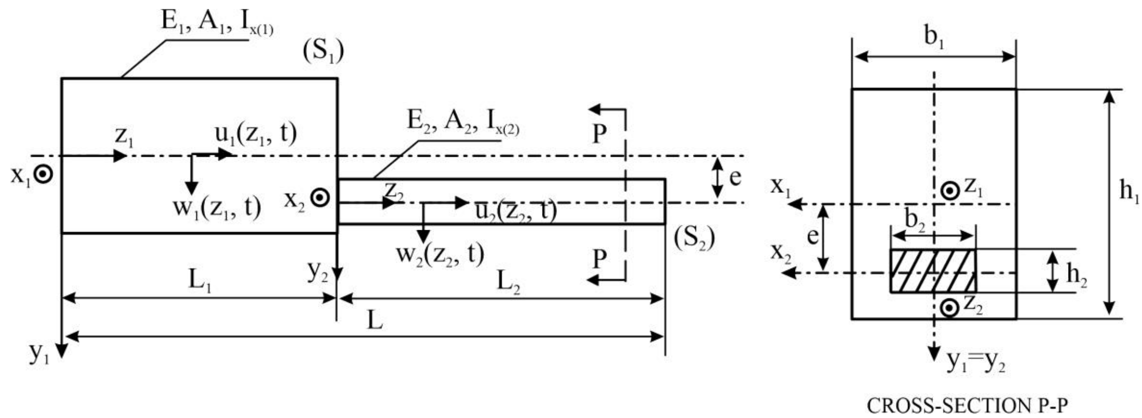

Figure 1: Stepped Euler-Bernoulli beam

Introducing (3) into (1) and (2) yields:

$$\frac{d^4 W_i(z_i)}{dz_i^4} - k_i^4 W_i(z_i) = 0,\ \ i=1,2, \tag{4}$$

$$\frac{d^2 U_i(z_i)}{dz_i^2} + p_i^2 U_i(z_i) = 0,\ \ i=1,2, \tag{5}$$

$$\frac{d^2 \mathrm{T}(\mathrm{t})}{dt^2} + \omega^2 \mathrm{T}(\mathrm{t}) = 0, \tag{6}$$

where:

$$k_i^4 = \frac{\rho_i A_i}{E_i I_{x(i)}}\omega^2, \qquad p_i^2 = \frac{\rho_i}{E_i}\omega^2, \ \ i=1,2. \tag{7}$$

Based on (7), the following relation can be established:

$$p_i = \sqrt{\frac{I_{x(i)}}{A_i}}k_i^2, \quad i = 1,2. \tag{8}$$

Combining (7) and (8) with the following expressions:

$$k_1 = k, \quad p_1 = \sqrt{\frac{I_{x(1)}}{A_1}}k^2 , \tag{9}$$

yields:

$$k_2 = \sqrt[4]{\frac{E_1 I_{x(1)} \rho_2 A_2}{E_2 I_{x(2)} \rho_1 A_1}}k, \quad p_2 = \sqrt{\frac{E_1 I_{x(1)} \rho_2}{E_2 A_1 \rho_1}}k^2, \tag{10}$$

$$\omega^2 = \frac{E_1 I_{x(1)}}{\rho_1 A_1}k^4. \tag{11}$$

Introducing now the following dimensionless quantities:

$$\bar{z}_i = \frac{z_i}{L}, \; \bar{W}_i(\bar{z}_i) = \frac{W(L\bar{z}_i)}{L}, \quad \bar{U}_i(\bar{z}_i) = \frac{U(L\bar{z}_i)}{L}, \quad \frac{d^j}{d^j} = \frac{1}{L^j}\frac{d^j}{d\bar{z}^j}, \; j = 1,2,\ldots, \tag{12}$$

the equations (4) and (5) as well as the relations (9) and (10) can be written in the following dimensionless forms:

$$\frac{d^4 \bar{W}_i(\bar{z}_i)}{d\bar{z}_i^4} - \bar{k}_i^4 \bar{W}_i(\bar{z}_i) = 0, \; i = 1,2, \tag{13}$$

$$\frac{d^2 \bar{U}_i(\bar{z}_i)}{d\bar{z}_i^2} + \bar{p}_i^2 \bar{U}_i(\bar{z}_i) = 0, \; i = 1,2, \tag{14}$$

$$\bar{k}_1 = \bar{k}, \; \bar{p}_1 = \bar{r}\bar{k}^2, \tag{15}$$

$$\bar{k}_2 = \sqrt[4]{\frac{\gamma_A \gamma_\rho}{\gamma_E \gamma_I}}\bar{k}, \quad \bar{p}_2 = \sqrt{\frac{\gamma_\rho}{\gamma_E}}\bar{r}\bar{k}^2, \tag{16}$$

where $\bar{k}_i = k_i L$, $\bar{p}_i = p_i L$, $\gamma_\rho = \rho_2 / \rho_1$, $\gamma_E = E_2 / E_1$, $\gamma_A = A_2 / A_1$, $\gamma_I = I_{x(2)} / I_{x(1)}$, $\bar{r} = \sqrt{I_{x(1)} / (A_1 L^2)}$, and:

$$\bar{k} = kL, \quad \bar{\omega} = \bar{k}^2 = \sqrt{\frac{\rho_1 A_1 L^4}{E_1 I_{x(1)}}}\omega \tag{17}$$

are the dimensionless frequency coefficient and the dimensionless natural angular frequency, respectively. General solutions of the equations (13) and (14) are given as follows [37, 38]:

$$\bar{W}_i(\bar{z}_i) = C_{1(i)}\cos(\bar{k}_i\bar{z}_i) + C_{2(i)}\sin(\bar{k}_i\bar{z}_i) + C_{3(i)}\cosh(\bar{k}_i\bar{z}_i) + C_{4(i)}\sinh(\bar{k}_i\bar{z}_i), \quad i = 1,2, \tag{18}$$

$$\bar{U}_i(\bar{z}_i) = C_{5(i)}\cos(\bar{p}_i\bar{z}_i) + C_{6(i)}\sin(\bar{p}_i\bar{z}_i), \quad i = 1,2, \tag{19}$$

where $C_{1(i)},\ldots,C_{6(i)}$ are integration constants.

3. BOUNDARY CONDITIONS AND THE FREQUENCY EQUATION

3.1. Boundary conditions at the left end of the stepped beam

For the clamped left end of the considered stepped beam the following boundary conditions hold:

$$\bar{U}_1(0) = 0, \quad \bar{W}_1(0) = 0, \quad \frac{d\bar{W}_1}{d\bar{z}_1}(0) = 0, \tag{20}$$

whereas for the pinned left end one has:

$$\bar{U}_1(0) = 0, \quad \bar{W}_1(0) = 0, \quad \frac{d^2\bar{W}_1}{d\bar{z}_1^2}(0) = 0. \tag{21}$$

[1] Introducing $\mathbf{C}_1 = \left[C_{1(1)} \dots C_{6(1)}\right]^T$ as a vector of integration constants corresponding to segment (S_1) and putting (18) and (19) into (20) and (21) yields the following matrix relation:

$$\mathbf{C}_1 = \mathbf{T}_0 \mathbf{C}_0 \tag{22}$$

where for the clamped end one has:

$$\mathbf{C}_0 = \left[C_{1(1)} \;\; C_{2(1)} \;\; C_{6(1)}\right]^T, \quad \mathbf{T}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

and for the pinned one:

$$\mathbf{C}_0 = \left[C_{2(1)} \;\; C_{4(1)} \;\; C_{6(1)}\right]^T, \quad \mathbf{T}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

3.2. Boundary conditions at the junction of the segments

In order to establish corresponding continuity conditions at the junction of the stepped beam segments, let us consider an infinitesimal part of the stepped beam at the step location as it is depicted in Figure 2.

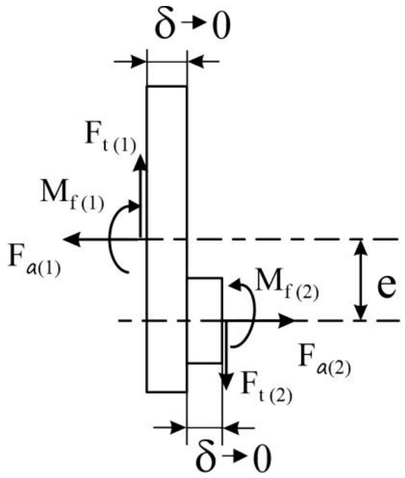

Figure 2: Free-body diagram of an infinitesimal part of the stepped beam at the junction of segments

Here, $F_{t(1)}$ and $F_{t(2)}$ are the shear forces defined as [31, 33]:

$$F_{t(1)} = -E_1 I_{x(1)} \frac{d^3 W_1}{dz_1^3}(L_1 - \delta), \qquad F_{t(2)} = -E_2 I_{x(2)} \frac{d^3 W_2}{dz_2^3}(0 + \delta), \tag{25}$$

$F_{a(1)}$ and $F_{a(2)}$ are the axial forces given as [31, 33]:

$$F_{a(1)} = E_1 A_1 \frac{dU_1}{dz_1}(L_1 - \delta), \qquad F_{a(2)} = E_2 A_2 \frac{dU_2}{dz_2}(0 + \delta), \tag{26}$$

and, finally, $M_{f(1)}$ and $M_{f(2)}$ are the bending moments defined as [31, 33]:

$$M_{f(1)} = -E_1 I_{x(1)} \frac{d^2 W_1}{dz_1^2}(L_1 - \delta), \qquad M_{f(2)} = -E_2 I_{x(2)} \frac{d^2 W_2}{dz_2^2}(0 + \delta). \tag{27}$$

Taking the length δ approaches to zero yields the following continuity conditions at the level of forces and moments:

$$F_{t(1)} = F_{t(2)} \Leftrightarrow \frac{d^3\bar{W}_1}{d\bar{z}_1^3}(\gamma_{L1}) = \gamma_E\gamma_I \frac{d^3\bar{W}_2}{d\bar{z}_2^3}(0), \tag{28}$$

$$F_{a(1)} = F_{a(2)} \Leftrightarrow \frac{\mathrm{d}\bar{U}_1}{d\bar{z}_1}(\gamma_{L1}) = \gamma_E\gamma_A \frac{\mathrm{d}\bar{U}_2}{d\bar{z}_2}(0), \tag{29}$$

$$M_{f(1)} = M_{f(2)} + F_{a(2)}\mathrm{e} \Leftrightarrow \frac{d^2\bar{W}_1}{d\bar{z}_1^2}(\gamma_{L1}) = \gamma_E\gamma_I \frac{d^2\bar{W}_2}{d\bar{z}_2^2}(0) - \frac{\gamma_E\gamma_A\bar{e}}{\bar{r}^2}\frac{\mathrm{d}\bar{U}_2}{d\bar{z}_2}(0), \tag{30}$$

where $\gamma_{L1} = L_1 / L$ and $\bar{e} = e / L$. Also, at the level of displacements one has the following continuity conditions:

$$W_1(L_1) = W_2(0) \Leftrightarrow \bar{W}_1(\gamma_{L1}) = \bar{W}_2(0), \tag{31}$$

$$\frac{dW_1}{dz_1}(L_1) = \frac{dW_2}{dz_2}(0) \Leftrightarrow \frac{d\bar{W}_1}{d\bar{z}_1}(\gamma_{L1}) = \frac{d\bar{W}_2}{d\bar{z}_2}(0), \tag{32}$$

$$U_1(L_1) - \mathrm{e}\frac{dW_1}{dz_1}(L_1) = U_2(0) \Leftrightarrow \bar{U}_1(\gamma_{L1}) - \bar{e}\frac{d\bar{W}_1}{d\bar{z}_1}(\gamma_{L1}) = \bar{U}_2(0). \tag{33}$$

The continuity conditions (28)-(33) generate the following matrix relation:

$$\mathbf{T}_{1L}\mathbf{C}_1 = \mathbf{T}_{1R}\mathbf{C}_2 \tag{34}$$

where $\mathbf{C}_2 = \left[C_{1(2)} \cdots C_{6(2)}\right]^T$ is the vector of integration constants corresponding to segment (S_2) and entries of the matrix $\mathbf{T}_{1L} \in R^{6\times6}$ are:

$$\boldsymbol{T}_{1L} = \begin{bmatrix} sin(\bar{k}\gamma_{L1}) & -cos(\bar{k}\gamma_{L1}) & sinh(\bar{k}\gamma_{L1}) & cosh(\bar{k}\gamma_{L1}) & 0 & 0 \\ 0 & 0 & 0 & 0 & -sin(\bar{r}\bar{k}^2\gamma_{L1}) & cos(\bar{r}\bar{k}^2\gamma_{L1}) \\ -cos(\bar{k}\gamma_{L1}) & -sin(\bar{k}\gamma_{L1}) & cosh(\bar{k}\gamma_{L1}) & sinh(\bar{k}\gamma_{L1}) & 0 & 0 \\ cos(\bar{k}\gamma_{L1}) & sin(\bar{k}\gamma_{L1}) & cosh(\bar{k}\gamma_{L1}) & sinh(\bar{k}\gamma_{L1}) & 0 & 0 \\ -sin(\bar{k}\gamma_{L1}) & cos(\bar{k}\gamma_{L1}) & sinh(\bar{k}\gamma_{L1}) & cosh(\bar{k}\gamma_{L1}) & 0 & 0 \\ \bar{e}\bar{k}\,sin(\bar{k}\gamma_{L1}) & -\bar{e}\bar{k}\,cos(\bar{k}\gamma_{L1}) & -\bar{e}\bar{k}\,sinh(\bar{k}\gamma_{L1}) & -\bar{e}\bar{k}\,cosh(\bar{k}\gamma_{L1}) & cos(\bar{r}\bar{k}^2\gamma_{L1}) & sin(\bar{r}\bar{k}^2\gamma_{L1}) \end{bmatrix}, \tag{35}$$

and of the matrix $\mathbf{T}_{1R} \in R^{6\times6}$:

$$\mathbf{T}_{1R} = \begin{bmatrix} 0 & -\gamma_E\gamma_I\sqrt[4]{\left(\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}\right)^3} & 0 & \gamma_E\gamma_I\sqrt[4]{\left(\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}\right)^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma_E\gamma_A\sqrt{\frac{\gamma_\rho}{\gamma_E}} \\ -\gamma_E\gamma_I\sqrt{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}} & 0 & \gamma_E\gamma_I\sqrt{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}} & 0 & 0 & -\frac{\gamma_E\gamma_A\bar{e}}{\bar{r}}\sqrt{\frac{\gamma_\rho}{\gamma_E}} \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & \sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}} & 0 & \sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{36}$$

Solving (34) for $\mathbf{C}_2$ yields:

$$\mathbf{C}_2 = \mathbf{T}_1\mathbf{C}_1 \tag{37}$$

where the matrix $\mathbf{T}_1 \in R^{6\times6}$ is determined by:

$$\mathbf{T}_1 = \mathbf{T}_{1R}^{-1}\mathbf{T}_{1L} \tag{38}$$

and represents the transfer matrix between the integration constants of segments (S_1) and (S_2).

3.3. Boundary conditions at the right end of the stepped beam

In this section the following types of the right end of the stepped beam will be considered: free end, clamped end, and pined end. The boundary conditions for free right end read:

$$\frac{d\bar{U}_2}{d\bar{z}_2}(\gamma_{L2})=0,\ \ \frac{d^2\bar{W}_2}{d\bar{z}_2^2}(\gamma_{L2})=0,\ \ \frac{d^3\bar{W}_2}{d\bar{z}_2^3}(\gamma_{L2})=0, \tag{39}$$

for clamped right end one has:

$$\bar{U}_2(\gamma_{L2})=0,\quad \bar{W}_2(\gamma_{L2})=0,\quad \frac{d\bar{W}_2}{d\bar{z}_2}(\gamma_{L2})=0, \tag{40}$$

and, finally, the corresponding boundary conditions for pinned right end are:

$$\bar{U}_2(\gamma_{L2})=0,\quad \bar{W}_2(\gamma_{L2})=0,\quad \frac{d^2\bar{W}_2}{d\bar{z}_2^2}(\gamma_{L2})=0, \tag{41}$$

where $\gamma_{L2}=L_2/L=1-\gamma_{L1}$. Introducing (18) and (19) into (39)-(41) yields the following matrix expression:

$$\mathbf{T}_2\mathbf{C}_2=0_{3\times1} \tag{42}$$

where $0_{3\times1}\in R^{3\times1}$ is a zero matrix and the matrix $\mathbf{T}_2\in R^{3\times6}$ has the following entries:

- free right end

$$\mathbf{T}_2=\begin{bmatrix} 0 & 0 & 0 & 0 & -\sin\left(\sqrt{\frac{\gamma_\rho}{\gamma_E}}\bar{r}\bar{k}^2\gamma_{L2}\right) & \cos\left(\sqrt{\frac{\gamma_\rho}{\gamma_E}}\bar{r}\bar{k}^2\gamma_{L2}\right) \\ -\cos\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & -\sin\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \cosh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \sinh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & 0 & 0 \\ \sin\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & -\cos\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \sinh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \cosh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & 0 & 0 \end{bmatrix} \tag{43}$$

- clamped right end

$$\mathbf{T}_2=\begin{bmatrix} 0 & 0 & 0 & 0 & \cos\left(\sqrt{\frac{\gamma_\rho}{\gamma_E}}\bar{r}\bar{k}^2\gamma_{L2}\right) & \sin\left(\sqrt{\frac{\gamma_\rho}{\gamma_E}}\bar{r}\bar{k}^2\gamma_{L2}\right) \\ \cos\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \sin\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \cosh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \sinh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & 0 & 0 \\ -\sin\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \cos\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \sinh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \cosh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & 0 & 0 \end{bmatrix} \tag{44}$$

- pinned right end

$$\mathbf{T}_2=\begin{bmatrix} 0 & 0 & 0 \\ \cos\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \sin\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \cosh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) \\ -\cos\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & -\sin\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & \cosh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) \end{bmatrix}$$

$$\left.\begin{matrix} 0 & \cos\left(\sqrt{\frac{\gamma_\rho}{\gamma_E}}\bar{r}\bar{k}^2\gamma_{L2}\right) & \sin\left(\sqrt{\frac{\gamma_\rho}{\gamma_E}}\bar{r}\bar{k}^2\gamma_{L2}\right) \\ \sinh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & 0 & 0 \\ \sinh\left(\sqrt[4]{\frac{\gamma_\rho\gamma_A}{\gamma_E\gamma_I}}\bar{k}\gamma_{L2}\right) & 0 & 0 \end{matrix}\right] \tag{45}$$

3.4. Derivation of the frequency equation

Substituting (22) and (37) into (42) implies a homogeneous system of equations for unknown components of the vector $\mathbf{C}_0$. This equations system can be written in the matrix form as follows:

$$\mathbf{TC}_0 = 0_{3\times1}, \tag{46}$$

where $\mathbf{T} \in R^{3\times3}$ represents overall transfer matrix given as:

$$\mathbf{T} = \mathbf{T}_2\mathbf{T}_1\mathbf{T}_0 . \tag{47}$$

Finally, the corresponding frequency equation for the problem analyzed reads:

$$f\left(\bar{k}\right) \equiv \det\mathbf{T} = 0 . \tag{48}$$

4. NUMERICAL EXAMPLES

In numerical calculations of this section, the stepped beam geometrical parameters given in [21] will be used as follows: $h_1 = 19.05\,\mathrm{mm}$, $h_2 = 5.49\,\mathrm{mm}$, $b_1 = b_2 = 25.4\,\mathrm{mm}$, $L_1 = 254\,\mathrm{mm}$, $L_2 = 140\,\mathrm{mm}$. Based on the theoretical considerations given in Sections 2 and 3, the effect of eccentricity e on dimensionless natural angular frequencies of the stepped beam for various combinations of materials of the beam segments is shown in Table 1.

Table 1: Values of the lowest four dimensionless natural angular frequencies for various combinations of materials of beam segments

Boundary conditions	γ_E	γ_ρ	$\bar{e}$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$
C-F	1	1	0	4.91792	11.5118	41.2372	63.3360
			$(h_1-h_2)/2L$	4.91721	11.5160	41.1452	63.2388
	210/70	7800/2702	0	3.56078	12.7782	38.3359	68.4161
			$(h_1-h_2)/2L$	3.55984	12.7772	38.0964	68.1344
	210/200	7800/5700	0	4.37158	10.8834	38.1153	59.7993
			$(h_1-h_2)/2L$	4.37092	10.8816	38.0313	59.6232
	70/210	2702/7800	0	6.04735	10.2739	44.1409	58.7212
			$(h_1-h_2)/2L$	6.04700	10.2732	44.1087	58.6712
	200/210	5700/7800	0	5.47852	12.1880	44.2492	67.5286
			$(h_1-h_2)/2L$	5.47778	12.1873	44.1545	67.4851
C-C	1	1	0	10.5456	41.6661	63.4959	126.956
			$(h_1-h_2)/2L$	10.9146	42.6224	63.8636	127.547
	210/70	7800/2702	0	12.1451	38.6258	68.5107	122.251
			$(h_1-h_2)/2L$	12.2614	39.4791	68.6165	122.755
	210/200	7800/5700	0	10.2694	38.5099	59.8737	117.221
			$(h_1-h_2)/2L$	10.6321	39.2554	60.4155	117.596
	70/210	2702/7800	0	9.36794	44.6403	58.9085	131.053
			$(h_1-h_2)/2L$	9.73649	45.1417	59.3185	131.361
	200/210	5700/7800	0	10.7400	44.6716	67.8257	135.238
			$(h_1-h_2)/2L$	11.1144	45.8394	68.0187	136.001
C-P	1	1	0	8.48564	33.1612	55.5437	111.304
			$(h_1-h_2)/2L$	9.03568	33.5446	56.4792	111.553
	210/70	7800/2702	0	8.52040	32.2248	57.7435	110.564
			$(h_1-h_2)/2L$	8.80366	32.7437	58.1751	110.898
	210/200	7800/5700	0	8.11873	30.2376	53.2552	101.200
			$(h_1-h_2)/2L$	8.65478	30.5024	54.2718	101.308

	70/210	2702/7800	0	2.90797	33.6863	54.0219	110.938
			$(h_1-h_2)/2L$	8.8912	33.7989	54.8195	111.008
	200/210	5700/7800	0	8.77674	36.4176	57.7648	121.942
			$(h_1-h_2)/2L$	9.33593	36.9565	58.5750	122.396
P-P	1	1	0	3.58639	27.5587	45.8283	101.296
			$(h_1-h_2)/2L$	3.99912	28.4686	46.1990	102.027
	210/70	7800/2702	0	4.97231	25.0462	50.0108	96.9531
			$(h_1-h_2)/2L$	5.08686	25.8629	50.1198	97.5953
	210/200	7800/5700	0	3.53750	25.3954	43.2920	93.5887
			$(h_1-h_2)/2L$	3.93047	26.1356	43.7984	94.0675
	70/210	2702/7800	0	2.29499	30.0517	42.0169	105.198
			$(h_1-h_2)/2L$	2.83477	30.5298	42.4227	105.562
	200/210	5700/7800	0	3.60704	29.7006	48.7503	108.041
			$(h_1-h_2)/2L$	4.03779	30.7882	48.9768	109.005

The limit value $\bar{e}=(h_1-h_2)/(2L)$ corresponds to the case of an eccentrically stepped beam with the flat bottom surface. The influence of the eccentricity $\bar{e}$ on the lowest four dimensionless frequency coefficients is examined in Figures 3, 4, 5, and 6. At that, the values of $\bar{e}$ is taken from the interval $0.001 \le \bar{e} \le (h_1-h_2)/(2L)$.

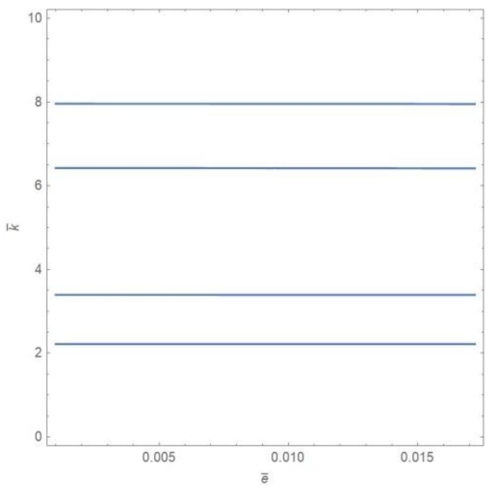

Figure 3: The effect of the eccentricity $\bar{e}$ on the lowest four dimensionless frequency coefficients of the clamped-free stepped beam

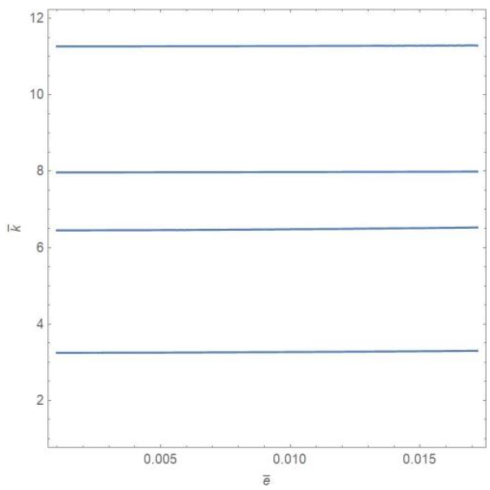

Figure 4: The effect of the eccentricity $\bar{e}$ on the lowest four dimensionless frequency coefficients of the clamped-clamped stepped beam

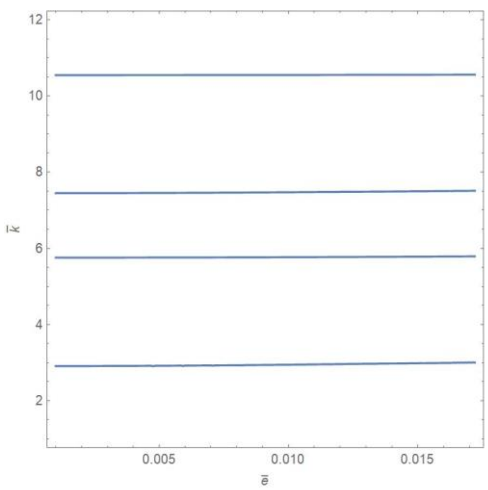

Figure 5: The effect of the eccentricity $\bar{e}$ on the lowest four dimensionless frequency coefficients of the clamped-pinned stepped beam

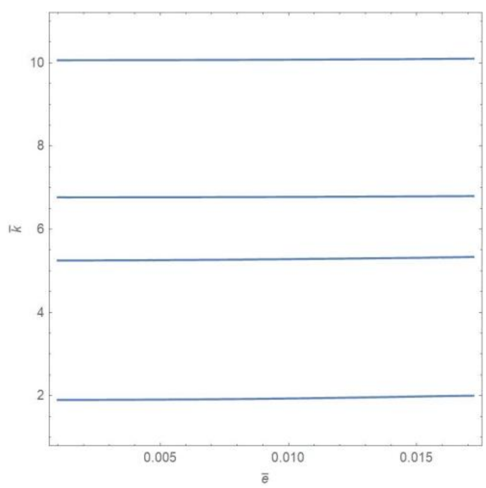

Figure 6: The effect of the eccentricity $\bar{e}$ on the lowest four dimensionless frequency coefficients of the pinned-pinned stepped beam

5. CONCLUSIONS

An analytical approach based on the transfer matrix method for free vibration analysis of stepped Euler-Bernoulli beams with coupled axial and bending vibrations has been presented. The mutual eccentric position of the beam segments longitudinal axes has been considered as the cause of coupling of axial and bending vibrations. The presented method can be also used in the cases of pure axial and pure bending vibrations of stepped beams. The numerical simulations show that the existence of eccentricity e causes small changes in the values of the first four natural frequencies. These changes are more pronounced in the case of the beam segments made of different materials. By changing the value of eccentricity e, the crossing and veering phenomena have not been detected.

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