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# A local-time-induced unique pointer basis

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There is a solution to the problem of asymptotic completeness in many-body scattering theory that offers a specific view of the quantum unitary dynamics which allows for the straightforward introduction of local time for every, at least approximately closed, many-particle system. In this approach, time appears as a hidden classical parameter of the unitary dynamics of a many-particle system. We show that a closed many-particle system can exhibit behaviour that is characteristic for open quantum systems and there is no need for the 'state collapse' or environmental influence. On the other hand, closed few-particle systems bear high quantum coherence. This local-time scheme encompasses concepts including 'emergent time', 'relational time' as well as the 'hybrid system' models with possibly induced gravitational uncertainty of time.

## 1. Introduction

A solution to the problem of asymptotic completeness in the many-body scattering theory offers a specific view of the quantum unitary dynamics. The important work of Enss [1,2] opened the door for new methods in solving the problem. On this basis, the later elaboration due to Kitada [3,4] allowed Kitada [5–7] to introduce the notion of local time, that is a dynamics generated by the Hamiltonian of the local system that can serve as a (local) 'clock'.

The notion of local time or 'multi-time' is not a new idea. Mainly motivated by relativity, a separate time coordinate for every particle in a composite system has been introduced (e.g. [8,9] and references therein). It is also shown that the 'timeless' Wheeler–DeWitt equation:

$$H(x)|\Psi\rangle = 0, \tag{1.1}$$



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follows if one assumes the existence of a preferred foliating family of space-like surfaces in space–time [10].

On closer inspection, such ad hoc schemes for local-time face serious obstacles in non-relativistic quantum context. For example, it can be shown that 'Multi-time Schrödinger equations cannot contain interaction potentials' [8]. Consequently, the following scenario appears to be inescapable: a non-relativistic Universe, seen as a collection of interacting subsystems, must have unique, global time that is common for all possible subsystems. In other words, it would seem that there is no room for the idea of local time in non-relativistic quantum theory regarding the relativistic considerations [11].

However, Kitada's [5] concept of local time is neither ad hoc nor does it suffer from such problems. Based on the non-relativistic many-body scattering theory, Kitada's approach defines 'local systems (clocks)' as approximately isolated from the other systems in the sense that a local clock of each local system is described by the Schrödinger Law. Prima facie, the assumption on independent local clocks may seem to be a reminiscent of the observation [8] that interactions (and the induced quantum entanglement) may ruin the idea of local time in non-relativistic quantum theory. However, in Kitada's scheme, a particle's time is *shared with some other particles* and is *not fixed* once for all. The particle's time is the time defined by the isolated local system that the particle is a part of. Dynamics of the total system (the Universe) can place the particle in different local systems and therefore in different local times. This provides a specific interpretation of equation (1.1)—on the level of the total universe, time does not exist, but on the local level, time does exist [3–7]—as well as makes the scheme well suited both for the purposes of quantum measurement [12–14] and decoherence [14–18] and for some models of open quantum systems [19,20] theory. A clear example of the latter which we know from measurement and decoherence is that the conglomeration of small few-particle systems can produce, at some point, a many-particle system that is approximately isolated from the other 'conglomerates'.

In quantum measurement, one deals with the many-particle systems that are assumed to be almost isolated (closed) systems. That is, an 'object of measurement + apparatus' ($O + A$) system, or 'object of measurement + apparatus + the apparatus' environment ($O + A + E$), is subjected to unitary Schrödinger dynamics disregarding existence of other such systems. In the standard decoherence theory, the composite system 'open system + environment' ($S + E$) is assumed to be subject to unitary Schrödinger dynamics despite the presence of other open systems and their environments. Finally, in the context of the open quantum systems theory, it is known that virtually every dynamics of an open system $S$ can be described by the Schrödinger Law for the extended system $S + E$ [20]. So, the macroscopic (many-particle) systems $O + A$, $O + A + E$ and $S + E$ *that are subject to the Schrödinger Law* appear as perfect candidates for application of, and some kind of a test for, the local-time scheme of Kitada.

With this motivation in mind, we hypothesize the following rule for the universally valid quantum theory: 'Every many-particle system that is (approximately) subject to the Schrödinger law, can be assigned a local time independently of other such systems', and we investigate the consequences for the description of the quantum-decoherence-like processes.

In this paper, we slightly extend the original proposal [5] by pointing out the uncertainty of local time for the local clocks. This introduces the local time effectively as a hidden classical parameter in the system's unitary Schrödinger dynamics. We introduce a Gaussian distribution for a time interval as the obvious choice and for a subsystem of a closed bipartite many-particle system we obtain *unique* 'pointer basis'. Our results come from the *macroscopic* domain but without a need for resorting to 'state collapse' or environmental influence.

Implications of the local-time scheme of Kitada are noteworthy. Hence we believe it is worth further investigation in the foundations of quantum theory as well as towards the original relativistic motivations with the view of the possible reduction of the gap between the quantum and relativistic theories.

The contents of this paper are as follows. In §2, we provide a brief account of the many-particle scattering theory and recapitulate Kitada's notion of local time. In §3, we point out uncertainty of time for bipartition of a many-particle system and set the quantitative criteria that revolve around

equation (3.2). In §4, we apply Kitada's scheme to the standard decoherence-like processes; §4*b* provides the main result of our paper. Section 5 provides some illustrative examples that exhibit technical simplicity, generality and clarity of the local-time scheme in the context of quantum measurement and decoherence. Section 6 is discussion of interpretational ramifications of the local-time scheme and §7 is conclusion.

## 2. Outlook of the many-body scattering theory and the notion of local time

### (a) Asymptotic completeness in the many-body scattering theory

A reader uninterested in details of the many-body scattering can skip to §2*b*. Scattering theory is essentially time-independent perturbation theory applied to the case of a continuous spectrum. The goal of scattering theory is to solve the full energy-eigenstate problem

$$(E - H_\circ - V)|\Psi\rangle = 0, \tag{2.1}$$

where $E > 0$ (unless otherwise specified), and $|\Psi\rangle$ is the eigenstate of the full Hamiltonian $H = H_0 + V$ with energy $E$. Already two-particle scattering is a hard problem. The many-body scattering poses even the more serious technical problems. It is due to Enss [1,2] that the method of clustering the composite system in conjunction with the so-called micro-local analysis method offers a systematic approach to the problem for both short-range and long-range interactions (denoted by $V$ in equation (2.1)). Subsequent development of the field of many-body scattering can be found, for example, in Sigal [21] and references therein.

The method of clustering consists in the following idea. A composite, closed system $\mathcal{S}$ that consists of $N$ non-identical particles can be differently structured [22], e.g. clustered in mutually non-intersecting clusters. For example, a tripartite system $\mathcal{S} = 1 + 2 + 3$ can be structured as $\mathcal{S}_1 = \{1, 2, 3\}$, $\mathcal{S}_2 = \{\{1\}, \{2, 3\}\}$, $\mathcal{S}_3 = \{\{2\}, \{1, 3\}\}$, $\mathcal{S}_4 = \{\{3\}, \{1, 2\}\}$, $\mathcal{S}_5 = \{\{1\}, \{2\}, \{3\}\}$, where the brackets '$\{*\}$' denote one cluster. So, the structures $\mathcal{S}_i$, $i = 2, 3, 4$ are different bipartitions of the total system $\mathcal{S}$, the $\mathcal{S}_5$ is a tripartite structure of the total system $\mathcal{S}$, while the $\mathcal{S}_1$ represents a formally unstructured system (only one cluster). For every cluster, a centre-of-mass (CM) and the relative-positions ($R$) degrees of freedom are introduced; the $R$ system's degrees of freedom are often chosen as the Jacobi relative coordinates. Then, bearing in mind the variety of the different possible structures, *all* the possible scattering scenarios are described by the scattering of the clusters' CM-systems.

For the $b$th structure (cluster decomposition) with $k$ clusters, the collective relative positions variable $x_b = \{x_{bi}, i = 1, 2, 3, \ldots, k\}$ and the related conjugate momentum, $p_b = \{p_{bi}, i = 1, 2, 3, \ldots, k\}$, are introduced for the clusters' centres-of-mass systems. The commutators $[x_{bi}, p_{b'j}] = \imath\hbar\delta_{ij}\delta_{bb'}$. The rest of the relative-positions variables are collectively denoted by $x^b$ with the conjugate momentums $p^b$. Then the Hilbert state space factorizes

$$\mathcal{H} = \mathcal{H}_{\mathrm{CM}} \otimes \mathcal{H}_b \otimes \mathcal{H}^b. \tag{2.2}$$

Of course, the above factorization equation is different for different structures, i.e. $\mathcal{H}_b \neq \mathcal{H}_{b'}$. By placing the reference frame in the total system's CM system, which is common for all structures, i.e. by choosing $X_{\mathrm{CM}} = 0$, the factorization equation (2.2) is reduced

$$\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}^b. \tag{2.3}$$

Therefore, observation of the scattering process reduces to observation of the inter-cluster Jacobi coordinates $x_{bi}$, for every structure $b$.

Another essential point in [1,2] comes from the fact that, as emphasized above, scattering refers to the continuous spectrum of the total system's Hamiltonian. So, the pure point-spectrum of the Hamiltonian should be removed from the consideration. The Enss solution is remarkable: he

considers the 'velocity operator' $v_b = m_b^{-1} p_b$, where $p_b$ is defined above and $m_b$ is the diagonal mass matrix with the diagonals being the intercluster reduced masses and introduces a kind of projection operator $\tilde{P}_b^{M_b^m}$, that can be found in appendix A. Then for every quantum state $|\Psi\rangle$ and for a wide class of the potentials $V$, equation (2.1), Enss proved

$$\left\| \left( \frac{x_b}{t_m} - v_b \right) \tilde{P}_b^{M_b^m} \exp\left( -\frac{\imath t_m H}{\hbar} \right) |\Psi\rangle \right\| \to 0, \tag{2.4}$$

as the time-index $m \to \pm\infty$, for *every* structure $b$ save for the 'structureless' one with only one cluster ($k = 1$). The norm $\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle}$, where $\langle\psi|\psi\rangle$ is the scalar product in the Hilbert state space $\mathcal{H}$, and the index $m$ is an integer taking positive or negative values including zero. The limit in equation (2.4) is in the standard sense of limit for a sequence of real norms indexed by $m$.

## (b) The notion of local time

Physical meaning of equation (2.4) is as follows. For a single free particle

$$(x - tv) \exp\left( -\frac{\imath tT}{\hbar} \right) = \exp\left( -\frac{\imath tT}{\hbar} \right) x, \tag{2.5}$$

where $x$ is the position and $v$ the velocity observable, while $T$ stands for the kinetic-energy observable and $t$ is an instant of time. Action of the operators in equation (2.5) on a wave function $\Psi(x)$ is as follows: if the support of $\Psi(x)$ is around some $x_\circ$ in the instant $t_\circ = 0$, then the support of the propagated wave function $\exp(-\imath tT/\hbar)\Psi(x)$ is localized around the point $x_\circ + vt$ in the instant of time $t$. Equation (2.4) essentially says that *the same holds* for the wave function of a *closed many-body* (many-particle) system with virtually arbitrary kinds of pair interactions in the system.

Equation (2.4) encompasses all the possible scattering scenarios for the closed system $\mathcal{S}$ as both the Hamiltonian, $H$, and the time instants, $t_m$, are *common for all structures*. Therefore, measurement of arbitrary $x_b$ and $v_b$ and obtaining their mean values, $\langle x_b \rangle$ and $\langle v_b \rangle$, provides, at least sketchily, the measurement of time for the total system [5–7]

$$\frac{\langle x_b \rangle}{\langle v_b \rangle} \sim t \tag{2.6}$$

in the asymptotic limit. This allows for the definition of the notion of $t$ as 'the reading of a clock $b$' in a way consistent with informal discussions of Relativity.

Scattering is a fundamental method of interaction for systems at all quantum scales. In analogy with equation (2.5), it is therefore reasonable to interpret equation (2.6) as a *notion* of time, which is common for all the structures, clusters and particles in the *closed* system $\mathcal{S}$, but *not necessarily* for some other *closed* systems, $\mathcal{S}'$, $\mathcal{S}''$, etc. The different rates of operation give rise to the intuitive picture of time as a characteristic of a local, i.e. approximately closed system.

While the concept of local time in certain schemes is an ad hoc idea [8,9], in the many-body scattering theory, this notion naturally fits with equation (2.4), which directly provides the following rules [5]: (a) systems with different Hamiltonians such as those with a different numbers of particles, or different kinds of particles, or different kinds of interactions between the particles are subject to different local times; (b) systems that mutually interact are subjected to the same time; (c) non-interacting systems need not have a common time; (d) non-identical many-body systems which do not interact and locally follow independent Schrödinger dynamics do not have a common time—which makes the universal time undefinable, as for equation (1.1); and (e) local time refer even to the mutually identical many-body systems, as long as they represent the mutually independent local systems.

In the remainder of this paper, we use the above points (a)–(e) as a matter of principle, i.e. as the new *universal rule* in quantum theory.

## 3. Uncertainty of local time

In certain processes, such as atomic collisions and chemical reactions, there may occur a change in the system's structure, $\mathcal{S}_b \to \mathcal{S}_{b'}$ [22,23]. In the case of more fundamental non-relativistic particles scattering experiments, a structure $\mathcal{S}_b$ typically remains unchanged. Then the measurement of the intercluster observable $x_b$ describes collisions of the particles for that structure.

Within the standard universally valid quantum mechanics, a closed system is defined by the unique state (in the Schrödinger picture):

$$|\Psi(t_\circ)\rangle = U(t_\circ)|\Psi(t=0)\rangle, \tag{3.1}$$

where $U(t) = \exp(-\imath tH/\hbar)$ and $H$ is the total system's Hamiltonian. Of course, equation (3.1) assumes unique, global physical time. If equation (3.1) models a measurement (or decoherence), then the measurement is assumed to be complete in an instant $t_\circ$, and the limit $t_\circ \to \infty$ is formally allowed.

According to point (e), §2*b*, even in the limit of zero metrological error, *there is a time uncertainty* $\Delta t$ in determining the *finite* $t_\circ$ that gives instead of equation (3.1)

$$\sigma = \int_{t_\circ-\Delta t}^{t_\circ+\Delta t} \rho(t)|\Psi(t)\rangle\langle\Psi(t)|\,\mathrm{d}t, \tag{3.2}$$

where for the time probability density $\rho(t)$:

$$\int_{t_\circ-\Delta t}^{t_\circ+\Delta t} \rho(t)\,\mathrm{d}t = 1, \int_{t_\circ-\Delta t}^{t_\circ+\Delta t} t\rho(t)\,\mathrm{d}t = t_\circ. \tag{3.3}$$

For the time probability density, we require: (1) to be symmetric on the narrow interval $[t_\circ - \Delta t, t_\circ + \Delta t]$, (2) regarding the decoherence-like processes, see equation (4.1), $t_0$ is the time instant in which decoherence is approximately complete, and therefore the limit $t_\circ \to \infty$ should be formally allowed and (3) to allow a proper limit $\rho(t) \to \delta(t)$, with the Dirac delta function $\delta(t)$, in order to be reducible to the standard case equation (3.1).

Physically, and also operationally, i.e. for an observer, the state equation (3.2) is *objective*—the so-called 'proper mixture'. Determining the time instant $t$ from the interval $[t_\circ - \Delta t, t_\circ + \Delta t]$ is equivalent with distinguishing between non-orthogonal states $|\Psi(t)\rangle$. Hence the no-cloning theorem [24] makes the task of distinguishing the time instants from the interval $[t_\circ - \Delta t, t_\circ + \Delta t]$ impossible *in principle* [25].

The time uncertainty $\Delta t$ does not introduce uncertainty of energy. Every term in equation (3.2) describes a unitary Schrödinger evolution with energy preservation: $\langle\Psi(t)|H|\Psi(t)\rangle = \langle\Psi(t=0)|H|\Psi(t=0)\rangle$. Then there is energy conservation also for the state equation (3.2): $\mathrm{tr}\,\sigma H = \mathrm{const}$.

### (a) The state equation (3.2) is mixed

By construction, the state equation (3.2) is mixed. Nevertheless, for the arbitrarily short interval $\Delta t \ll t_\circ$, from equations (3.3) and (3.4):

$$\sigma \approx \int_{t_\circ-\Delta t}^{t_\circ+\Delta t} \rho(t)\left(I - \frac{\imath(t-t_\circ)}{\hbar}H\right)|\Psi(t_\circ)\rangle\langle\Psi(t_\circ)|\left(I + \frac{\imath(t-t_\circ)}{\hbar}H\right)\mathrm{d}t \approx |\Psi(t_\circ)\rangle\langle\Psi(t_\circ)| \tag{3.4}$$

with an error of the order of $(\delta t/\hbar)^2$ and the standard deviation $\delta t$. For $t_\circ \gg 1$ (cf. the above point (2)), the interval $\Delta t$ need not be that short while it can still fulfil $\Delta t \ll t_\circ$.

On the other hand, for $\Delta t > \tau_{\min} = \max\{\pi\hbar/2\Delta H, \pi\hbar/2(\langle H\rangle_{t=0} - E_\mathrm{g})\}$, where $\Delta H$ is the standard deviation and $E_\mathrm{g}$ stands for the Hamiltonian ground energy, there are three time instants, $t_\circ - \Delta t$, $t_\circ$ and $t_\circ + \Delta t$, which pertain to mutually (approximately) orthogonal states

[26,27]. While $\Delta t$ can be very small in some physical units, it can still be 'large' so as, for the coarse-grained time axis with the width $\Delta t$, the state equation (3.2) reads

$$\sigma = p_-|\Psi(t_\circ - \Delta t)\rangle\langle\Psi(t_\circ - \Delta t)| + p_\circ|\Psi(t_\circ)\rangle\langle\Psi(t_\circ)| + p_+|\Psi(t_\circ + \Delta t)\rangle\langle\Psi(t_\circ + \Delta t)|. \tag{3.5}$$

For such time interval $\Delta t$, the states in equation (3.5) can be mutually distinguished.

In this paper, we reduce our attention to the proper small intervals $\Delta t$ that allow the limit $\Delta t \to 0$, i.e. only slight deviation from equation (3.1), while not leading either to equation (3.4) or to equation (3.5).

In accordance with equations (3.3) and (3.4) and due to the above points (1)–(3), we choose a Gaussian time probability distribution:

$$\rho(t) = \sqrt{\frac{\lambda}{\pi}}\exp(-\lambda(t - t_\circ)^2), \tag{3.6}$$

which in the limit $\lambda \to \infty$ provides the standard case equation (3.1). Therefore, we choose the smallest possible $\lambda$ so as $\tau_{\min}/2 > \Delta t > \lambda^{-1}$ and

$$\int_{t_\circ - \Delta t}^{t_\circ + \Delta t} \rho(t)\,\mathrm{d}t \approx \int_{-\infty}^{\infty} \rho(t)\,\mathrm{d}t = 1. \tag{3.7}$$

Equations (3.6) and (3.7) provide the estimate $\Delta t \gtrsim \lambda^{-1/2}$ that implies $\Delta t > \lambda^{-1}$ and the above constraint reduces to $\tau_{\min} > 2\Delta t$.

The choice of a Gaussian density equation (3.6) is by no means the only one possible. Nevertheless, it facilitates the analysis and allows a comparison of the different models of the composite $O + A$ system—see §5 for some relevant models. The possible extensions of our considerations towards non-Gaussian or even model-dependent density $\rho(t)$ may be physically sound but are beyond the scope of this paper.

We are interested in the description of quantum decoherence which includes finite-dimensional systems. In this context, typically, the pure discrete (point) energy spectrum and bound states are considered: the exact spectral form $H = \sum_n h_n|n\rangle\langle n|$ describes a closed system confined to a finite region of space and involving a finite number of particles.

Then for arbitrary initial pure state $|\Phi\rangle = \sum_n c_n|n\rangle$ equation (3.1) reads

$$U(t_\circ)\sum_n c_n|n\rangle = \sum_n c_n \exp\left(-\frac{\imath t_\circ h_n}{\hbar}\right)|n\rangle. \tag{3.8}$$

Now equation (3.2) takes the form

$$\sigma = \sum_n |c_n|^2|n\rangle\langle n| + \sum_{n\neq n'} c_n c_{n'}^* \exp\left(-\frac{\imath t_\circ(h_n - h_{n'})}{\hbar}\right)\exp\left(-\frac{(h_n - h_{n'})^2}{4\hbar^2\lambda}\right)|n\rangle\langle n'|. \tag{3.9}$$

In calculating equation (3.9), we used the Gaussian integral $\int_{-\infty}^{\infty}\exp(-ax^2/2 + \imath Jx)\,\mathrm{d}x = (2\pi/a)^{1/2} \times \exp(-J^2/2a)$, where $a > 0$ and $J$ are real numbers with $J$ being conjugate variable of $x$. By very definition equation (3.2), the state $\sigma$ is hermitean, positive and with unit trace.

From equation (3.9)

$$\mathrm{tr}\,\sigma^2 = \sum_{n,n'} |c_n|^2|c_{n'}|^2 \exp\left(-\frac{(h_n - h_{n'})}{2\hbar^2\lambda}\right) < 1, \tag{3.10}$$

which clearly exhibits the state $\sigma$ is mixed.

## (b) Few-particle versus many-particle systems

The terms $\exp(-(h_n - h_{n'})^2/4\hbar^2\lambda)$ appearing in equation (3.9) can in general vary from almost 0 to 1. There can be plenty of close energy values and thus plenty of terms in the sum equation (3.9) can equal or be very close to 1. For poor energy spectrum, which is characteristic

for some small (few-particle) systems, equation (3.10) can be very close to 1, i.e. there can be pure states in close vicinity of the mixed $\sigma$ state. This is readily seen for the standard state equation (3.1) and equation (3.8): the fidelity [25], $\mathcal{F} = \mathrm{tr}\sqrt{|\Psi\rangle\langle\Psi|\sigma|\Psi\rangle\langle\Psi|} = \sqrt{\langle\Psi|\sigma|\Psi\rangle} = \sqrt{\sum_{n,n'} |c_n|^2|c_{n'}|^2 \exp(-(h_n - h_{n'})/4\hbar^2\lambda)}$.

Therefore, high quantum coherence in the total $O + A$ system can be expected as a consequence of the constraint $\tau_{\min}/2 > \Delta t$ and equation (3.7). To this end appear the following two questions. First, whether mixed state equation (3.2) regarding few-particle systems, such as for example the EPR pairs, could be in conflict with phenomenology. And, the second, whether one can safely use pure states in the vicinity of the mixed state $\sigma$ for many-particle systems.

The first question appears in the context of the decoherence theory and the open system theory: how can we reproduce validity of the Schrödinger Law, i.e. quantum coherence, on the microscopic level? The often offered answer is pragmatic: the small systems are very well isolated and the environmental influence is *almost* negligible *in practice*, e.g. [13]. Hence the local-time scheme goes along with the standard decoherence theory in describing the few-particle systems [13,14]: the mixing of states can be weak and the small system can be considered to be in pure state for the most of the practical purposes.

On the other hand, bearing in mind that the energy spectrum for many-particle systems is dense, the typical macroscopic measurements bury the exact eigenvalues and provide a seemingly continuous spectrum–the very basis of the 'continuous approximation' that is widely used, for example, in condensed matter physics. Then equation (3.2) reads

$$\sigma = \int \mathrm{d}E\,\mathrm{d}E'\Psi(E)\Psi^*(E')\exp\left(-\frac{\imath t_\circ(E-E')}{\hbar}\right)\exp\left[-\frac{(E-E')^2}{4\hbar^2\lambda}\right]|E\rangle\langle E'|. \tag{3.11}$$

Equation (3.11) resembles the well-known expressions for the continuous-variable (CV) systems decoherence: while there is high coherence over all in the state $\sigma$, equation (3.11), there is substantial loss of coherence for certain energy values, for which $\exp(-(E-E')^2/4\hbar^2\lambda) \ll 1$. This situation is typical for virtually all CV open systems [13,14,16,18]. Bearing in mind that this cannot *in principle* be achieved by pure states, we obtain the answer to the above-posed second question: no, the use of the 'close' pure states in general cannot be useful.

Coarse graining of the energy spectrum can reduce coherence in the closed system. By definition, coarse graining decreases the number of energy eigenvalues as well as the number of Gaussian terms, $\exp(-(h_n - h_{n'})^2/4\hbar^2\lambda)$, with smaller terms $|h_n - h_{n'}|$ in the exponent. For certain few-particle systems with poor spectrum, typically, this procedure will not work. Therefore, while the few-particle systems can be expected to exhibit approximate quantum behaviour, the many-particle systems can exhibit quantal versus classical-like behaviour—the latter being a reminiscent of the conjecture that this is not merely a matter of the system's spatial size or mass but rather of the energy scale [28]. Some examples can be found in §5.

## (c) Local-time as a dynamical map

It cannot be overemphasized that the mixed state equation (3.2) refers to a *closed* system. Compared to equation (3.1), equation (3.2) emphasizes the following map:

$$|\Psi(t=0)\rangle\langle\Psi(t=0)| \to \sigma(t_\circ) = \int_{t_\circ-\Delta t}^{t_\circ+\Delta t} \rho(t)|\Psi(t)\rangle\langle\Psi(t)|\,\mathrm{d}t.$$

It is readily extendible to the following dynamical map:

$$\mathcal{S}(t=0) = \sum_i p_i|\Psi_i(t=0)\rangle\langle\Psi_i(t=0)| \to \mathcal{S}(t_\circ) = \sum_i p_i\sigma_i(t_\circ),$$

where every $\sigma_i(t_\circ)$ is of the form of equation (3.2). Since the $\sigma$, equation (3.2), is Hermitean, positive and with unit trace, this dynamical map is positive. This map is an instance of the 'random unitary evolution' that is known for the finite-dimensional systems to be completely positive [29].

## 4. A local-time scheme for decoherence-like processes

In this section, our analysis is a deductive application of the rules established in §§2*b* and 3 for the case of strong interaction in the $O+A$ system. We also briefly analyse quantum measurement and reproduce some well-known results.

### (a) Strong interaction in the $O+A$ system

The points (a)–(e) in §2*b* set the clear-cut scenario of the decoherence-like situations. Before interaction, the systems $O$ and $A$ are described by the Hamiltonian $H_O+H_A$. According to point (b), the systems may or may be not subjected to the same time. However, interaction in the $O+A$ system introduces the new Hamiltonian, $H_O+H_A+H_{\rm int}$, where $H_{\rm int}$ is the interaction Hamiltonian. According to point (b), now both the $O$ and $A$ systems are subject to the same local time. According to point (a), the time for the $O+A$ system is not the same as for the $O$, i.e. the $A$ system before interaction. So, the start of the interaction locally defines the initial time instant, $t$, for the *newly formed many-body* $O+A$ *system* and sets the 'clock' implemented by this system to the value $t=0$. As long as the system $O$ and the apparatus $A$ are precisely defined, the time instant $t=0$ is assumed to be uniquely defined. In the terms of the standard theory, the local $t=0$ corresponds to an instant $t_\circ$ that is assumed to be arbitrary but fixed and measured by a clock at the observer's disposal.

In quantum decoherence (and also in measurement), typically, interaction in the $O+A$ system is assumed to dominate the system's dynamics [13–18]. Physically, it means that the self-Hamiltonian can be neglected: $H=H_O+H_A+H_{\rm int}\approx H_{\rm int}$.

We consider a pure initial tensor-product state $|\phi\rangle_O|\chi\rangle_A$. The separable spectral form for the interaction Hamiltonian with the real eigenvalues $h_{\alpha\beta}$ [13–18,30]:

$$H_{\rm int}=\sum_{\alpha,\beta}h_{\alpha\beta}P^O_\alpha\otimes\Pi^A_\beta, \tag{4.1}$$

where appear the projectors, $P$ and $\Pi$, on the respective factor spaces.

Then the standard unitary dynamics gives the pure state

$$|\Psi(t)\rangle=\sum_\alpha b_\alpha|\alpha\rangle_O|\chi_\alpha(t)\rangle_A, \tag{4.2}$$

where

$$|\chi_\alpha(t)\rangle_A=\sum_\beta d_\beta\exp\left(-\frac{\imath t h_{\alpha\beta}}{\hbar}\right)|\beta\rangle_A, \tag{4.3}$$

with $b_\alpha|\alpha\rangle_O=P^O_\alpha|\phi\rangle_O$ and $d_\beta|\beta\rangle_A=\Pi^A_\beta|\chi\rangle_A$; $\sum_\alpha|b_\alpha|^2=1=\sum_\beta|d_\beta|^2$.

Substituting equation (4.2) into equation (3.2)

$$\sigma=\sum_\alpha|b_\alpha|^2|\alpha\rangle_O\langle\alpha|\otimes\rho^A_\alpha(t_\circ)+\sum_{\alpha\neq\alpha'}b_\alpha b^*_{\alpha'}|\alpha\rangle_O\langle\alpha'|\otimes\rho^A_{\alpha\alpha'}(t_\circ), \tag{4.4}$$

where $t_\circ$ is the time instant for which decoherence is (at least approximately) complete.

In equation (4.4),

$$\rho^A_\alpha=\sum_{\beta,\beta'}d_\beta d^*_{\beta'}\exp\left(-\frac{\imath t_\circ(h_{\alpha\beta}-h_{\alpha\beta'})}{\hbar}\right)\exp\left(-\frac{(h_{\alpha\beta}-h_{\alpha\beta'})^2}{4\hbar^2\lambda}\right)|\beta\rangle_A\langle\beta'| \tag{4.5}$$

and

$$\rho^A_{\alpha\alpha'}=\sum_{\beta,\beta'}d_\beta d^*_{\beta'}\exp\left(-\frac{\imath t_\circ(h_{\alpha\beta}-h_{\alpha'\beta'})}{\hbar}\right)\exp\left(-\frac{(h_{\alpha\beta}-h_{\alpha'\beta'})^2}{4\hbar^2\lambda}\right)|\beta\rangle_A\langle\beta'|. \tag{4.6}$$

It is easy to see, that $\rho^A_\alpha$s are hermitean and positive with unit trace.

**Lemma 4.1.** (i) *The density matrices* $\rho_\alpha^A$ *are mutually approximately orthogonal for most of the large values of* $t_\circ$, *symbolically* $\lim_{t_\circ\to\infty}\rho_\alpha^A\rho_{\alpha'}^A\approx 0, \forall\alpha\neq\alpha'$; (ii) *the trace of* $\rho_{\alpha\alpha'}^A$*s equals approximately zero for most of the large values of* $t_\circ$, *symbolically* $\lim_{t_\circ\to\infty}\mathrm{tr}_A\,\rho_{\alpha\alpha'}^A\approx 0, \forall\alpha\neq\alpha'$.

*Proof.* (i) From equation (4.5), the matrix elements

$$(\rho_\alpha^A\rho_{\alpha'}^A)_{\beta\beta''}=d_\beta d_{\beta''}^*\exp\left(-\frac{\imath t_\circ(h_{\alpha\beta}-h_{\alpha'\beta''})}{\hbar}\right)\sum_{\beta'}|d_{\beta'}|^2\exp\left(-\frac{\imath t_\circ(h_{\alpha'\beta'}-h_{\alpha\beta'})}{\hbar}\right)$$
$$\times\exp\left\{-\frac{[(h_{\alpha\beta}-h_{\alpha\beta'})^2+(h_{\alpha'\beta'}-h_{\alpha'\beta''})^2]}{4\hbar^2\lambda}\right\}$$
$$\equiv d_\beta d_{\beta''}^*\exp\left(-\frac{\imath t_\circ(h_{\alpha\beta}-h_{\alpha'\beta''})}{\hbar}\right)\zeta\,\chi. \tag{4.7}$$

In the last row of equation (4.7), we simplify notation and introduce: $0<\epsilon_{\beta'}\equiv\exp\{-[(h_{\alpha\beta}-h_{\alpha\beta'})^2+(h_{\alpha'\beta'}-h_{\alpha'\beta''})^2]/4\hbar^2\lambda\}\leq 1$, $\zeta\equiv\sum_{\beta'}|d_{\beta'}|^2\epsilon_{\beta'}$, $p_{\beta'}\equiv|d_{\beta'}|^2\epsilon_{\beta'}/\zeta$ and $\omega_{\beta'}\equiv(h_{\alpha\beta'}-h_{\alpha'\beta'})/\hbar$. Since $\sum_{\beta'}p_{\beta'}=1$, $\chi\equiv\sum_{\beta'}p_{\beta'}\exp(-\imath t_\circ\omega_{\beta'})$ is the well-known 'correlation amplitude' [15]. For sufficiently long-time interval $[t,t+T]$, such that $t_\circ\in[t,t+T]$, for $\alpha\neq\alpha'$, the correlation amplitude satisfies [15]: (a) the time average on the interval $\lim_{T\to\infty}\langle\chi\rangle_T=0$, and (b) the standard deviation on the interval $\lim_{T\to\infty}\langle|\chi|^2\rangle_T=0$ for typical models of the many-particle $A$ system. Bearing in mind that $\zeta\leq 1$, the point (i) is proved.

(ii) From equation (4.6),

$$\mathrm{tr}_A\,\rho_{\alpha\alpha'}^A=\sum_\beta|d_\beta|^2\exp\left(-\frac{\imath t_\circ(h_{\alpha\beta}-h_{\alpha'\beta})}{\hbar}\right)\exp\left(-\frac{(h_{\alpha\beta}-h_{\alpha'\beta})^2}{4\hbar^2\lambda}\right), \tag{4.8}$$

where the right-hand side is of the $\chi$-function form considered in (i) above—that completes the proof of the point (ii). ∎

In lemma 4.1, we resort to the results on the almost periodic functions presented in [15]. Temporal behaviour of the almost periodic functions is rather subtle [15,31,32] and, in general, requires separate careful analysis. Here we have in mind the cases essentially described in [15] that do not apply to the few-particle systems while $t_\circ$ appears to be of the order of 'decoherence time'.

Lemma 4.1 (i) implies $\rho_\alpha^A=\sum_m p_{\alpha m}|m\rangle_A\langle m|$, $\sum_m p_{\alpha m}=1, \forall\alpha$, and therefore the first term in equation (4.4) in the formal limit $t_\circ\to\infty$:

$$\sum_{\alpha,m}|b_\alpha|^2p_{\alpha m}|\alpha\rangle_O\langle\alpha|\otimes|m\rangle_A\langle m|, \tag{4.9}$$

which is the so-called 'classical–classical' state with zero two-way discord [33], $D^{\leftrightarrow}(O|A)=0$, i.e. without quantum correlations. Note that the states $|\alpha\rangle_O$ diagonalize $H_{\mathrm{int}}$, while, in general, this is not the case with the $|m\rangle_A$ states in equation (4.9).

With the use of notation of lemma 4.1, equation (4.4) gives rise to

$$\lim_{t_\circ\to\infty}\rho^O=\lim_{t_\circ\to\infty}\mathrm{tr}_A\,\sigma=\sum_\alpha|b_\alpha|^2|\alpha\rangle_O\langle\alpha|,\quad \rho^A=\mathrm{tr}_O\,\sigma=\sum_\alpha|b_\alpha|^2\rho_\alpha^A, \tag{4.10}$$

which are the states at the observer's disposal; only for the few-particle systems, the observer may have access to the total system's state equation (4.4).

## (b) Unique pointer basis

Orthogonality of the $\rho^A_\alpha$s, lemma 4.1 (i), implies that they have orthogonal support. Then from equation (4.10), the mutual information $I(O:A)$ [25] can easily be calculated (in the formal limit $t_\circ \to \infty$):

$$I(O:A) = S(\rho^A) - \sum_\alpha |b_\alpha|^2 S(\rho^A_\alpha) = H(O), \tag{4.11}$$

where $S(\rho) = -\mathrm{tr}\,\rho \ln \rho$ is the von Neumann entropy and $H(O) = -\sum_\alpha b_\alpha \ln b_\alpha$ is the Shannon entropy of the $O$'s state; the last equality in equation (4.11) is a direct consequence of Theorem 11.10 in [25]. In the context of decoherence theory, equation (4.11) exhibits that the environment carries classically distinguishable records about the open system's states $|\alpha\rangle_O$. In this sense, we can see that the quantum environment is performing a measurement on the open system [34].

For the pure state case, e.g. equation (3.1), of the total system arises a problem as the Schmidt form of the state need not be unique. This happens only if $\rho_O$, equation (4.10), has degenerate spectrum, $|b_\alpha|^2$, as a consequence of the choice of the initial state of the $O$ system. Then equation (4.11) might simultaneously apply to mutually non-commuting observables—this is known as the 'preferred-basis problem' [13]. In the remainder of this section, we show that this is not the case for the mixed state $\sigma$, equation (4.4). Thus we learn an important technical lesson: even a tiny mixedness in a bipartite many-particle system can remove the ambiguity known for the Schmidt form of pure states of the total system.

From equation (4.4), it follows for an alternative basis $|\nu\rangle_O$: $\sigma = \sum_{\nu,\nu'} |\nu\rangle_O\langle\nu'| \otimes R^A_{\nu\nu'}$; $R^A_{\nu\nu'} = \sum_{\alpha,\alpha'} b_\alpha b^*_{\alpha'} c_{\alpha\nu} c^*_{\alpha'\nu'} \rho^A_{\alpha\alpha'}$ and $c_{\alpha\nu} = {}_O\langle\nu|\alpha\rangle_O$. Owing to lemma 4.1 (i), $\rho^A_{\alpha\alpha'}\rho^A_{\alpha''\alpha'''} = \delta_{\alpha'\alpha''}\rho^A_{\alpha\alpha'}\rho^A_{\alpha'\alpha'''}$ so it follows:

$$\mathrm{tr}\, R^A_{\nu\nu'} = \sum_\alpha |b_\alpha|^2 c_{\alpha\nu} c^*_{\alpha\nu'} = {}_O\langle\nu| \left( \sum_\alpha |b_\alpha|^2 |\alpha\rangle_O\langle\alpha| \right) |\nu'\rangle_O \tag{4.12}$$

and

$$\begin{aligned}(R^A_\nu R^A_{\nu'})_{\beta\beta'} &= \sum_{\alpha,\alpha'} b_\alpha b^*_{\alpha'} c_{\alpha\nu} c^*_{\alpha'\nu'} \sum_{\alpha''} |b_{\alpha''}|^2 c^*_{\alpha''\nu} c_{\alpha''\nu'} (\rho^A_{\alpha\alpha''}\rho^A_{\alpha''\alpha'})_{\beta\beta'} \\ &= \sum_{\alpha,\alpha'} b_\alpha b^*_{\alpha'} c_{\alpha\nu} c^*_{\alpha'\nu'}\ {}_O\langle\nu'| \left( \sum_{\alpha''} |b_{\alpha''}|^2 (\rho^A_{\alpha\alpha''}\rho^A_{\alpha''\alpha'})_{\beta\beta'} |\alpha''\rangle_O\langle\alpha''| \right) |\nu\rangle_O. \end{aligned} \tag{4.13}$$

For degenerate spectrum of $\rho_O$, equation (4.10), that is for at least two equal $|b_\alpha|^2$s, we can choose an alternative basis for which $\mathrm{tr}\, R^A_{\nu\nu'} = 0, \forall \nu \neq \nu'$, when the point (ii) of lemma 4.1 applies to the new basis $|\nu\rangle_O$. E.g. for $|b_1|^2 = |b_2|^2$, equation (4.12) is fulfilled for the orthonormalized basis $\{|\nu_1\rangle_O, |\nu_2\rangle_O, |\alpha\rangle_O, \alpha = 3, 4, 5, \ldots\}$ for $|\nu_i\rangle_O = \sum_{\alpha=1}^2 c_{i\alpha}|\alpha\rangle_O$, $i = 1, 2$. However, as long as $\langle\nu| * |\nu'\rangle \approx 0$ in equation (4.12), the matrix element in equation (4.13) $\langle\nu'| * |\nu\rangle \not\approx 0$, and *vice versa*.

On the other hand, lemma 4.1 (i) requires that the condition $(R^A_\nu R^A_{\nu'})_{\beta\beta'} \approx 0$ is satisfied for all combinations of the indices $\nu \neq \nu', \beta, \beta'$ as well as for most of the large values of $t_\circ$. As lemma 4.1 is concerned with the many-particle systems, there are thus a huge number of equations that should be satisfied simultaneously. For example, for fixed $\nu, \nu'$, and for $n$ qubits in the $A$ system, there are $2^{n-1}(2^n + 1)$ equations that should be simultaneously satisfied for most of the large values of $t_0$. These arguments yield: there is not any alternative basis $|\nu\rangle_O$ with $c_{\alpha\nu} \neq \delta_{\alpha\nu}$ for which both points, (i) and (ii) of lemma 4.1, could be valid for every combination of the indices $\nu \neq \nu', \beta$ and $\beta'$ as well as for most of the large values of $t_\circ$ of local time of a many-particle $O + A$ system.

Now lemma 4.1 and equations (4.4)–(4.13) *uniquely* determine the pointer basis $\{|\alpha\rangle_O\}$ as well as the 'superselection sectors' $P^O_\alpha$ and the 'pointer observable' $A^O = \sum_\alpha a_\alpha P^O_\alpha$. In contrast to the case of the pure state of the total system, degeneracy in the $O$ system's state equation (4.10) does *not* give rise to ambiguity in regard of what is 'measured' in the local-time scheme. This conclusion holds independently of the initial state of the $O$ system as well as of the interaction-energy spectrum and the number of particles in the $A$ system, $N$ (except that $N \gg 1$). This brings the

*main result* of this paper: a bipartition of a *closed*, finite-dimensional or infinite-dimensional, *many-particle* system can exhibit behaviour that is characteristic for *open* systems. Neither 'objective wave packet collapse' nor the environmental influence is required.

### (c) Analysis of quantum measurement

Many-particle (macroscopic) systems are always in inevitable interaction with their environments. In this section, we extend the analysis of the previous sections to the case of measurement, when the $A$ system is the 'apparatus' and we introduce the apparatus' environment ($E$), which does not interact with the $O$ system. This scenario clearly distinguishes the two standard 'phases' in quantum measurement: in the standard measurement theory, interaction of the $O$ and $A$ systems gives rise to 'premeasurement' [13–15], i.e. to formation of quantum entanglement in the $O+A$ system. The second phase of measurement is decoherence of the apparatus that is induced by the environment [13,14].

In the local-time scheme, the two phases of measurement are clearly distinguished. According to the point (a) of §2*b*, they refer to different many-particle systems, $O+A$ and $O+A+E$ and their related local times, respectively. Regarding premeasurement, in an instant $t_\circ$ of local time for the $O+A$ system, the total system's state is (approximately) given in a Schmidt canonical form, $\sum_i b_\alpha|\alpha\rangle_O|\alpha\rangle_A$. Now, in the local-time scheme, the second phase considers another, newly formed many-particle system, $O+A+E$, which dynamically evolves in accordance with *its own* local time and re-sets its own time to the instant $t=0$.

The correlation between the $O$ and $A$ systems is preserved by the environment [13–17] and carries the information that constitutes the measurement performed on $O$ by the $A$ system. Of course, this requires robustness of the apparatus' states $|\alpha\rangle_A$ as the very basic requirement of successful measurement [13–17]. For the tripartite system $O+A+E$, the Schrödinger dynamics gives

$$U(t)\sum_{\alpha,j} b_\alpha d_j|\alpha\rangle_O|\alpha\rangle_A|j\rangle_E = \sum_{\alpha,j} b_\alpha d_j \exp\left(-\frac{\imath t h_{\alpha j}}{\hbar}\right)|\alpha\rangle_O|\alpha\rangle_A|j\rangle_E, \tag{4.14}$$

while assuming strong interaction between the $A$ and $E$ systems; without loss of generality, we can ignore degeneracy in the interaction between $A$ and $E$. Then the state equation (4.4) takes the following form:

$$\sigma = \sum_\alpha |b_\alpha|^2|\alpha\rangle_O\langle\alpha|\otimes|\alpha\rangle_A\langle\alpha|\otimes\rho^E_\alpha + \sum_{\alpha\neq\alpha'} b_\alpha b^*_{\alpha'}|\alpha\rangle_O\langle\alpha'|\otimes|\alpha\rangle_A\langle\alpha'|\otimes\rho^E_{\alpha\alpha'}. \tag{4.15}$$

It can be easily shown that the operators $\rho^E_\alpha$ and $\rho^E_{\alpha\alpha'}$ are exactly of the form of equations (4.5) and (4.6), respectively, and lemma 4.1 applies. Consequently, the conclusions are analogous: the environment monitors the composite system $O+A$, while not affecting the correlations formed in premeasurement in the $O+A$ system.

Bearing lemma 4.1 in mind, now equation (4.15) implies

$$\rho^{O+A} = \mathrm{tr}_E\,\sigma = \sum_\alpha |b_\alpha|^2|\alpha\rangle_O\langle\alpha|\otimes|\alpha\rangle_A\langle\alpha|, \quad \rho^O = \mathrm{tr}_A\,\rho^{O+A} = \sum_\alpha |b_\alpha|^2|\alpha\rangle_O\langle\alpha|. \tag{4.16}$$

So the local-time scheme straightforwardly reproduces the basic result equation (4.16) of the environmental influence on the 'object+apparatus' system [13–15,17].

## 5. Some models of quantum decoherence and measurement

In this section, we analyse some relevant analytically solvable models described by 'pure decoherence' [18,34,35] Hamiltonian equation (4.1). We observe technical simplicity and clarity of the local-time scheme. In accordance with §3, we choose the highest possible value for $\Delta t$ and the smallest possible value for $\lambda$. Coarse graining of the pointer observable ($A^O$) values is considered without a change of values of the $\Delta t$ and $\lambda$ parameters; for motivation see §6.

## (a) A pair of spin-$\frac{1}{2}$ particles

Consider a pair of spin-$\frac{1}{2}$ particles (qubits) and interaction $H_{\rm int} = CS_{1z}S_{2z}$. This is a separable interaction [30], cf. equation (4.1), with the eigenstates $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ and eigenvalues (in the units of the Planck constant, $\hbar = 1$) $h_{++} = C/4 = h_{--}, h_{+-} = -C/4 = h_{-+}$, while the ground energy $E_{\rm g} = -C/4$.

Let us consider the mixed state equation (4.4) for this case:

$$\rho_+^{(2)} = \begin{pmatrix} |d_+|^2 & d_+d_-^* \, {\rm e}^{-\imath t_\circ C/2 - C^2/16\lambda} \\ d_+^*d_- \, {\rm e}^{\imath t_\circ C/2 - C^2/16\lambda} & |d_-|^2 \end{pmatrix},$$

$$\rho_-^{(2)} = \begin{pmatrix} |d_+|^2 & d_+d_-^* \, {\rm e}^{\imath t_\circ C/2 - C^2/16\lambda} \\ d_+^*d_- \, {\rm e}^{-\imath t_\circ C/2 - C^2/16\lambda} & |d_-|^2 \end{pmatrix}$$

while

$$\rho_{+-}^{(2)} = \begin{pmatrix} |d_+|^2 \, {\rm e}^{-\imath t_\circ C/2} \, {\rm e}^{-C^2/16\lambda} & d_+d_-^* \\ d_+^*d_- & |d_-|^2 \, {\rm e}^{\imath t_\circ C/2} \, {\rm e}^{-C^2/16\lambda} \end{pmatrix}.$$

Now it easily follows:

$$\left.\begin{aligned} (\rho_+^{(2)}\rho_-^{(2)})_{++} &= |d_+|^2 \, {\rm e}^{-\imath t_\circ C/2}[|d_+|^2 \, {\rm e}^{\imath t_\circ C/2} + |d_-|^2 \, {\rm e}^{-\imath t_\circ C/2} \, {\rm e}^{-C^2/8\lambda}] \\ \text{and} \qquad {\rm tr}_2 \, \rho_{+-}^{(2)} &= {\rm e}^{-C^2/16\lambda}[|d_+|^2 \, {\rm e}^{-\imath t_\circ C/2} + |d_-|^2 \, {\rm e}^{\imath t_\circ C/2}] = {\rm e}^{-C^2/16\lambda} \cos \frac{Ct_\circ}{2}, \end{aligned}\right\} \tag{5.1}$$

i.e. lemma 4.1 is not fulfilled for this case.

The choice $d_\pm = 2^{-1/2}$ satisfies the condition $\langle H_{\rm int}\rangle_{t=0} = 0$ and $\Delta H_{\rm int} = C/4 = -E_{\rm g}$. Then there is the unique time bound, $\tau_{\rm min}/2 = \pi/C$, and for $C = 1$ we can choose $\Delta t = 3$ and $\lambda = 1$, with the very well satisfied equality equation (3.7). Thus, $\exp[-C^2/16\lambda] = \exp(-\frac{1}{16}) \approx 0.939$ and the small off-diagonal term, $\exp[-(h_{++} - h_{--})^2/4\lambda] = \exp(-\frac{1}{16}) \approx 0.939$. Therefore,

$$\sigma \approx |\Psi\rangle\langle\Psi|, \tag{5.2}$$

with the error less than 0.062, (the error decreases with the increase of $\lambda$), and with $|\Psi\rangle = [c_+ \exp(-\imath t_\circ/4)|++\rangle + c_- \exp(-\imath t_\circ/4)|--\rangle + c_+ \exp(\imath t_\circ/4)|+-\rangle + c_- \exp(\imath t_\circ/4)|-+\rangle]/\sqrt{2}$–that is equation (3.1) for this case; $|c_+|^2 + |c_-|^2 = 1$. So quantum coherence is very high in the local-time scheme for 'microscopic' systems—and is analogous to the approximate isolation (and coherence) of the microscopic systems in the context of open quantum systems as anticipated in §3*b*.

## (b) Four spin-$\frac{1}{2}$ particles: a case study

A spin-$\frac{1}{2}$ system is in interaction with mutually non-interacting spin-$\frac{1}{2}$ systems (qubits): $H_{\rm int} = S_{1z}(S_{2z} + S_{3z} + S_{4z})$. The system $2+3+4$ is the 1 system's environment. The interaction is separable [30], equation (4.1), and the eigenstates and eigenvalues (in the units $\hbar = 1$) can be denoted $|\pm i\rangle$ and $h_{\pm i}$, respectively. The indices $\pm$ refer to the 1 system, while the index $i = 1, 2, \ldots, 8$ denotes the set of mutually orthogonal tensor-product states, $|m_2n_3p_4\rangle$, $m, n, p = \pm$, which constitute an orthonormalized basis for the $2+3+4$ system. The eigenvalues $h_{\pm\beta}$ and degeneracies $g_\beta$ are as follows: $h_{\pm1} = \pm\frac{3}{4} = h_{\mp4}, h_{\pm2} = \pm\frac{1}{4} = h_{\mp3}$ and $g_1 = 1 = g_4$, while $g_2 = 3 = g_3$.

Again we choose $\langle H_{\rm int}\rangle = 0$ that is satisfied e.g. with equal distribution of eigenstates $|m_2n_3p_4\rangle$ for the initial environment's state, $8^{-1/2}, \forall i = 1 - 8$. Then, bearing in mind the degeneracies, with the use of notation of equations (4.2) and (4.3), $|d_1|^2 = \frac{1}{8} = |d_4|^2$ and $|d_2|^2 = \frac{3}{8} = |d_3|^2$. For arbitrary

initial state of the 1 system, $\tau_{\min}/2 = \pi/4\Delta H_{\rm int} = \pi/4(\langle H_{\rm int}\rangle - E_{\rm g}) = \pi/3$. Hence we can choose $\Delta t = 1$ and $\lambda = 2$ in order to provide a very good approximation for equation (3.7). This gives, for example,

$$\begin{aligned}(\rho_+^{(234)}\rho_-^{(234)})_{11} = |d_1|^2 \exp\left(-\frac{3\imath t_\circ}{2}\right) \quad &\left[|d_1|^2 \exp\left(\frac{3\imath t_\circ}{2}\right)\right.\\ &+ |d_2|^2 \exp\left(\frac{\imath t_\circ}{2} - \frac{1}{16}\right) + |d_3|^2 \exp\left(-\frac{\imath t_\circ}{2} - \frac{1}{4}\right)\\ &\left.+|d_4|^2 \exp\left(-\frac{3\imath t_\circ}{2} - \frac{9}{16}\right)\right]\end{aligned}$$

and

$$\begin{aligned}{\rm tr}_{234}\,\rho_{+-}^{(234)} &= |d_1|^2 \exp\left(-\frac{3\imath t_\circ}{2} - \frac{9}{32}\right) + |d_2|^2 \exp\left(-\frac{\imath t_\circ}{2} - \frac{1}{32}\right)\\ &+ |d_3|^2 \exp\left(\frac{\imath t_\circ}{2} - \frac{1}{32}\right) + |d_4|^2 \exp\left(\frac{3\imath t_\circ}{2} - \frac{9}{32}\right)\\ &= \frac{1}{4}\cos\left(\frac{3t_\circ}{2}\right)\exp\left(-\frac{9}{32}\right) + \frac{3}{4}\cos\left(\frac{t_\circ}{2}\right)\exp\left(-\frac{1}{32}\right).\end{aligned} \tag{5.3}$$

Needless to say, owing to small number of terms in the sums in equation (5.3), lemma 4.1 is not satisfied. Nevertheless, comparison of equations (5.1) and (5.3) clearly exhibits that increase in the size of the environment gives a better satisfied lemma 4.1. Both traces in equations (5.1) and (5.3) are periodic functions (with the periods approx. $2\pi$ and $4\pi$) and the increase in the number of terms in the sum leads to the almost periodic functions, lemma 4.1. Physically, equation (5.3) reveals environment's periodic memory, with small period, about the object's state—that is not a good measurement or decoherence of the 1 system.

The real exponential terms $\exp[-(h_{\alpha\beta} - h_{\alpha'\beta'})^2/4\lambda]$ for the above set of energy eigenvalues and for small $\lambda = 2$ have the smallest value $\exp(-\frac{9}{32}) = 0.755$ and the largest value $\exp(-\frac{1}{32}) = 0.969$. Compared to the previous model, there is less quantum coherence in the total system. So

$$\sigma \not\approx |\Psi'\rangle\langle\Psi'| \tag{5.4}$$

where $|\Psi'\rangle = \sum_{\alpha=\pm}\sum_{\beta=1}^{4} b_\alpha d_\beta \exp(-\imath t_\circ h_{\alpha\beta})|\alpha\beta\rangle$ is the standard state equation (3.1). The fidelity satisfies $0.869 = \sqrt{0.755} < \mathcal{F} = \sqrt{\langle\Psi'|\sigma|\Psi'\rangle} < \sqrt{0.969} = 0.984$. After a straightforward but lengthy computation, the exact fidelity amounts to 0.894. Decrease of quantum coherence relative to the model of §5*a* supports and illustrates the general notions provided in §3*b*: the larger the environment the less quantum coherence in the total system.

## (c) Decoherence and measurement of a single qubit: the qubit environment

We consider the well-studied, analytically solvable model of 'decoherence of a single qubit' [15]. This also models the Stern–Gerlach experiment, if the environment is modelled as the set of molecules in the plate that can be either decayed or non-decayed by the atoms caught by the plate.

The interaction Hamiltonian for the pair $O + A$, where the $O$ system is the single qubit is separable

$$H_{\rm int} = (a_+|+\rangle_O\langle+| + a_-|-\rangle_O\langle-|) \otimes \sum_{k=1}^{N} g_k(|+\rangle_{Ak}\langle+| - |-\rangle_{Ak}\langle-|)\Pi_{k\neq k'} I_{k'}, \tag{5.5}$$

with $a_+ = -a_- = 1$ and with $N \gg 1$.

Initial state $|\Psi\rangle = (a|+\rangle_O + b|-\rangle_O)\Pi_{k=1}^{N}(a_k|+\rangle_{Ak} + b_k|-\rangle_{Ak})$ gives for an instant of time [15]

$$|\Psi(t)\rangle = a|+\rangle_O|\chi_+(t)\rangle_A + b|-\rangle_O|\chi_-(t)\rangle_A, \tag{5.6}$$

where (for $\hbar = 1$)

$$|\chi_\pm(t)\rangle_A = \Pi_{k=1}^{N}(a_k\,{\rm e}^{-\imath a_\pm g_k t}|+\rangle_{Ak} + b_k\,{\rm e}^{\imath a_\pm g_k t}|-\rangle_{Ak}). \tag{5.7}$$

Equation (5.7) can be written as follows:

$$|\chi_{\pm}(t)\rangle_A = \sum_{j_1\cdots j_N=\pm} c_{j_1}\ldots c_{j_N}\Pi_{k=1}^{N}\,\mathrm{e}^{-\imath t a_{\pm g_k\alpha_k}}|j_k\rangle_A, \tag{5.8}$$

where $\alpha_k = \nu_k - \mu_k$ and $\nu, \mu = 0, 1$ with the following rule: if $j_k = +$, then $\nu_k = 1$ and $\mu_k = 0$, while for $j_k = -$, $\nu_k = 0$ and $\mu_k = 1$, with independent constants for different indices $k$.

Setting $\Pi_{k=1}^{N}\,\mathrm{e}^{-\imath t a_{\pm g_k\alpha_k}}|j_k\rangle_A = \mathrm{e}^{-\imath t a_{\pm}\sum_k g_k\alpha_k}\Pi_{k=1}^{N}|j_k\rangle_A$, the mixed state equation (3.2)

$$\sigma = |a|^2|+\rangle_O\langle+|\otimes\rho_+^A + |b|^2|-\rangle_O\langle-|\otimes\rho_-^A + ab^*|+\rangle_O\langle-|\otimes\rho_{+-}^A + a^*b|-\rangle_O\langle+|\otimes\rho_{-+}^A. \tag{5.9}$$

In equation (5.9),

$$\left.\begin{aligned}\rho_{\pm}^A(t_\circ) &= \sum_{j_1\ldots j'_N} c_{j_1}\ldots c^*_{j'_N}\,\mathrm{e}^{-\imath t_\circ a_{\pm}\sum_k g_k(\alpha_k-\alpha'_k)}\,\mathrm{e}^{-(a_{\pm}\sum_k g_k(\alpha_k-\alpha'_k))^2/4\lambda}\Pi_k|j_k\rangle_A\langle j'_k|\\ \text{and}\qquad \rho_{+-}^A(t_\circ) &= \sum_{j_1\ldots j'_N} c_{j_1}\ldots c^*_{j'_N}\,\mathrm{e}^{-\imath t_\circ\sum_k g_k(a_+\alpha_k - a_-\alpha'_k)}\\ &\quad\times\mathrm{e}^{-(\sum_k g_k(a_+\alpha_k-a_-\alpha'_k))^2/4\lambda}\Pi_k|j_k\rangle_A\langle j'_k|,\end{aligned}\right\} \tag{5.10}$$

where $\alpha_k$ refers to $j_k$ and $\alpha'_k$ to $j'_k$.

From equation (5.10),

$$\left.\begin{aligned}\rho_+^A(t_\circ)\rho_-^A(t_\circ) &= \sum_{j_1\ldots j_N j'_1\ldots j'_N} c_{j_1}\ldots c_{j_N}c^*_{j'_1}\ldots c^*_{j'_N}\,\mathrm{e}^{-\imath t_\circ\sum_k g_k(a_+\alpha_k-a_-\alpha'_k)}\Pi_k|j_k\rangle_A\langle j'_k|\\ &\quad\times\left(\sum_{j''_1\ldots j''_N}|c_{j''_1}|^2\ldots|c_{j''_N}|^2\,\mathrm{e}^{-\imath t_\circ(a_--a_+)\sum_k g_k\alpha''_k}\,\mathrm{e}^{-((a_+\sum_k g_k(\alpha_k-\alpha''_k))^2+(a_-\sum g_k(\alpha''_k-\alpha'_k))^2)/4\lambda}\right)\\ \text{and}\quad \mathrm{tr}_A\,\rho_{+-}^A &= \sum_{j_1\ldots j_N}|c_{j_1}|^2\ldots|c_{j_N}|^2\,\mathrm{e}^{-\imath t_\circ(a_+-a_-)\sum_k g_k\alpha_k}\,\mathrm{e}^{-((a_+-a_-)\sum_k g_k\alpha_k)^2/4\lambda}.\end{aligned}\right\} \tag{5.11}$$

The term in the parenthesis and the trace $\mathrm{tr}_A\,\rho_{+-}^A$ are of the form of the $\chi$-function defined in the proof of lemma 4.1(i)—see below equation (4.7). Therefore, lemma 4.1 applies for the case studied: symbolically, $\lim_{t_\circ\to\infty}\rho_+^A\rho_-^A = 0$ and $\lim_{t_\circ\to\infty}\mathrm{tr}_A\,\rho_{+-}^A = 0$ for $N \gg 1$.

In order to compare with Zurek's [15], we deal with the random values for $g_k \in (0, 1)$ and $|a_k| \approx |b_k|$, $\forall k$; the latter gives rise to $\langle H_{\mathrm{int}}\rangle \approx 0$. It easily follows $\Delta H_{\mathrm{int}} = \sqrt{\sum_k g_k^2}$. For randomly chosen $g_k = k/N$ and with equal probability $1/N$ for every $g_k$, $\Delta H_{\mathrm{int}} = \sqrt{N^{-3}\sum_{k=1}^{N}k^2} = 3^{-1/2} > -E_{\mathrm{g}} = N^{-2}\sum_{k=1}^{N}k = \frac{1}{2}$, for $N \gg 1$. Therefore, $\tau_{\min}/2 = \pi/2 > 1.57$. So we choose $\Delta t = 1.56$ and the smallest value $\lambda = 1$ that provide very good approximation for equation (3.7).

Exponential factors appearing in the $\rho_{+-}^A$ in equation (5.10)

$$\mathrm{e}^{-(\sum_k g_k(a_+\alpha_k\pm a_-\alpha'_k))^2/4\lambda} = \mathrm{e}^{-(\sum_k g_k(\alpha_k\pm\alpha'_k))^2/4}. \tag{5.12}$$

Since $\max\{\alpha_k \pm \alpha'_k\} = 2$, the smallest exponential factor is $\exp(-\frac{1}{4}) = 0.779$. All other terms are with the nominator in the exponent of the form $(\pm\sum_{k=1}^{M}g_k \mp \sum_{k=M+1}^{N}g_k)^2 = (N^{-2}[\pm\sum_{k=1}^{M}k \mp \sum_{k=M+1}^{N}k])^2$. Numerical estimates reveal that such terms are not less than 0.94. In order to compare with the model of §5*b*, we set $N = 3$ (and placing $\lambda = 2$ and the eigenvalues $\pm1$ instead of $\pm\frac{1}{2}$) and obtain similar results. So we find that there is high quantum coherence for both models of §5*b* and of this section.

Without further ado, let us consider the object's spectrum $a_i \in \{-2, -1, 1, 2\}$–which can describe the four spin-$\frac{1}{2}$ particles total-spin values; the $a_i$ values substitute the above $a_{\pm}$ values. For a pair of values, e.g. 2 and $-1$, and for random $g_k$s (see above) while $N \gg 1$, the smallest Gaussian factor $\exp[-(\sum_k g_k(2\alpha_k - \alpha'_k))^2/4] = \exp(-\frac{9}{16}) \approx 0.57$, while the largest one amounts to $\exp(-\frac{1}{16}) = 0.939$. For the pair 2, $-2$, the smallest term $\exp(-1) = 0.368$ while the largest amounts to 1. Now consider

the coarse graining of this spectrum by introducing the new set of values, $a'_j \in \{-2, 0, 2\}$. For the values 2 and 0, (with $\lambda = 1$), there is the unique value of $\exp(-\frac{1}{4}) = 0.778$, while the terms pertaining to the pair 2, −2, remain intact.

So we obtain a rough idea about decrease in coherence due to the coarse graining of spectrum of the pointer observable, and consequently of energy in the composite system: the number of large Gaussian terms decreases. Needless to say, due to the poor spectrums, this is not possible for the microscopic objects of §5*a*,*b* and equation (5.6). In turn, we also realize: finer measurements—e.g. of the spectrum $a_i$ instead of the coarse-grained values $a'_j$—can, in principle, provide observation of coherence, i.e. of quantum correlations in the total system.

## (d) Position measurement

The classic von Neumann model [36] that implements the Heisenberg idea of position measurement is described by strong interaction $H_{\rm int} = Cx_O \otimes P_A$ between the one-dimensional object $O$ and the apparatus $A$; the conjugate momentum/position observables $p_O$ and $X_A$, respectively. The model is readily generalized for measurement of any continuous observable $Q_O$ as well as to the three-dimensional models [37]. Similar results are obtained for the interaction $H_{\rm int} = x_O \otimes X_A$. For the collective position observable $X_A = \sum_j \kappa_j x_{Aj}$, the object $O$ undergoes quantum Brownian motion [19] that does not depend on the strength of interaction.

Let us consider the composite system initially spatially contained in the linear dimensions $[-L, L]$ and the initial state $|\phi\rangle_O|\chi\rangle_A$ as a tensor product of two wavepackets with the position and momentum spreads $\sigma_{x_O} \equiv \sigma_1$ and $\sigma_{P_A} \equiv \sigma_2$, while for convenience $\langle H_{\rm int}\rangle = 0$. For the analogous interval for the apparatus momentum $[-P, P]$ the ground energy $E_{\rm g} = -LP \ll 1$. If the spreads $\sigma_1 \sim 1$ and $\sigma_2 \sim 1$, then (in the units $\hbar = 1$ and for $C = 1$) $\tau_{\min}/2 = \max\{\pi/4\sigma_1\sigma_2, \pi/4LP\} = \pi/4$, while $\Delta t = 0.78$ and $\lambda = 3$ well satisfy equation (3.7).

Then the state equation (4.4) reads

$$\sigma = \int {\rm d}x\, {\rm d}x' |x\rangle_O\langle x'| \otimes \rho_A(x, x'), \tag{5.13}$$

with

$$\rho_A(x, x') = \int {\rm d}P\, {\rm d}P' \phi(x)\phi^*(x')\chi(P)\chi^*(P') \exp(-\imath t_\circ(xP - x'P')) \times \exp\left(-\frac{(xP - x'P')^2}{12}\right) |P\rangle_A\langle P'|. \tag{5.14}$$

From equation (5.14), one easily obtains validity of lemma 4.1, due to direct applicability of the Riemann–Lebesgue lemma (cf. e.g. Proposition 5.2.1 in [15]) in our case: $\lim_{t_\circ\to\infty} \int {\rm d}P|\chi(P)|^2 \exp(-\imath t_\circ(x' - x)P) \exp\{-[(xP' - xP)^2 + (x'P - x'P'')^2]/12\} = 0$ for $x \neq x'$.

The fidelity $\sqrt{\int {\rm d}x\, {\rm d}x'\, {\rm d}P\, {\rm d}P' |\phi(x)|^2|\phi(x')|^2|\chi(P)|^2|\chi(P')|^2 \exp(-(xP - x'P')^2/12)}$ reveals very high coherence for the object's state—there are plenty of close $x$ and $x'$. Nevertheless, there are still very small values for the Gaussian factors for which $|x - x'| \gg 12$—that is, well within the chosen domain of $L \gg 1$.

Coarse graining of the pointer-observable $x_O$ continuous spectrum (while keeping the parameter $\lambda$ fixed) reduces the number of the Gaussian terms, which almost equal 1. If the width of the spatial interval is $\Delta x$, then one can choose the wavepackets with the spread $\Delta x$ as the approximate (non-orthogonal) normalizable 'pointer basis' states. Formally, for a set of approximately orthogonal minimum-uncertainty (the 'coherent') states $|\psi_{ij}\rangle_O$, such that ${}_O\langle\psi_{ij}|\psi_{i'j'}\rangle_O \approx \delta_{ii'}\delta_{jj'}$, one obtains ${}_O\langle\psi_{ij}|x_O|\psi_{i'j'}\rangle_O \approx x_i\delta_{ii'}\delta_{jj'}$. Then the exact interaction is almost

diagonal for the $|\psi_{ij}\rangle_O$ states: ${}_O\langle\psi_{ij}|H_{\rm int}|\psi_{i'j'}\rangle_O \approx 0$, $\forall i \neq i', j, j'$. Furthermore, the unitary operator generated by the interaction is also almost diagonalizable for these states. The proof reduces to computing the ${}_O\langle\psi_{ij}|x_O^n|\psi_{i'j'}\rangle_O$ terms. For $\psi_{ij}(x) = (2\pi\sigma_i)^{-1/2}\exp(-(x-x_i)^2/2\sigma_i^2 + \imath x p_j)$,

$$ {}_O\langle\psi_{ij}|x_O^n|\psi_{i'j'}\rangle_O = (2\pi\sigma_i\sigma_{i'})^{-1}\exp\left(-\frac{(x_i-x_{i'})^2}{2(\sigma_i^2+\sigma_{i'}^2)}\right)\mathcal{I}_n \tag{5.15}$$

where $\mathcal{I}_n = \int_{-\infty}^{\infty} {\rm d}x x^n \exp(-(x-x_\circ)^2/2\sigma^2 - \imath x(p_j - p_{j'}))$; $\sigma^2 = \sigma_i^2\sigma_{i'}^2/(\sigma_i^2+\sigma_{i'}^2)$ and $x_\circ = (x_{i'}\sigma_i^2 + x_i\sigma_{i'}^2)/(\sigma_i^2+\sigma_{i'}^2)$. The Gaussian term in equation (5.15) proves the claim: $\|{}_O\langle\psi_{ij}|U(t_\circ)|\psi_{i'j'}\rangle_O\| \propto \exp(-(x_i-x_{i'})^2/4) \ll 1$.

Since $\sum_{i,j}|\psi_{ij}\rangle_O\langle\psi_{ij}| < I$, there are plenty of 'coherent states' in the vicinity of every $|\psi_{ij}\rangle_O$ that contribute to degeneracy of the interaction. Hence for the set of the values $x_i$ (out of the continuous interval of the position values $x \in (-\infty, \infty)$) one obtains substantial decrease of the Gaussian factors, while the coherent states $|\psi_{ij}\rangle_O$ constitute a set of approximate pointer basis states for the exact continuous pointer observable $x_O$. The more rigorous methods [36,38] give rise to redefinition of the exact pointer observable and interaction and hence of the $\Delta t$ and $\lambda$ parameters that we are not interested in—see §6.

## (e) Walls–Collet–Milburn measurement model

The open system $O$ and the apparatus $A$ are taken to be harmonic oscillators defined by the respective annihilation operators, $a$ and $b$ (the modes) and with the separable interaction [39]

$$ H_{OA} = \frac{\hbar}{2}a^\dagger a(\epsilon^* b + \epsilon b^\dagger). \tag{5.16}$$

There is also the apparatus environment $E$, which is a thermal bath of harmonic oscillators with the interaction

$$ H_{AE} = b\sum_j \kappa_j^* c_j^\dagger + b^\dagger\sum_j \kappa_j c_j \tag{5.17}$$

with the environmental annihilation operators (the modes) $c_j$. The thermal bath can be 'purified' and appears as a subsystem of a larger system, which is initially in a pure state that we are concerned with cf. equation (3.1), and will continue to be denoted by $E$.

According to §4*c*, equation (5.16) describes pre-measurement, §4*c*, that gives rise to the final state of the $O+A$ system [19]

$$ |\Psi(t)\rangle_{OA} = \sum_n c_n|n\rangle_O\left|\frac{n\epsilon t}{2}\right\rangle_A, \tag{5.18}$$

where $a^\dagger a|n\rangle_O = n|n\rangle_O$ and the apparatus states are 'coherent states' (the minimum uncertainty Gaussian states). Setting $t = t_\circ \to \infty$, the apparatus states are approximately orthogonal [14] and in the instant of time $t_\circ$, pre-measurement is complete.

The second phase of the measurement, cf. §4*c*, is described by the interaction equation (5.17). By following Agarwal [40], the interaction equation (5.17) is obtained via the so-called rotating-wave approximation [19,20] that reveals the Schrödinger-picture, original interaction to be of the separable form [39,40]:

$$ H_{AE} = X_A\left[\sum_j \kappa_j^* c_j + \sum_j \kappa_j c_j^\dagger\right], \tag{5.19}$$

where $X_A$ is the apparatus position observable. Equation (5.19) is of interest within the local-time scheme.

Equation (5.19) is actually the model considered in §5*d*: the environment $E$ measures the apparatus' position $X_A$. So we conclude that the second phase of the measurement—according to §4*c*—is an (almost ideal) 'non-demolition' measurement [15,30] that distinguishes the $X_A$

observable as the pointer observable with the approximate pointer basis $|n\epsilon t/2\rangle_A$ for the apparatus. Needless to say, the object's exact pointer observable is $a^\dagger a$ and the exact pointer basis states $|n\rangle_O$. As in equation (4.16), the related density matrices

$$\rho_{O+A} = \sum_n |c_n|^2 |n\rangle_O\langle n| \otimes \left|\frac{n\epsilon t}{2}\right\rangle_A \left\langle\frac{n\epsilon t}{2}\right|, \quad \rho_O = \sum_n |c_n|^2 |n\rangle_O\langle n|. \tag{5.20}$$

## 6. Discussion

In local-time scheme, the few-particle systems sustain high quantum coherence. However, for bipartition of a many-particle closed system we obtain effects that are characteristic for open systems, without a need for the state collapse (reduction) or environmental influence. Within the local-time scheme, 'local system' and 'local operations' are defined via the set of local time in a closed system. If certain pair interactions are of similar strength, then the composite system can be subject to the unique time, cf. equation (2.4). The recipe for determining the local time is conceptually rather simple as everything is written in the total system's Hamiltonian: the degrees of freedom that are relatively strongly coupled and (approximately) unitary evolve in time constitute a subsystem, i.e. a 'local system' that is defined by its own local time that flows differently than for some other local systems. Those findings come from the macroscopic domain in the context of the full-quantum mechanical analysis. As distinct from the Copehnagen interpretation, the local-time scheme does not assume or require 'classical apparatus'.

It is remarkable that the local-time scheme is technically simple. It straightforwardly reproduces (§5) some basic results of the standard decoherence and measurement theory. Amount of quantum coherence in the total system depends on the system's state that is reflected by the values of the $\Delta t$ and $\lambda$ parameters. On the other hand, coarse graining of the energy-observable spectrum and/or of the pointer-observable spectrum gives rise to a decrease in quantum coherence as it is found in some other contexts [36,38,41–44].

In the context of our considerations, operational approach to coarse graining [36,38,41–44] requires a change in the values of the parameters $\Delta t$ and $\lambda$ and therefore in the time bound $\tau_{\min}$, §3*a*. In the example of the microlocal analysis [38], one introduces quasi-projectors and thus redefines the position observable $x$ and consequently the interaction considered in §5*d*. The introduction of the new sets of eigenvalues and (approximate) eigenspaces inevitably gives rise to a change in the bound $\tau_{\min}$—as it can be easily shown. Not doing so, as we can see in §5, highlights the observation of §3*b*, that refining the measurement, i.e. operational accessibility of the exact, 'microscopic', eigenvalues, can in principle give rise to observation of quantum effects in the many-particle systems.

The local-time scheme is easily adapted to reproduce the basic assumptions of diverse approaches to quantum foundations involving emergent, relational and fundamental time. First, the scheme admits considerations (interpretation) that physical time is *emergent*, i.e. not physically fundamental. To this end, time is a construct from the fundamental quantum dynamics, e.g. presented by equation (2.4). In this scheme, space–time quantization may be undefinable. Second, the local-time scheme provides relational character [45] of common local time for interacting particles (subsystems), cf. the point (b) of §2*b*. Finally, if time is fundamental, the introduction of time uncertainty, §3, can be interpreted differently. To this end, removal of the integration from equation (3.2) provides the state $\rho(t)|\Psi(t)\rangle\langle\Psi(t)|$, which introduces time as a classical system, $T$, which extends the quantum system $O + A$. Then the total system $T + O + A$ appears, at least formally, as a 'hybrid system' [46] (and references therein) that might link quantum and relativistic theories in a new way [47]. We observe that the local-time scheme is richer, both conceptually and interpretationally, as well as being reducible to certain existing theories and interpretations of quantum theory. To this end, the local-time scheme points out a new, fresh foundation of quantum theory, along with some recent approaches [48] that also, but

not equivalently, perform in the context of the universally valid Schrödinger law. Mathematically elaborate microscopic models of realistic physical situations, cf. e.g. [48] (and references therein), are highly welcome in this context.

Our conclusions do not directly apply to the weak-interaction scenarios (e.g. the weak-measurement and some Markovian open systems dynamics) that require separate considerations. Mutual relations between the local times remains intact in this paper (but see [5] for a proposal). Finally, interpretation of equation (3.2) in terms of single system of an ensemble of identical systems in connection to the above described deeper physical nature of time might provide a fresh look into the long-standing problem of quantum measurement theory. To this end, research is in progress.

## 7. Conclusion

The local-time scheme of Kitada straightforwardly derives some basic results of quantum decoherence and measurement theory yet for the isolated (closed) many-particle system. At the same time, high quantum coherence is provided for the few-particle systems. Non-necessity of state collapse (reduction) and environmental influence, technical simplicity as well as interpretational ramifications regarding the deeper physical nature of time exhibit that the scheme is worth a pursuit in foundations and interpretation of quantum theory and measurement.

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## Appendix A

The typical scattering situation is described as follows. In the laboratory reference system, there is a fixed many-particle target and, e.g., a few-particle projectile directed to the target. Detected projectile is assumed to be far from the target that is described by the limit of infinite time, $t \to \infty$.

The many-body scattering is truly complex task. It regards all the possible decompositions of the scattering particles. More precisely: of interest are all the possible scattering of particles that can be composed of the initially introduced particles. And for every possible such scenario, one should discard the bound states. Description of this complex picture amounts to the problem known as the 'problem of asymptotic completeness' in quantum many-body scattering theory. The important work of Enns opened the door for a solution of this problem.

Consider an $N$-particle, isolated (closed) system $\mathcal{S}$ with the Hamiltonian $H$ and the Hilbert state space $\mathcal{H}$. Denote the individual particles position and momentum operators by $x_i$ and $p_i$, respectively: $[x_i, p_j] = \imath\hbar\delta_{ij}$, $i, j = 1, 2, 3, \ldots, N$. The total system can be divided into clusters where a cluster can consist of arbitrary number of the constituent particles. Let $\mathcal{S}_b = \{\mathcal{C}_i, i = 1, 2, \ldots, k\}$ be the $b$th structure (cluster decomposition) of the total system $\mathcal{S}$ with the number $k$ of clusters. We call 'elementary' the structure in which every particle is one cluster, $\mathcal{S}_e = \{\{1\}, \{2\}, \ldots, \{N\}\}$—the corresponding number of clusters is, of course, $k = N$.

Consider a structure $\mathcal{S}_b$ with $k$ clusters, $\mathcal{S}_b = \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$, with $N_i$ particles in the $i$th cluster; $\sum_i N_i = N$. For every cluster introduce the centre of mass and the Jacobi relative positions: $X^b_{CMi}$ and $x_l^{\mathcal{C}_{bi}}$, respectively, where $l = 1, 2, 3, \ldots, N_i - 1$. Then the intracluster variable is defined for the structure $\mathcal{S}_b$, $x^b = \{x^{\mathcal{C}_{b1}}, x^{\mathcal{C}_{b2}}, \ldots, x^{\mathcal{C}_{bk}}\}$. For the set of the clusters' centres of mass, the Jacobi variables transformation introduce the total system's centre of mass and the intercluster Jacobi relative variable, $\{x_{b1}, x_{b2}, \ldots, x_{bk}\}$. The related conjugate Jacobi (momentum) variables, $p^l$ and $p_l$, and the commutators $[x_i^l, p_j^{l'}] = \imath\hbar\delta_{ij}\delta_{ll'}$, and analogously for the intracluster variables. In the position representation: $\nabla_l \equiv -\imath\hbar\partial/\partial x_i^l$ is canonically conjugate to the position multiplicative

variable $x_i^l$. In the position representation: $x^l \in \mathcal{R}^{3(N-k)}$ and $x_l \in \mathcal{R}^{3(k-1)}$. Then the total Hilbert state space, in the standard functional analysis notation, $\mathcal{H} = L^2(\mathcal{R}^{3N})$, can be factorized

$$\mathcal{H} = \mathcal{H}_{\mathrm{CM}} \otimes \mathcal{H}_b \otimes \mathcal{H}^b, \tag{A 1}$$

which is equation (2.2) in the main text. By omitting the total CM system from consideration, equation (A 1) reduces to

$$\mathcal{H} = \mathcal{H}_b \otimes \mathcal{H}^b, \tag{A 2}$$

that is equation (2.3) in the main text. For $b \neq b'$, $\mathcal{H}_b \neq \mathcal{H}_{b'}$ and $\mathcal{H}^b \neq \mathcal{H}^{b'}$, while $\mathcal{H}_b \otimes \mathcal{H}^b = \mathcal{H}_{b'} \otimes \mathcal{H}^{b'}$.

The Hamiltonian for the total system $\mathcal{S}$ and for the 'elementary' structure (with $x_{ij} = x_i - x_j$)

$$H = \sum_{i=1}^{N} T_i + \sum_{i \neq j=1}^{N} V(|x_{ij}|), \tag{A 3}$$

where $T$ stands for the kinetic term, and the potentials $V$ are the pairwise interactions.

For the $b$th structure, bearing in mind the factorization equation (A 2), the Hamiltonian reads [3,4]

$$H = T_b \otimes I^b + I_b \otimes H_\circ^b + V^{(b)}; \tag{A 4}$$

in equation (A 4), $T$ stands for the kinetic term, $H_\circ$ for the 'self-Hamiltonian' and $V^{(b)}$ encapsulates all the interaction terms for the two factor spaces of the $b$th structure.

Removing the bound states from consideration is managed as follows. The projector $P_b$ is introduced for the pure point spectrum of $H_\circ^b$. Let us now introduce the 'small' projectors $P_b^M$, $M = 1, 2, 3, \ldots$ such that: (i) $P_b^M P_b = P_b^M$ and (ii) $s - \lim_{M\to\infty} P_b^M = P_b$. From those projectors, the following operator is built: $\tilde{P}_b^{\hat{M}_b^m} = P_b^{M_k} \hat{P}_b^{\hat{M}_b^m}$, where the limit $m \to \pm\infty$ is equivalent with $M \to \infty$ and the number of clusters in the $b$th structure is $k$. Without entering the details, the projector $\hat{P}_b^{\hat{M}_b^m}$ projects onto the pure continuous spectrum of the Hamiltonian for the structure $b$. Then Enss was able to prove a theorem that can be concisely presented by equation (2.4). This subtle procedure and the proof can be found in [1–4].

## References


1. Enss V. 1978 Asymptotic completeness for quantum mechanical potential scattering I. Short-range potentials. *Commun. Math. Phys.* **61**, 285–291. (doi:10.1007/BF01940771)
2. Enss V. 1986 Introduction to asymptotic observables for multiparticle quantum scattering. In *Schrödinger Operators, Aarhus 1985*, vol. 1218 (ed. E Balslev). Lecture Notes in Mathematics, pp. 61–92. Berlin, Germany: Springer.
3. Kitada H. 1991 Asymptotic completeness of N-body wave operators I. Short-range quantum systems. *Rev. Math. Phys.* **3**, 101. (doi:10.1142/S0129055X91000047)
4. Kitada H. 1993 Asymptotic completeness of N-body wave operators II. A new proof for the short-range case and the asymptotic clustering for long-range systems. In *Functional analysis and related topics, 1991*, vol. 1540 (ed. H Komatsu). Lecture Notes in Mathematics, pp. 149–189. Berlin, Germany: Springer.
5. Kitada H. 1994 Theory of local times. *Il Nuovo Cimento* **109 B**, 281.
6. Kitada H, Fletcher L. 1996 Local time and the unification of physics, Part I: local time. *Apeiron* **3**, 38.
7. Kitada H, Fletcher L. 1997 Comments on the problem of time. Preprint (http://arxiv.org/abs/gr-qc/9708055).
8. Petrat S, Tumulka R. 2014 Multi-time Schrödinger equations cannot contain interaction potentials. *J. Math. Phys.* **55**, 032302. (doi:10.1063/1.4867524)
9. Petrat S, Tumulka R. 2014 Multi-time equations, classical and quantum. *Proc. R. Soc. A* **470**, 20130632. (doi:10.1098/rspa.2013.0632)
10. Hartle JB. 1991 Time and prediction in quantum cosmology. In *Conceptual problems of quantum gravity* (eds A Ashtekar, J Stachel), pp. 172–203. Boston, MA: Birkhauser.

11. Petrat S, Tumulka R. 2014 Multi-time wave functions for quantum field theory. *Ann. Phys.* **345**, 17–54. (doi:10.1016/j.aop.2014.03.004)
12. Wheeler JA, Zurek WH (eds). 1984 *Quantum theory and measurement*. Princeton NJ: Princeton University Press.
13. Schlosshauer M. 2004 Decoherence, the measurement problem, and interpretations of quantum mechanics. *Rev. Mod. Phys.* **76**, 1267. (doi:10.1103/RevModPhys.76.1267)
14. Zurek WH. 2003 Decoherence, einselection, and the quantum origins of the classical. *Rev. Mod. Phys.* **75**, 715. (doi:10.1103/RevModPhys.75.715)
15. Zurek WH. 1981 Pointer basis of quantum apparatus: Into what mixture does the wave packet collapse? *Phys. Rev. D* **24**, 1516. (doi:10.1103/PhysRevD.24.1516)
16. Zurek WH. 1982 Environment-induced superselection rules. *Phys. Rev. D* **26**, 1862. (doi:10.1103/PhysRevD.26.1862)
17. Joos E, Zeh HD, Kiefer C, Giulini DJW, Kupsch J, Stamatescu I-O. 2003 *Decoherence and the appearance of a classical world in quantum theory*, 2nd edn. Berlin, Germany: Springer.
18. Riedel CJ, Zurek WH, Zwolak M. 2013 The objective past of a quantum universe—Part 1: redundant records of consistent histories. Preprint (http://arxiv.org/abs/1312.0331v1).
19. Breuer H-P, Petruccione F. 2002 *The theory of open quantum systems*. Oxford, UK: Oxford University Press.
20. Rivas Á, Huelga SF. 2011 *Open quantum systems: an introduction*. Briefs in Physics. Berlin, Germany: Springer.
21. Sigal IM. 1996 Asymptotic completeness. *AMS Transl.* **175**, 183.
22. Jeknić-Dugić J, Arsenijević M, Dugić M. 2013 *Quantum structures: a view of the quantum world*. Saarbrücken: LAP Lambert Academic Publishing.
23. Dugić M, Arsenijević M, Jeknić-Dugić J. 2013 Quantum correlations relativity for continuous variable systems. *Sci. Chi. Phys. Mech. Astron.* **56**, 732.
24. Wootters WK, Zurek WH. 1982 A single quantum cannot be cloned. *Nature* **299**, 802–803. (doi:10.1038/299802a0)
25. Nielsen MA, Chuang IL. 2000 *Quantum computation and quantum information*. Cambridge, UK: Cambridge University Press.
26. Margolus N, Levitin LB. 1999 The maximum speed of dynamical evolution. *Physica D* **120**, 188–195. (doi:10.1016/S0167-2789(98)00054-2)
27. Dugić M, Ćirković MM. 2002 Quantum information processing: the case of vanishing interaction energy. *Phys. Lett. A* **302**, 291.
28. Leggett AJ. 1980 Macroscopic quantum systems and the quantum theory of measurement. *Prog. Theor. Phys. Suppl.* **69**, 80–100. (doi:10.1143/PTPS.69.80)
29. Audenaert KMR, Scheel S. 2007 On random unitary channels. *New J. Phys.* **10**, 023011. (doi:10.1088/1367-2630/10/2/023011)
30. Dugić M. 1997 On diagonalization of a composite-system observable. Separability. *Phys. Scr.* **56**, 560.
31. Besicovitch AS. 1963 *Almost periodic functions*. Dover: Cambridge University Press.
32. Kac M. 1943 On the distribution of values of trigonometric sums with linearly independent frequencies. *Am. J. Math.* **65**, 609–615. (doi:10.2307/2371869)
33. Modi K, Brodutch A, Cable H, Paterek T, Vedral V. 2012 The classical-quantum boundary for correlations: discord and related measures. *Rev. Mod. Phys.* **84**, 1655. (doi:10.1103/RevModPhys.84.1655)
34. Zwolak M, Quan HT, Zurek WH. 2009 Quantum Darwinism in a hazy environment. *Phys. Rev. Lett.* **103**, 110402. (doi:10.1103/PhysRevLett.103.110402)
35. Zwolak M, Riedel CJ, Zurek WH. 2014 Amplification, redundancy, and the quantum chernoff information. *Phys. Rev. Lett.* **112**, 140406. (doi:10.1103/PhysRevLett.112.140406)
36. Von Neumann J. 1955 *Mathematical foundations of quantum mechanics*. Princeton, NJ: Princeton University Press.
37. Dugić M. 1998 On the investigation of separability of a bipartite system observable. *J. Res. Phys.* **27**, 141.
38. Omnés R. 1994 *The interpretation of quantum mechanics*. Princeton, NJ: Princeton University Press.
39. Walls DF, Collet MJ, Milburn GJ. 1985 Analysis of a quantum measurement. *Phys. Rev. D* **32**, 3208. (doi:10.1103/PhysRevD.32.3208)
40. Agarwal GS. 1971 Brownian motion of a quantum oscillator. *Phys. Rev. A* **4**, 739. (doi:10.1103/PhysRevA.4.739)

41. Kofler J, Brukner Č. 2007 Classical world arising out of quantum physics under the restriction of coarse-grained measurements. *Phys. Rev. Lett.* **99**, 180403. (doi:10.1103/PhysRevLett.99.180403)
42. Wang T, Ghobadi R, Raeisi S, Simon C. 2013 Precision requirements for observing macroscopic quantum effects. *Phys. Rev. A* **88**, 062114. (doi:10.1103/PhysRevA.88.062114)
43. Jeong H, Lim Y, Kim MS. 2014 Coarsening measurement references and the quantum-to-classical transition. *Phys. Rev. Lett.* **112**, 010402. (doi:10.1103/PhysRevLett.112.010402)
44. Sekatski P, Gisin N, Sangouard N. 2014 How difficult it is to prove the quantumness of macroscropic states? Preprint (http://arxiv.org/abs/1402.2542v1).
45. Gambini R, Porto RA, Pullin J. 2014 A relational solution to the problem of time in quantum mechanics and quantum gravity induces a fundamental mechanism for quantum decoherence. *New J. Phys.* **6**, 45. (doi:10.1088/1367-2630/6/1/045)
46. Diósi L. 2014 Hybrid quantum-classical master equations. Preprint (http://arxiv.org/abs/1401.0476).
47. Kafri D, Taylor JM, Milburn GJ. 2014 A classical channel model for gravitational decoherence. *New J. Phys.* **16**, 065020. (doi:10.1063/1.4867524)
48. Figari R, Teta A. 2014 *Quantum dynamics of a particle in a tracking chamber*. Briefs in Physics. Berlin, Germany: Springer.