COMPARISON OF NUMERICAL INTEGRATION METHODS IN THE LINEAR DYNAMIC ANALYSIS

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COMPARISON OF NUMERICAL INTEGRATION METHODS IN THE LINEAR DYNAMIC ANALYSIS

Danijela Miloradović[1], Gordana Bogdanović[2], Lozica Ivanović[3], Vladimir Geroski[4], Marija Rafailović[5]

Summary: *Direct integration methods for ordinary differential equations encountered in analysis of dynamic problems of vibration are studied and compared. Examples considered are a single degree of freedom oscillator and a multi degree of freedom linear model of vehicle. For one degree of freedom system, numerical solutions are compared with analytical solution. The time-integrators used include both explicit and implicit methods of Runge-Kutta, Newmark, Wilson, the central difference method. The fourth-order Runge-Kutta method has been the preferred numerical integration scheme for solving linear single degree of freedom system or two degree of freedom systems. This method is very accurate, but requires very small time-steps and four equation solutions per time-step. These drawbacks hinder the solution of problems in multi degree of freedom systems, therefore implicit methods are considered for multi degree of freedom. Methods are compared in terms of accuracy and ease of formulation.*
Key words: dynamic analysis, numerical integration methods, direct integration methods, vehicle dynamics

1. INTRODUCTION

The behaviour of many dynamic systems undergoing time-dependent changes can be described by ordinary differential equations. When the solution to the differential equations of motion of a dynamic system cannot be obtained in closed form, a numerical procedure is applied. All the numerical integration methods have two basic characteristics. First, they do not satisfy the differential equations at all time t, but only at discrete time intervals, Δt. Secondly, within each time interval Δt, a specific type of variation of the displacement X, velocity $\dot{X}$, and acceleration $\ddot{X}$ is assumed. Thus,

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several numerical integration schemes are available depending on the type of variation assumed for X, $\dot{X}$ and $\ddot{X}$ within each time interval Δt. A brief description of these methods is presented for linear dynamic response analysis and tested for single degree of freedom system, and then applied for vehicle dynamics response.

2. SINGLE DEGREE OF FREEDOM

Consider a linear movement of the point M, the mass m (Fig. 1) according to the fixed horizontal smooth surface. Point M is subjected to a harmonic function $\vec{F} = F_0 \cos(\Omega t)\vec{i}$. The equation of motion can be written as [1]:

$$\ddot{x} + 2\delta\dot{x} + \omega^2 x = h\cos(\Omega t) \tag{2.1}$$

The equation of motion is linear differential equation with constant coefficients, so solution contains two components, homogeneous solution x_h and particular solution x_p. Characteristic equation of differential equation (2.1) is in the form:

$$\lambda^2 + 2\delta\lambda + \omega^2 = 0 \tag{2.2}$$

and roots of characteristic equation are:

$$\lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega^2} \tag{2.3}$$

Character of movement of point M is determined by roots of characteristic equation (2.3). There are three posibilities. If $\delta < \omega$, damping is small, and roots of (2.3) are conjugate complex. If $\delta > \omega$, damping is large, and roots of (2.3) are real and distinct numbers. If $\delta = \omega$, critical damping is applied, and roots of (2.3) are equal. Particular solution of (2.1) is assumed in form:

$$x_p = M\cos(\Omega t) + N\sin(\Omega t) \tag{2.4}$$

If it is introduced that $M = \cos\gamma$ and $N = \sin\gamma$, then particular solution is in the form:

$$x_p = C\cos(\Omega t - \gamma) \tag{2.5}$$

and constant of integration is determined by $C = f_{st} / \sqrt{(1-\Lambda^2)^2 + 4\Psi^2\Lambda^2}$, where f_{st} is static deformation of spring, ψ is unitless damping coefficient, and Λ is disturbance coeficient. Then solutions of differential equation (1.1) are:

$$\delta < \omega \qquad x = e^{-\delta t}\left[C_1\cos(pt) + C_2\sin(pt)\right] + C\cos(\Omega t - \gamma)$$

$$\delta > \omega \qquad x = e^{-\delta t}\left[C_1 e^{qt} + C_2 e^{-qt}\right] + C\cos(\Omega t - \gamma)$$

$$\delta = \omega \qquad x = e^{-\delta t}\left[C_1 t + C_2\right] + C\cos(\Omega t - \gamma)$$
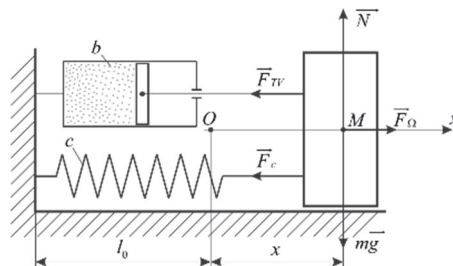

Fig. 1 *Forced oscillation with damping*

3. DIRECT NUMERICAL INTEGRATION METHODS

The general equation of damped multi degree of freedom dynamical systems, which is linear, can be expressed in the following general form:

$$[M]\{\ddot{X}\}+[C]\{\dot{X}\}+[K]\{X\}=\{F(t)\} \tag{3.1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices of the system, $F(t)$ is the applied force vector. For a linear dynamic system, matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are independent of time and therefore remain unchanged during the integration procedure.

In a direct integration method, the system of equations of motion is integrated successively by using a step by step numerical procedure. Method is direct because there is no need to transform equation of motion prior to integration. There are two main approaches used in the direct integration method: explicit and implicit schemes. In an explicit scheme (Runge-Kutta and central difference method), the response quantities are expressed in terms of previously determined values of displacement, velocity and acceleration. In an implicit scheme (Newmark and Wilson theta method), the difference equations are coupled with the equations of motion, and the displacements are calculated directly by solving the equations. Detailed algorithm and procedures for these methods are given in [2] and [3].

For central difference method, acceleration and velocity are given by:

$$\{\dot{X}_t\}=\frac{1}{2\Delta t}\Big[\{X_{t+\Delta t}\}-\{X_{t-\Delta t}\}\Big]$$
$$\{\ddot{X}_t\}=\frac{1}{\Delta t^2}\Big[\{X_{t+\Delta t}\}-2\{X_t\}+\{X_{t-\Delta t}\}\Big] \tag{3.2}$$

The Wilson θ method assumes that the acceleration of the system varies linearly between two instants of time. If τ is the increase in time t between t and $t+\theta\Delta t$ $(0\leq\tau\leq\theta\Delta t)$, then for time interval t to $t+\theta\Delta t$, it can be assumed that:

$$\ddot{X}_{t+\tau}=\ddot{X}_t+\frac{\tau}{\theta\Delta t}\left(\ddot{X}_{t+\theta\Delta t}-\ddot{X}_t\right) \tag{3.3}$$

Expression for velocity and displacement can be obtained by integrating (4.2):

$$\dot{X}_{t+\tau}=\dot{X}_t+\ddot{X}_t\tau+\frac{\tau^2}{2\theta\Delta t}\left(\ddot{X}_{t+\theta\Delta t}-\ddot{X}_t\right)$$
$$X_{t+\tau}=X_t+\dot{X}_t\tau+\frac{1}{2}\ddot{X}_t\tau^2+\frac{\tau^3}{6\theta\Delta t}\left(\ddot{X}_{t+\theta\Delta t}-\ddot{X}_t\right) \tag{3.4}$$

The Newmark Beta integration method is also based on the assumption that the acceleration varies linearly between two instants of time. Two parameters α and β are used in this method, which can be changed to suit the requirements of a particular problem and to obtain desired integration accuracy and stability. The expressions for velocity and displacements are given by:

$$\dot{X}_{t+\Delta t}=\dot{X}_t+\Big[(1-\beta)\ddot{X}_t+\beta\ddot{X}_{t+\Delta t}\Big]\Delta t$$
$$X_{t+\Delta t}=X_t+\dot{X}_t\Delta t+\left[\left(\frac{1}{2}-\alpha\right)\ddot{X}_t+\alpha\ddot{X}_{t+\Delta t}\right]\Delta t^2 \tag{3.5}$$
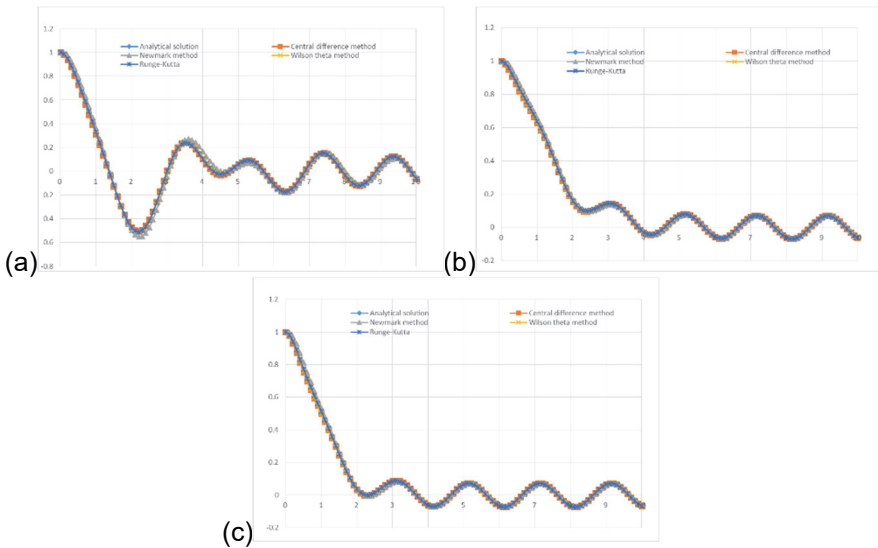
(a)

(b)

(c)

Fig 2. *Comparison of Numerical Methods with Analytical solution when (a) roots are conjugate complex, (b) roots are real and distinct and (c) roots are equal*

4. VERTICAL MODEL OF VEHICLE DYNAMICS

In order to be able to study the oscillations of a vehicle in a vertical plane with a linear movement on a straight road, without the affection of side-winds, a spatial model of a vehicle with frontal indipendent and rear dependent elastic suspension system was adopted which is shown in Figure 3. The model was developed by Danijelas research of non-linear oscillations of vehicles influenced by the unevenness of the road. Detailed model of vehicle dynamics can be found at [4] and [5].
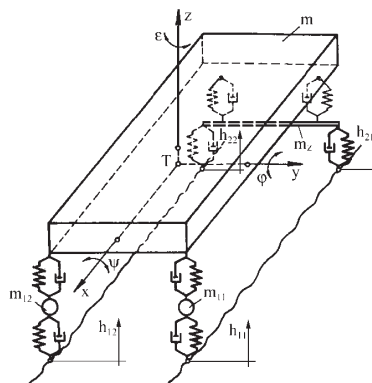

Fig. 3 *Adopted model of vertical dynamics of the vehicle*

The differential equations of motion for the model vehicle are derived from the Lagrange's equation of second order by taking into account the adoption of generalized

coordinates. The vehicle model is observed as a system composed of rigid bodies subjected to conservative, dissipative, and other arbitrary non-conservative forces, where the Lagrange's equation of second order has a form:

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_i}\right) - \frac{\partial E_k}{\partial q_i} = -\frac{\partial E_p}{\partial q_i} - \frac{\partial \Phi}{\partial \dot{q}_i} + Q_{q_i}^*, \quad i = 1, n \tag{4.1}$$

Equation of vehicle dynamics in matrix form [4]:

$$[M]\begin{Bmatrix} \ddot{z} \\ \ddot{z}_{11} \\ \ddot{z}_{12} \\ \ddot{z}_2 \\ \ddot{\psi} \\ \ddot{\varphi} \\ \ddot{\psi}_2 \end{Bmatrix} + [B]\begin{Bmatrix} \dot{z} \\ \dot{z}_{11} \\ \dot{z}_{12} \\ \dot{z}_2 \\ \dot{\psi} \\ \dot{\varphi} \\ \dot{\psi}_2 \end{Bmatrix} + [C]\begin{Bmatrix} z \\ z_{11} \\ z_{12} \\ z_2 \\ \psi \\ \varphi \\ \psi_2 \end{Bmatrix} == \begin{Bmatrix} 0 \\ b_{p1}\dot{h}_{11} + c_{p1}h_{11} \\ b_{p1}\dot{h}_{12} + c_{p1}h_{12} \\ b_{p2}\dot{h}_{21} + c_{p2}h_{21} + b_{p2}\dot{h}_{22} + c_{p2}h_{22} \\ 0 \\ 0 \\ b_{p2}s_2(\dot{h}_{21} - \dot{h}_{22}) + c_{p2}s_2(h_{21} - h_{22}) \end{Bmatrix}.$$

Vehicle dynamics model is numerically simulated with four numerical methods. Road is simulated as a harmonic function with the amplitude of $\pm 10\,[\text{mm}]$. Responses of vehicle are shown in Figure 4.
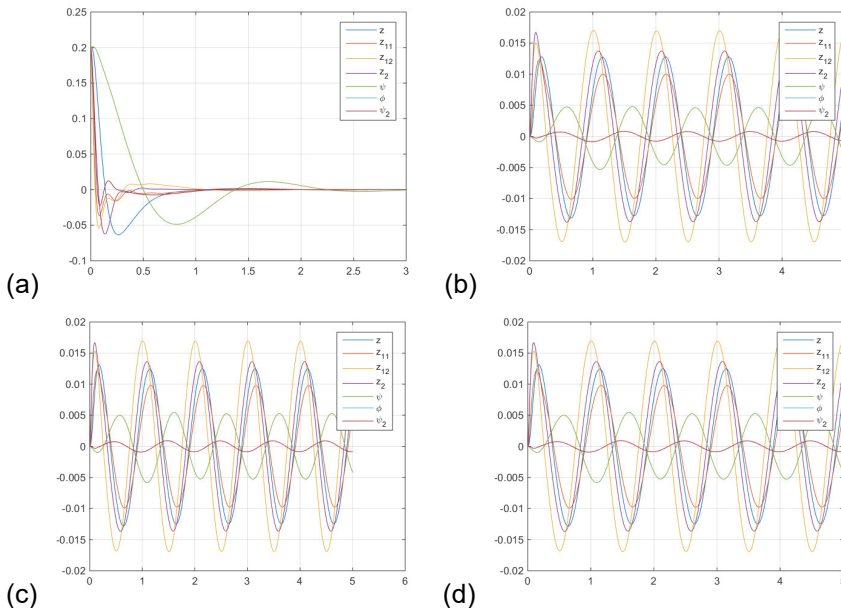

Fig. 4 *(a) Free vibration of vehicle, (b) response of vehicle with harmonic unevenness of road-Newmark method, (c) response of vehicle with harmonic unevenness of road-Central difference method, (d) response of vehicle with harmonic unevenness of road-Wilson theta method*

5. CONCLUSION

In this paper were used both implict and explicit integration scheme for vehicle response under linear condition. For single degree of freedom, tested methods have shown that they are very accurate. For multi degree of freedom, like vehicle model, with seven degree of freedom, there are small differences, except for Runge-Kutta, which shown to be very unstable even with very small integration step size. Still, Runge-Kutta method in single degree of freedom had the closest solution to analytical solution. Newmark method had the solution with a greater value of damping than its real value, but result is still very accurate, and Newmark method can be implemented to be unconditionaly stable, so there is big advantage in multi degree of freedom dynamical analysis. Wilson theta method can be also unconditionaly stable, when $\theta \geq 1.37$.

For systems with one or two degree of freedom, it is recommended to use Runge-Kutta method, but for more than two degree of freedom it is recommended to use other schemes, and if stability is the issue then implicit scheme is the preffered solution.

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