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# Dynamical emergence of Markovianity in local time scheme

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Recently we pointed out the so-called local time scheme as a novel approach to quantum foundations that solves the preferred pointer-basis problem. In this paper, we introduce and analyse in depth a rather non-standard dynamical map that is imposed by the scheme. On the one hand, the map does not allow for introducing a properly defined generator of the evolution nor does it represent a quantum channel. On the other hand, the map is linear, positive, trace preserving and unital as well as completely positive, but is not divisible and therefore non-Markovian. Nevertheless, we provide quantitative criteria for dynamical emergence of time-coarse-grained Markovianity, for exact dynamics of an open system, as well as for operationally defined approximation of a closed or open many-particle system. A closed system never reaches a steady state, whereas an open system may reach a unique steady state given by the Lüders–von Neumann formula; where the smaller the open system, the faster a steady state is attained. These generic findings extend the standard open quantum systems theory and substantially tackle certain cosmological issues.

## 1. Introduction

Recently we pointed out the so-called local time scheme (LTS) [1] as a novel non-interpretational, minimalist approach to quantum foundations. In LTS, dynamics is a primitive that asymptotically defines local time for a single closed ('local') quantum system [1,2].

In general, quantum systems are subjected to different local times and hence there is not uniquely defined time for an [statistical] ensemble of such systems as well as for the Universe as a whole.

Instead of the universal ('global') time we learn about a single closed system's local time as a hidden classical parameter of the system's dynamics. This 'multi-time' scheme establishes a dynamical change in a single-system's local time when the system sufficiently strongly interacts with another system that is not subjected to the same local time. The scheme [1] naturally differentiates between the few- and many-particle systems, routinely and technically simply describes quantum measurement, resolves the 'preferred pointer-basis problem' for an ensemble of bipartitions of closed many-particle systems and provides a plausible interpretation of the Wheeler–DeWitt equation.

Understanding time is a deep issue of physics and philosophy. Nevertheless, resolving this issue is not yet necessary in the LTS. Moreover, mathematical formalization [1] of the concept of local time is an unexpected tool for distinguishing the few- and many-particle systems and hence may be expected to concern the whole set of the foundational, interpretation-related issues in quantum theory such as quantum measurement, microscopic origin of the phenomenological 'arrow of time' and the problem of the 'transition from quantum to classical', which, in turn, is recognized as the ultimate basis of the new technologies [3–5].

In contrast to LTS, quantum foundational research is often fragmented and typically goes to interpretations [6–8] or completely discards some foundational problems in a purely operational manner [9]. Crosstalk is exceptional and has only recently started in more systematic and comprehensive forms [10–14]. The outcome of this new endeavour is difficult to predict.

In this paper, we do not tackle the interpretation-related issues but rather pursue the so-far-useful methodology of ref. [1] to address the following question: Whether or not the scheme introduces some new elements or insights into the standard theory of open quantum systems [15–18]? As a remote goal we recognize value addressing the following question: What might be the consequences of non-unique time in regard of certain basic issues in cosmology?

As a step forward in investigating the implications of the idea of local time, in the spirit of ref. [1], our considerations are minimalist and resort to the mathematical aspects that constitute the basis for the future investigation of the interpretational implications of LTS. Bearing in mind the importance of Markovian processes [15–17], we investigate the (non)Markovian character of the dynamics imposed by LTS. To this end, we stick to the following definition that is essentially taken over from ref. [17]:

**Definition 1.1.** A quantum system is said to undergo Markovian dynamics if its dynamics is described by a family of dynamical maps  $\{\mathcal{E}_{(t_2,t_1)}, t_2 \geq t_1\}$  such that, for every  $t_2 \geq t_1$ ,  $\mathcal{E}_{(t_2,t_1)}$  is a completely positive (CP) map and fulfils the composition law  $\mathcal{E}_{(t_3,t_1)} = \mathcal{E}_{(t_3,t_2)}\mathcal{E}_{(t_2,t_1)}$ ,  $t_3 \geq t_2 \geq t_1$ .

The following are the main findings of this paper: (i) for a closed many-particle system, the map is CP but neither divisible nor differentiable; (ii) regarding an observer not capable of resolving the close energy values of a closed many-particle system, the (approximate) map is divisible and dynamically acquires complete positivity and hence time-coarse-grained Markovianity (TCGM); (iii) depending on the system's energy, an observer can in principle detect low-energy (e.g. low temperature) Markovian dynamics, the unitary-like Markovian behaviour for relatively high energies (e.g. high temperature) and non-Markovian dynamics for the rest of quantum states of a closed many-particle system; (iv) regarding an open system in a proper strong interaction with a many-particle environment, the exact map is CP and dynamically acquires divisibility and hence TCGM; (v) an approximate map for the open system is divisible and dynamically acquires complete positivity and, therefore, TCGM; (vi) a closed system never reaches a steady state, while an open system may reach unique steady state, which is given by the Lüders–von Neumann formula (i.e. by the von Neumann's projection postulate) in quantum measurement; (vii) the smaller the open system the faster is reached the steady state.

In the context of the new fundamental dynamical law (which takes the place and the role of the standard Schrödinger law), equation (2.2), these findings are generic. Hence, the possible

foundational character of the LTS not only in regard of open quantum systems theory but also regarding certain cosmological issues.

In §2 and with the aid of appendix A, we briefly overview and discuss certain subtle points in LTS. On this basis, we investigate mathematical characteristics of the LTS dynamical map, in §3 for a closed many-particle system, and in §4 for an open system in contact with a many-particle environment. In §5, we provide quantitative criteria for Markovian dynamics for a closed many-particle system. In §6, we compare the obtained results with the related counterparts from the standard open systems theory. Section 7 places an emphasis on certain cosmological issues. Section 8 concludes the paper.

## 2. Outlines of LTS

Formally, LTS introduces a variation in the standard unitary dynamics of a closed quantum system.

The standard unitary dynamics

$$|\Psi(t_o)\rangle = U(t_o)|\Psi(0)\rangle \quad (2.1)$$

is exchanged by a statistically weighted the ‘final’ instant of time  $t_o$ :

$$\sigma(t_o) = \int_{t_o-\Delta t}^{t_o+\Delta t} dt \rho(t) |\Psi(t)\rangle \langle \Psi(t)| = \int_{t_o-\Delta t}^{t_o+\Delta t} dt \rho(t) U(t) |\Psi(0)\rangle \langle \Psi(0)| U^\dagger(t). \quad (2.2)$$

We do not see any alternative to equation (2.2) within the *minimal* extension [1] of the standard theory, which adopts equation (2.1).

The origin of equation (2.2), which is the fundamental dynamical law in LTS, is an idea proposed within the quantum many-body scattering theory. While investigating the problem of asymptotic completeness in the many-body scattering theory, Kitada [2] noticed that unitary dynamics allows for an operational *definition* of time. The standard ‘time instant’ becomes a (one-dimensional) classical parameter that is determined by the unitary (Schrödinger) dynamics of a closed (or approximately closed) quantum system in the asymptotic limit; such systems are routinely assumed in quantum decoherence and measurement as well as in the open systems theory [15–18]. Hence the possibility that different Hamiltonians determine different times, each local time being a characteristic of a ‘local’ (approximately closed) system. LTS is a kind of ‘multi-time’ theory that does not suffer from some known drawbacks of fixed local times [1,19].

The fact that local time is defined in the *asymptotic* limit [2] leads to equation (2.2), since the *finite* value  $t_o$  for a single local system is not uniquely defined but only within some interval  $\Delta t$ . In order not to approximate equation (2.2) by equation (2.1), the interval  $\Delta t$  for the continuous parameter  $t_o$  should not be arbitrarily small. Hence the requirement  $t_o \gg \Delta t$  implies that  $t_o$  cannot be arbitrarily small either, while the formal limit  $t_o \rightarrow \infty$  should be allowed.

On the other hand, for some values of  $\Delta t$ , certain states  $|\Psi(t)\rangle$  in equation (2.2) can be mutually orthogonal [20,21]. As this seems to contradict equation (2.1), a proper upper bound,  $\tau_{\min}$ , for  $\Delta t$  has been recognized that yields the constraint  $\tau_{\min} > \Delta t$ . The choice of a Gaussian probability density  $\rho(t) = \sqrt{\lambda/\pi} \exp(-\lambda(t-t_o)^2)$  gives rise to the constraint  $\tau_{\min} > \Delta t > \lambda^{-1/2}$ . The condition  $\tau_{\min} = \max\{\pi\hbar/2\Delta H, \pi\hbar/2(\langle H \rangle - E_g)\}$ , where  $\Delta H$  is the standard deviation and  $E_g$  stands for the Hamiltonian ground energy, gives rise to the estimation  $\lambda > C^2$ , where  $C$  represents the energy scale of the system’s Hamiltonian. It is the basic characteristic of LTS: there is energy conservation for the state equation (2.2) and hence time independence of both  $\Delta H$  and  $\langle H \rangle$ . Non-orthogonality of states  $|\psi(t)\rangle$  in equation (2.2) makes the local time instants from the interval  $[t_o - \Delta t, t_o + \Delta t]$  mutually indistinguishable. Hence, local time as a hidden classical parameter of the system’s dynamics.

Within LTS, a single system that is subjected to a local time  $t_1$  may interact with another single system that is subjected to a local time  $t_2$ . Assume that their *sufficiently strong* interaction starts in an instant of time  $t_{1o}$  for the first system, and in an instant  $t_{2o}$  of local time for the second system. The start of the interaction defines a new closed system,  $1+2$ , whose Hamiltonian

determines the composite system's local time, with the new initial 'instant of time'  $t=0$  for the combined dynamics. For the previously non-interacting systems undergoing independent unitary dynamics subjected to their independent local times, the tensor-product initial state is  $|\psi(t_{1o})\rangle_1|\phi(t_{2o})\rangle_2 \equiv |\Psi(t=0)\rangle$ , which appears in equation (2.2). This description may raise the following three concerns regarding consistency and coherence of LTS. First, the meaning of 'sufficiently strong' interaction (that underlies the meaning of 'approximately closed' ('local' system) is not obvious. Second, in realistic situations, a smooth dynamical transition from 'weak' to 'sufficiently strong' interaction is expected and should be properly described. Finally, provided the answers to these questions, it is natural to ask about uniqueness of the 'initial instant'  $t=0$  for the combined system. Putting  $t_o=0$  in equation (2.2) reveals non-uniquely defined 'initial time instant' for an ensemble of 1 + 2 systems. On the other hand, the local time instants  $t_{io}$ ,  $i=1,2$ , can locally satisfy  $t_{io} \rightarrow \infty$  as per equation (2.2), i.e. from the point of view of the local 1 and 2 systems, these instants are not special and are subject to the time uncertainty equation (2.2). In ref. [1] we analysed the standard scenario of unique instant of time in an ensemble presented by equation (2.1) and hence adopted equation (2.2) without variations. The first two questions are addressed in appendix A. Non-unique initial instant of time is the subject of lemma 3.3.

### 3. Closed system dynamics

Dynamics introduced by equation (2.2) is linear. Therefore, its generalization for mixed states

$$\sigma(t_o) = \int_{t_o-\Delta t}^{t_o+\Delta t} dt \rho(t) U(t) \sigma(0) U^\dagger(t), \quad (3.1)$$

with a dynamical map  $\mathcal{E}$

$$\sigma(t_o) = \mathcal{E}_{(t_o,0)}[\sigma(0)] \quad (3.2)$$

that we are interested in.

Equation (3.1) is of the form of the so-called random-unitary maps (RUMs) [22]. However, as distinct from the standard RUMs, of the general form  $\sum_i p_i U_i(t) \sigma(0) U_i^\dagger(t)$ ,  $\sum_i p_i = 1, \forall t$ , and unitary  $U_i, \forall i$ , the map equation (3.1) varies the time instant(s).

Inclusion of the unitary operator spectral form,  $U(t) = \sum_n \exp(-itE_n/\hbar) P_n$ , where  $E_n$  and  $P_n$  stand for the Hamiltonian eigenvalues and orthogonal eigenprojectors, gives for equation (3.1):

$$\sigma(t_o) = \sum_{m,n} \exp\left(\frac{-it_o(E_m - E_n)}{\hbar}\right) \exp\left(\frac{-(E_m - E_n)^2}{4\lambda\hbar^2}\right) P_m \sigma(0) P_n. \quad (3.3)$$

Integration in equation (3.1) gives equation (3.3) via the use of the following Gaussian integral:  $\int_{-\infty}^{\infty} \exp(-ax^2/2 + iJx) dx = (2\pi/a)^{1/2} \times \exp(-J^2/2a)$ , where  $a > 0$  and  $J$  are real numbers with  $J$  being conjugate variable of  $x$ .

From the mutually equivalent equations (3.1)–(3.3) directly follow: (i) the map is linear, positive and trace preserving; (ii) the map is unital, i.e.  $\mathcal{E}[I] = I$ ; (iii) the r.h.s. of equation (3.1) is of the so-called Kraus form that guarantees that the map is CP. Owing to the non-standard character of the map to be emphasized below, we give an alternative proof of complete positivity that employs the Jamiolkowski criterion [23]. According to this criterion, a map  $\mathcal{E}$  is CP if and only if the extended map  $\mathcal{I} \otimes \mathcal{E}$  is positive while acting on a maximally entangled state  $|\psi\rangle$  for an extended system:

$$\langle \phi | (\mathcal{I} \otimes \mathcal{E}[|\psi\rangle\langle\psi|]) | \phi \rangle \geq 0, \quad \forall |\phi\rangle. \quad (3.4)$$

By putting  $|\psi\rangle = \sum_i |i\rangle|i\rangle/d$  and arbitrary state  $|\phi\rangle = \sum_{j,k} c_{jk} |j\rangle|k\rangle$  into equation (3.4), the criterion reads

$$\frac{1}{d} \sum_{i,j,l,l'} c_{ij}^* c_{j'l'} \langle l | (\mathcal{E}[|i\rangle\langle j|]) | l' \rangle \geq 0. \quad (3.5)$$

For the map equation (3.1), the criterion equation (3.5) easily gives

$$\int_{t_0-\Delta t}^{t_0+\Delta t} dt \rho(t) \left| \sum_{i,l} c_{il}^* \langle l | U(t) | i \rangle \right|^2 > 0. \quad (3.6)$$

However, the map is rather non-standard as we are going to demonstrate. For the initial  $t = 0$  as well as for arbitrary value of the time parameter  $t$ , the map satisfies

$$\mathcal{E}_{(t,t)} \neq \mathcal{I}, \quad \forall t \quad (3.7)$$

where the  $\mathcal{I}$  stands for the identity (unity) map. Putting  $t_0 = 0$  in equation (3.1) does not return, as apparently it should, the initial state  $\sigma(0)$ . Hence, as distinct from the standard theory, equation (2.1), in which  $t_0 \in (-\infty, \infty)$ , in LTS, equation (2.2),  $t_0 \in (0, \infty)$ .

The in-equality (3.7) is striking: the map describes a *non-differentiable process*, that is incapable of deriving a differential form for the equation (3.3). Actually, derivation of a differential form of the open system dynamical law *assumes* [15–17] non-validity of equation (3.7), i.e. validity of  $\mathcal{E}_{(t,t)} = \mathcal{I}, \forall t$ , in order to have a mathematically properly defined ‘generator’ of evolution (often denoted  $\mathcal{L}_t$ ). Therefore, we proceed with the analysis of the ‘integral’ form of the dynamical law equation (3.3).

### (a) Non-divisibility of the map

**Lemma 3.1.** *The map (3.1) can be combined with the unitary dynamics:*

$$\mathcal{E}_{(t_0,0)}[\sigma(0)] = \mathcal{U}_{(t_0,t'')}[\mathcal{E}_{(t'',t')}[\mathcal{U}_{(t',0)}[\sigma(0)]]], \quad t_0 \geq t'' \geq t' > 0, \quad (3.8)$$

where  $\mathcal{U}$  denotes the unitary dynamics equation (2.1).

*Proof.* From equation (3.3) we can directly write

$$\begin{aligned} \sigma(t_0) &= \mathcal{E}_{(t_0,t')}[\mathcal{U}_{(t',0)}[\sigma(0)]] \\ &= \sum_{m,n} \exp\left(\frac{-i(t_0 - t')(E_m - E_n)}{\hbar}\right) \exp\left(\frac{-(E_m - E_n)^2}{4\lambda\hbar^2}\right) P_m \sigma(t') P_n. \end{aligned} \quad (3.9)$$

where  $\sigma(t')$  refers to the unitary evolution as well as

$$\sigma(t_0) = \mathcal{U}_{(t_0,t')}[\mathcal{E}_{(t',0)}[\sigma(0)]] = \sum_{m,n} \exp\left(\frac{-i(t_0 - t')(E_m - E_n)}{\hbar}\right) P_m \sigma(t') P_n \quad (3.10)$$

with  $\sigma(t')$  given by equation (3.3).

Now equation (3.9) gives

$$\begin{aligned} \sigma(t_0) &= \sum_{m,n} \exp\left(\frac{-i(t_0 - t')(E_m - E_n)}{\hbar}\right) \exp\left(\frac{-(E_m - E_n)^2}{4\lambda\hbar^2}\right) \\ &\quad \times P_m \left[ \sum_{p,q} \exp\left(-\frac{it'(E_p - E_q)}{\hbar}\right) P_p \sigma(0) P_q \right] P_n. \end{aligned} \quad (3.11)$$

From equation (3.10):

$$\begin{aligned} \sigma(t_0) &= \mathcal{U}_{(t_0,t')}[\mathcal{E}_{(t',0)}[\sigma(0)]] = \sum_{m,n} \exp\left(\frac{-i(t_0 - t')(E_m - E_n)}{\hbar}\right) \\ &\quad \times P_m \left[ \sum_{p,q} \exp\left(\frac{-it'(E_p - E_q)}{\hbar}\right) \exp\left(\frac{-(E_p - E_q)^2}{4\lambda\hbar^2}\right) P_p \sigma(0) P_q \right] P_n. \end{aligned} \quad (3.12)$$

Orthogonality of the projectors,  $P_m P_n = \delta_{mn} P_m$ , with the ‘Kronecker delta’,  $\delta_{mn}$ , gives equation (3.3) for both equations (3.11) and (3.12). Combination of equations (3.11) and (3.12) completes the proof. ■

Lemma 3.1 reveals that the time uncertainty, originally linked with  $t_o$  in equation (3.1), formally applies to every *single* value  $t \in (0, t_o]$  but not to any time interval. Thereby we learn that the LTS time-uncertainty equation (2.2) cannot be interpreted as or be reducible to action of a ‘quantum channel’ [3,22].

**Lemma 3.2.** *The map cannot be divided according to the law (3.2):*

$$\mathcal{E}_{(t_o,0)}[\sigma(0)] \neq \mathcal{E}_{(t_o,t')}[\mathcal{E}_{(t',0)}[\sigma(0)]], \quad t_o \geq t' > 0. \quad (3.13)$$

*Proof.* Owing to equation (3.3), the r.h.s. of equation (3.13) reads

$$\begin{aligned} \sigma(t_o) &= \sum_{m,n} \exp\left(\frac{-it_o(E_m - E_n)}{\hbar}\right) \exp\left(\frac{-(E_m - E_n)^2}{4\lambda\hbar^2}\right) \\ &\quad \times P_m \left[ \sum_{p,q} \exp\left(\frac{-it'(E_p - E_q)}{\hbar}\right) \exp\left(\frac{-(E_p - E_q)^2}{4\lambda\hbar^2}\right) P_p \sigma(0) P_q \right] P_n \\ &= \sum_{m,n} \exp\left(\frac{-it_o(E_m - E_n)}{\hbar}\right) \exp\left(\frac{-2(E_m - E_n)^2}{4\lambda\hbar^2}\right) P_m \sigma(0) P_n. \end{aligned} \quad (3.14)$$

Now decomposition of the interval  $(0, t_o]$  into  $k$  subintervals, with the aid of equation (3.14), gives instead of equation (3.3):

$$\sum_{m,n} \exp\left(\frac{-it_o(E_m - E_n)}{\hbar}\right) \exp\left(\frac{-k(E_m - E_n)^2}{4\lambda\hbar^2}\right) P_m \sigma(0) P_n \rightarrow \sum_m P_m \sigma(0) P_m \quad (3.15)$$

as  $k \rightarrow \infty$  that completes the proof. ■

Lemma 3.2 exhibits non-validity of the assumptions of definition 1.1 and hence reveals the non-Markovian character of the map. In the spirit of the standard wisdom [16,17], the total map,  $\mathcal{E}_{(t_o,0)}$ , and the initial map,  $\mathcal{E}_{(t',0)}$  (which are given by equation (3.1)), are CP. However the ‘intermediate’ map,  $\mathcal{E}_{(t_o,t')}$ , is not CP and therefore equation (3.1) does not apply to  $\mathcal{E}_{(t_o,t')}$  according to lemma 3.2.

In addition, we note that equation (3.3) can be transformed to a Kraus form that is alternative to the integral form of equation (3.1). Complete positivity  $\langle \chi | \sigma(t_o) | \chi \rangle \geq 0, \forall | \chi \rangle, t_o$ , cf. equation (3.6), implies the matrix  $\mathcal{A} = (\exp(-(E_m - E_n)^2/4\lambda\hbar^2))$  is positive semi-definite. Then diagonalization of the  $\mathcal{A}$  matrix by a unitary  $\mathcal{U} = (u_{km})$  matrix gives  $A_{mm} = \sum_k \gamma_k u_{km}^* u_{kn}$  with the eigenvalues  $\gamma_k \geq 0, \forall k$ , while  $A_{mm} = 1, \forall m$ . Therefore,

$$\sigma(t_o) = \sum_k K_k(t_o) \sigma(0) K_k^\dagger(t_o), \quad (3.16)$$

where the Kraus operators  $K_k(t_o) = \sqrt{\gamma_k} \sum_m u_{km}^* \exp(-it_o E_m/\hbar) P_m$  and the equality  $\sum_k K_k^\dagger(t_o) K_k(t_o) = I$  is satisfied. However, due to equation (3.7), existence of a Lindblad differential form of equation (3.16) is excluded.

**Lemma 3.3.** *Non-uniqueness of the initial  $t = 0$  in equation (3.1) does not change the character or the form of the map.*

*Proof.* A change of local time occurs due to sufficiently strong interaction of two systems not being subjected to the same local time. Then uncertainty of the initial ‘instant’  $t = 0$  regards the uncertainty of duration of the approximately independent dynamics of the two systems. Then

equation (3.3) gives

$$\sigma(t_o) = \sum_{m,n} \exp\left(\frac{-it_o(E_m - E_n)}{\hbar}\right) \exp\left(\frac{-(E_m - E_n)^2}{4\lambda\hbar^2}\right) P_m \sigma'(0) P_n. \quad (3.17)$$

where

$$\sigma'(0) = \int_{-\delta t}^{\delta t} \rho'(t) U'(t) \sigma(0) U'^{\dagger}(t) dt. \quad (3.18)$$

For non-interacting, or weakly interacting, systems the unitary operator  $U'(t) = \sum_p \exp(-itE'_p/\hbar) p_p$  and  $\sigma(0) = |\Psi(0)\rangle\langle\Psi(0)|$ , while  $|\Psi(0)\rangle = |\psi(t_{1o})\rangle_1 |\phi(t_{2o})\rangle_2$  and  $\rho'(t) = \sqrt{\lambda'/\pi} \exp(-\lambda't^2)$ , equation (3.18) reads (e.g. by putting  $t_o = 0$  in equation (3.3)):

$$\sigma'(0) = \sum_{p,q} \exp\left(\frac{-(E'_p - E'_q)^2}{4\lambda'\hbar^2}\right) p_p \sigma(0) p_q. \quad (3.19)$$

The two uncertainties,  $\Delta t$  and  $\delta t$ , determine the two Gaussian factors,  $\lambda$  and  $\lambda'$ , respectively. In order to comply with equation (2.1),  $\delta t$  should not exceed  $\Delta t$ . Hence, cf. §2,  $\lambda'$  cannot be smaller than  $\lambda$ ; we could have introduced two  $\lambda'$ 's for every subsystem separately but this would change nothing. If we introduce the fixed (cf. appendix A) energy scales  $c$  and  $C$  for the two Hamiltonians, the condition that interaction dominates the total system's dynamics yields  $c \ll C$  (in the units  $\hbar = 1$ ). Bearing in mind (§2) that  $\lambda' > \lambda > C^2$ , the exponential factors in equation (3.19) can all be estimated  $\exp(-c^2/4C^2) \approx 1$ , and hence  $\sigma'(0) \approx \sigma(0)$ . ■

Lemma 3.3 quantitatively, i.e. via the condition  $c/C \ll 1$ , removes the ambiguity regarding  $t = 0$  in LTS: for given  $c$ , strong interaction ( $C \gg c$ ) gives rise to practically indistinguishable dynamics equation (3.3) and equation (3.17). Interestingly, there is a question of handling the same for the standard theory, equation (2.1). Indeed, initial preparation of a quantum ensemble may be not pure as generally assumed and described by equation (2.1). Then equation (2.2) with  $t_o = 0$  applies to the standard theory of unique time and hence produces an observable, although probably weak, dissent with equation (2.1). A plausible answer might be that  $\delta t$  should be so small as to provide  $U(\delta t) = I + O(\delta t)$ , which gives equation (2.1) approximately correct. Needless to say, this option also applies to our considerations again leading to  $\sigma'(0) \approx \sigma(0)$ . Hence LTS appears to be more robust to variations in the 'initial instant' than the standard theory of unique time.

## (b) An approximation of the map

Numerical values of the real Gaussian factors in equation (3.3) depend on the energy-spectrum  $\{E_m\}$  and also on the initial state  $|\Psi(0)\rangle$ . Consider a coarse graining of the set of energies: every value  $E_m$  is assigned a set of numerically (or operationally) close values  $E_{v_m}$ . The standard procedure would be to introduce a new set of energy eigenvalues by setting  $E_m = E_{v_m}$  and the related eigenprojectors that redefine the system's Hamiltonian, and then to start over from equation (2.1), i.e. from equation (3.1). However, we will proceed in the following, more flexible operational fashion that is closer (but not identical) in spirit to [24].

We directly and independently adapt the Gaussian terms appearing in equation (3.3) via the numerical estimates:

$$\exp\left(-\frac{(E_k - E_{k'})^2}{4\lambda\hbar^2}\right) \ll 1, \quad k \in \{m, v_m\}, k' \in \{m', v'_m\}, \forall m \neq m'. \quad (3.20)$$

Being interested in large values of  $t_o$ , we set  $E_{v_m} - E_{v'_m} \approx \delta_m > 0, \forall m, v, v'$  and assume  $\exp(-\delta_m^2/4\lambda\hbar^2) \approx 1, \forall m$ , that can follow from the numerical values or determine the measurement

errors, and obtain

$$\sigma(t_\circ) \approx \sum_m P_m \sigma(0) P_m + \sum_m \exp\left(-\frac{it_\circ \delta_m}{\hbar}\right) P_m \sigma(0) \Pi^{(m)} + h.c. \quad (3.21)$$

Numerical details behind equation (3.21) can be found in §5. Equation (3.20) is typically not applicable to the few-particle systems, which are already known [1] to bear high quantum coherence, i.e. approximately to be dynamically described by equation (2.1).

In equation (3.21), for typically non-orthogonal but commuting projectors  $\Pi^{(m)} = \sum_{v_m} P_{v_m}$  uniquely defined by the chosen coarse graining: (i)  $P_m \Pi^{(m)} = 0, \forall m$ , while the commutator  $[P_m, \Pi^{(m')}] = 0, \forall m, m'$ ; (ii) if  $P_m \Pi^{(m')} \neq 0$  for some  $m$  and  $m'$ , then  $P_{m'} \Pi^{(m)} = 0$  for the same pair of indices  $m$  and  $m'$ ; (iii) it is allowed that  $\delta_m = \delta_{m'}$  for some  $m \neq m'$ .

The map equation (3.21) is linear, unital, trace preserving and applies equation (3.7) as well as Lemma 3.1.

For the  $d$ -independent coarse graining, i.e. for the coarse-graining parameter  $g = \max\{\text{tr} \Pi^{(m)}\}$  such that  $\lim_{d \rightarrow \infty} g = g$ :

**Lemma 3.4.** *The approximate map (3.21) is CP for almost all large values of  $t_\circ$ , i.e. formally, the map is CP in the limit  $t_\circ \rightarrow \infty$ .*

*Proof.* We start from the l.h.s. of equation (3.5), for equation (3.21) gives

$$\begin{aligned} & \frac{1}{d} \sum_m \left| \sum_{i,l} c_{il}^* \langle l | P_m | i \rangle \right|^2 \\ & + \frac{1}{d} \sum_m \exp\left(-\frac{it_\circ \delta_m}{\hbar}\right) \left( \sum_{i,l} c_{il}^* \langle l | P_m | i \rangle \right) \left( \sum_{j,l'} c_{jl'} \langle j | \Pi^{(m)} | l' \rangle \right) + c.c. \end{aligned} \quad (3.22)$$

Since the basis  $|i\rangle$  appearing in equation (3.5) can be chosen so as  $\langle l | P_m | i \rangle = 0, \forall l \neq i$  and  $\langle j | \Pi^{(m)} | l' \rangle = 0, \forall j \neq l'$ , the second term in equation (3.22):

$$\frac{1}{d} \sum_m \left( \sum_{i=1}^{g_m} c_{ii}^* \right) \left( \sum_{v_i=1}^{g^{(m)}} c_{v_i v_i} \right), \quad (3.23)$$

where  $g_m = \text{tr} P_m$  and  $g^{(m)} = \text{tr} \Pi^{(m)}$ . As we are interested in the largest possible value for the sum equation (3.23), we introduce the real  $c_{ii} = p_i \geq 0$  and  $c_{v_i v_i} = p_{v_i} \geq 0$ ; the largest value follows for  $\sum_{i=1}^d p_i^2 = 1$ , which pertains to the very special (normalized) states  $|\phi\rangle = \sum_i p_i |i\rangle |i\rangle$  in equation (3.4). For all other choices of the complex  $c_{ii}$  ( $c_{v_i v_i}$ ), the second (third) term in equation (3.22) is modulo smaller.

Regarding equation (3.23)

$$\sum_{m=1}^N \chi_m \equiv \sum_{m=1}^N \left( \sum_{i=1}^{g_m} p_i \right) \left( \sum_{v_i=1}^{g^{(m)}} p_{v_i} \right) \leq g_{\max} g N, \quad g_{\max} = \max\{g_m, m = 1, 2, \dots, N\}. \quad (3.24)$$

With the notation  $C \equiv \sum_{m=1}^N (\sum_{i=1}^{g_m} p_i)^2 > 0$ , we are interested to prove

$$1 + \frac{2g g_{\max} N}{C} \sum_{m=1}^N \cos(\delta_m t_\circ) \frac{\chi_m}{g g_{\max} N} \geq 0 \quad (3.25)$$

for almost all large values of  $t_\circ$  and for sufficiently large  $N$ .

From equation (3.24) follows  $\sum_{m=1}^N \chi_m / g g_{\max} N \leq 1$  and hence the sum in equation (3.25) is an almost periodic function [25–27]. For sufficiently long time interval  $[t, t + T]$ , and a large number  $N$  (that increases with the increase in the number of particles in the system [27]) such that  $t_\circ \in [t, t + T]$  typically [18,27]: (a) the time average  $\lim_{T \rightarrow \infty} \langle \sum_m \cos(\delta_m t_\circ) (\chi_m / g g_{\max} N) \rangle_T = 0$  and (b) the standard deviation  $\lim_{T \rightarrow \infty} \langle | \sum_m \cos(\delta_m t_\circ) (\chi_m / g g_{\max} N) |^2 \rangle_T = 0$ . On the other hand,



$C = \sum_{m=1}^N (g_m \bar{p}_m)^2$ , where  $\bar{p}_m$  is the average value for the set  $\{p_i, i=1, 2, \dots, g_m\}$  and so  $C \geq g_{\min}^2 \sum_{m=1}^N \bar{p}_m^2$ . So, the term  $2gg_{\max}N/C \leq 2g_{\max}g/\sum_{m=1}^N \bar{p}_m^2$ . Noting that  $g_{\max} < g$  and the fact that  $\sum_{m=1}^N \bar{p}_m^2/N = (\Delta\bar{p})^2 + \langle \bar{p} \rangle^2$  never decreases with the increase of  $N$ , the term  $2gg_{\max}N/C$  does not increase with the increase of  $N$ ;  $\Delta\bar{p}$  and  $\langle \bar{p} \rangle$  are the standard deviation and the average value for the set  $\{\bar{p}_m\}$ . Bearing in mind that the probability distribution for the sum in equation (3.25) goes to the Dirac delta-function as  $N \rightarrow \infty$  [27], the probability that the sum over  $m$  in equation (3.25) takes a value less than  $-C/2gg_{\max}N$  approaches zero in the limit  $N \rightarrow \infty$  [27]. Therefore, we proved equation (3.25). Multiplying equation (3.25) by  $C/d$ , for a not-very-large coarse graining constant  $g$ , from equation (3.22):

$$\frac{1}{d} \sum_{m=1}^N \left( \sum_{i=1}^{g_m} p_i \right)^2 + 2g \sum_{m=1}^N \cos(\delta_m t_\circ) \frac{\chi_m}{gd} \geq 0, \quad (3.26)$$

that is, for most of large values of  $t_\circ$  and for sufficiently large  $N$  (large number of particles in the system) the approximate map equation (3.21) is CP. ■

Lemma 3.4 exhibits *dynamical* emergence of complete positivity of the approximate map equation (3.21). The larger value of  $g$  the larger the ‘minimum’ value of  $t_\circ$  for which equation (3.26) is fulfilled. For exceedingly large  $g$ , the minimum value of  $t_\circ$  may be very large and also equation (3.26) less likely to be satisfied for some instants  $t > t_\circ$ .

Regarding the divisibility of the map equation (3.21), we have the following

**Lemma 3.5.** *The approximate map  $\mathcal{E}^{(\text{app})}$  equation (3.21) is divisible:*

$$\mathcal{E}_{(t_\circ, 0)}^{(\text{app})}[\sigma(0)] = \mathcal{E}_{(t_\circ, t')}^{(\text{app})}[\mathcal{E}_{(t', 0)}^{(\text{app})}[\sigma(0)]], \quad t_\circ \geq t' > 0. \quad (3.27)$$

*Proof.* Equation (3.21) gives

$$\sigma(t_\circ) \approx \sum_m P_m \sigma(t') P_m + \sum_m \exp\left(-\frac{i(t_\circ - t')\delta_m}{\hbar}\right) P_m \sigma(t') \Pi^{(m)} + \text{h.c.} \quad (3.28)$$

Substituting equation (3.21) with  $t'$  instead of  $t_\circ$  into equation (3.28), due to the above point (i) equation (3.28) reads

$$\begin{aligned} \sigma(t_\circ) \approx & \sum_m P_m \sigma(0) P_m + \sum_m \exp\left(-\frac{it_\circ \delta_m}{\hbar}\right) P_m \sigma(0) \Pi^{(m)} \\ & + \sum_{m,n} \exp\left(-\frac{i(t_\circ - t')\delta_m}{\hbar}\right) \exp\left(\frac{it' \delta_n}{\hbar}\right) P_m \Pi^{(n)} \sigma(0) P_n \Pi^{(m)} + \text{h.c.} \end{aligned} \quad (3.29)$$

Owing to the above point (ii), the last term in equation (3.29) equals zero, thus equation (3.29) taking the form of equation (3.21). ■

Now it is obvious that lemmas 3.4 and 3.5 give rise to the conclusion: as distinct from the exact map equation (3.3), the approximate map equation (3.21) *dynamically acquires Markovianity*, definition 1.1. Owing to lemma 3.4, Markovian character of the map equation (3.21) requires the lower time bound, i.e. coarse graining of the time interval  $(0, t_\circ]$ , thus resembling the so-called Born approximation in the standard theory [15–17]. In accordance with §2, the initial state for the  $S + E$  system is assumed to be tensor-product.

## 4. Open system dynamics

We are interested in a bipartite decomposition of the total system of §3 that consists of two subsystems,  $S$  and  $E$ , in which interaction between  $S$  and  $E$  dominates the total system’s dynamics, which is the situation described at the end of §2.

Then the unitary operator  $U(t) \approx \exp(-itH_{\text{int}}/\hbar)$  and we consider the ‘pure decoherence’ interaction given by the ‘separable’ spectral form [27,28]:

$$H_{\text{int}} = \sum_{\alpha,\beta} E_{\alpha\beta} P_{\alpha} \otimes \Pi_{\beta}, \quad (4.1)$$

where the orthogonal projectors  $P_{\alpha}$  refer to the  $S$  system, the projectors  $\Pi_{\beta}$  refer to the  $E$  system, while the interaction eigenvalues  $E_{\alpha\beta}$  are all real such that  $E_{\alpha\beta} = E_{\gamma\delta}$  if and only if  $\alpha = \beta$  and  $\gamma = \delta$ . In accordance with §2, the initial state for the  $S + E$  system is tensor-product that gives rise to the composite system’s state of the form of equation (3.3).

### (a) The exact map

With the use of equation (4.1), after some algebra from equation (3.3) follows:

$$\begin{aligned} \rho_S(t_{\circ}) &= \text{tr}_E \sigma(t_{\circ}) = \sum_{\alpha,\gamma} P_{\alpha} \rho_S(0) P_{\gamma} \times \sum_{\beta} \exp\left(-it_{\circ} \frac{E_{\alpha\beta} - E_{\gamma\beta}}{\hbar}\right) \exp\left(-\frac{(E_{\alpha\beta} - E_{\gamma\beta})^2}{4\lambda\hbar^2}\right) (\text{tr}_E \Pi_{\beta} \rho_E(0)) \\ &\equiv \sum_{\alpha,\gamma} B_{\alpha\gamma}(t_{\circ}) P_{\alpha} \rho_S(0) P_{\gamma}. \end{aligned} \quad (4.2)$$

Similarly, for the successive time intervals  $(0, t']$ ,  $[t', t_{\circ}]$  and for  $\sigma(t')$  given by equation (3.3):

$$\begin{aligned} \rho_S(t_{\circ}) &= \text{tr}_E \sigma(t_{\circ}) = \sum_{\alpha,\beta,\gamma,\delta} P_{\alpha} [\text{tr}_E \Pi_{\beta} \sigma(t') \Pi_{\delta}] P_{\gamma} \\ &\quad \times \exp\left(-i(t_{\circ} - t') \frac{E_{\alpha\beta} - E_{\gamma\delta}}{\hbar}\right) \exp\left(-\frac{(E_{\alpha\beta} - E_{\gamma\delta})^2}{4\lambda\hbar^2}\right) \\ &= \sum_{\alpha,\gamma} P_{\alpha} \rho_S(0) P_{\gamma} \times \sum_{\beta} \exp\left(-it_{\circ} \frac{E_{\alpha\beta} - E_{\gamma\beta}}{\hbar}\right) \exp\left(-2\frac{(E_{\alpha\beta} - E_{\gamma\beta})^2}{4\lambda\hbar^2}\right) (\text{tr}_E \Pi_{\beta} \rho_E(0)) \\ &\equiv \sum_{\alpha,\gamma} B'_{\alpha\gamma}(t_{\circ}) P_{\alpha} \rho_S(0) P_{\gamma}. \end{aligned} \quad (4.3)$$

From equation (4.2) it easily follows that the map is linear, positive, trace preserving and unital. Like the exact map, §3, the map fulfills equation (3.7). As the matrix  $(B_{\alpha\gamma}(t_{\circ}))$  is positive semi-definite (cf. equation (3.16)), the map is CP for every  $t_{\circ} > 0$ .

On the other hand, *prima facie* equation (4.3) exhibits non-divisibility of the map: decomposition of the interval  $(0, t_{\circ}]$  into  $k$  subintervals leads, in analogy with equation (3.15), to

$$\begin{aligned} \sum_{\alpha,\gamma} P_{\alpha} \rho_S(0) P_{\gamma} \times \sum_{\beta} \exp\left(-it_{\circ} \frac{E_{\alpha\beta} - E_{\gamma\beta}}{\hbar}\right) \exp\left(-k\frac{(E_{\alpha\beta} - E_{\gamma\beta})^2}{4\lambda\hbar^2}\right) (\text{tr}_E \Pi_{\beta} \rho_E(0)) \\ \rightarrow \sum_{\alpha} P_{\alpha} \rho_S(0) P_{\alpha} \end{aligned} \quad (4.4)$$

as  $k \rightarrow \infty$ . However, the map dynamically acquires divisibility as stated by the following lemma.

**Lemma 4.1.** *The map (4.2) has unique steady state and acquires divisibility for almost all large values of  $t_{\circ}$ , i.e. formally, the map is divisible in the limit  $t_{\circ} \rightarrow \infty$ :*

$$\lim_{t_{\circ} \rightarrow \infty} \rho_S(t_{\circ}) = \sum_{\alpha} P_{\alpha} \rho_S(0) P_{\alpha}. \quad (4.5)$$

*Proof.* For equation (4.2)

$$\rho_S(t_{\circ}) = \sum_{\alpha,\gamma} B_{\alpha\gamma}(t_{\circ}) P_{\alpha} \rho_S(0) P_{\gamma} \quad (4.6)$$

where

$$B_{\alpha\gamma}(t_{\circ}) = \zeta_{\alpha\gamma} \sum_{\beta} p_{\beta}^{(\alpha\gamma)} \exp\left(-it_{\circ} \frac{E_{\alpha\beta} - E_{\gamma\beta}}{\hbar}\right) \quad (4.7)$$

while the real  $p_{\beta}^{(\alpha\gamma)} = (\text{tr}_E \Pi_{\beta} \rho_E(0)) \exp(-(E_{\alpha\beta} - E_{\gamma\beta})^2 / 4\lambda\hbar^2) / \zeta_{\alpha\gamma} > 0$ ,  $\sum_{\beta} p_{\beta}^{(\alpha\gamma)} = 1$  and  $\zeta_{\alpha\gamma} \equiv \sum_{\beta} \exp(-(E_{\alpha\beta} - E_{\gamma\beta})^2 / 4\lambda\hbar^2) (\text{tr}_E \Pi_{\beta} \rho_E(0)) < 1$ . The sum over  $\beta$  in equation (4.7) is an almost periodic function [25,26] that appears also for the so-called correlation amplitude in quantum decoherence theory [27].

Now, in analogy with the proof of lemma 3.4, for sufficiently long time interval  $[t, t + T]$ , such that  $t_{\circ} \in [t, t + T]$ , for  $\alpha \neq \gamma$ , for large number of summands (many-particle environment  $E$ ) in equation (4.7): (a) the time average on the interval  $\lim_{T \rightarrow \infty} \langle B \rangle_T = 0$ , and (b) the standard deviation on the interval  $\lim_{T \rightarrow \infty} \langle |B|^2 \rangle_T = 0$  for typical models [13] of the many-particle  $E$  system. Since  $\zeta < 1$ , equation (4.5) is proved, with simultaneous observation that  $t_{\circ}$  is of the order of ‘decoherence time’ denoted  $\tau_{\text{dec}}$  [13,15]. ■

Lemma 4.1 is formally equivalent with lemma 4.1(i) of Ref. [1]: For most of the large values of  $t_{\circ}$ , the r.h.s. of equation (4.7) is negligible (for  $\alpha \neq \gamma$ ) already for the time intervals of the order of ‘decoherence time’ with the unique steady state on the r.h.s. of equation (4.5) that is trivially divisible, while for arbitrarily large  $t_{\circ}$  there is unavoidable recurrence of the initial values [18, 27]. The larger the environment  $E$  the shorter ‘decoherence time’ and the longer the recurrence time-interval.

As distinct from lemma 4.1(i) of ref. [1], equations (4.4) and (4.5) reveal how works the subsystem’s dynamical map: divisibility (and therefore Markovianity) does not apply for arbitrary time instants, but [in contrast to equation (3.15)] for most of the sufficiently large values of  $t_{\circ}$ , again exhibiting a need for the coarse graining of ‘time axis’.

## (b) The approximate map

From both equation (3.21) and equation (4.2), while bearing in mind equation (4.1), i.e. the exchange  $P_m \rightarrow P_{\alpha} \Pi_{\beta}$ , the subsystem’s approximate map:

$$\rho_S(t_{\circ}) \approx \sum_{\alpha} P_{\alpha} \rho_S(0) P_{\alpha} + \sum_{\alpha} \left( \sum_{\beta} \exp\left(-\frac{it_{\circ} \delta_{\alpha\beta}}{\hbar}\right) \text{tr}_E \Pi_{\beta} \rho_E(0) \right) P_{\alpha} \rho_S(0) \Pi^{(\alpha)} + \text{h.c.} \quad (4.8)$$

where  $\Pi^{(\alpha)} = \sum_{\nu_{\alpha}} P_{\nu_{\alpha}}$ ,  $\delta_{\alpha\beta} \approx E_{\alpha\beta} - E_{\nu_{\alpha}\beta}$  and in analogy with equation (3.20):

$$\exp\left(-\frac{(E_{\alpha\beta} - E_{\gamma\beta})^2}{4\lambda\hbar^2}\right) \ll 1, \quad (4.9)$$

for certain set of the  $\gamma$  indices for every fixed  $\alpha$ . As well as equation (3.20), equation (4.9) is typically not applicable to the few-particle systems.

The subsystem’s projectors  $P_{\alpha}$  and  $\Pi^{(\alpha)}$  in equation (4.8) can be easily shown to satisfy the algebra established for the  $P_m$  and  $\Pi^{(m)}$  projectors, which appear in equation (3.21). The map is obviously linear, positive, trace preserving and unital while satisfying equation (3.7).

The sums over  $\beta$  in equation (4.8) are almost-periodic functions that are essentially discussed in the proof of lemma 4.1. Hence for the time intervals of the order of decoherence time, the approximate map equation (4.8) returns the (approximate) steady state established by lemma 4.1. Thereby equation (4.9) regards the coarse-grained eigenvalues (for certain choices of  $\alpha$  and  $\gamma$ ) of the ‘measured’ subsystem’s observable whose eigenprojectors are exactly the  $P_{\alpha}$ ’s appearing in equation (4.1) [28]. In the context of the decoherence theory, such observable is often referred to as ‘pointer observable’ [27].

As  $\rho_S(t_{\circ}) \approx \sum_{\alpha} P_{\alpha} \rho_S(0) P_{\alpha}$  for most of large values of  $t_{\circ}$ , one may tempt to think that the map acquires divisibility and also complete positivity in the same time interval. However, this is naive as we are going to demonstrate—the approximations useful for the quantum states need not be as useful for the system’s dynamical map. In the rest of this section we more precisely determine the lower time bound for ‘decoherence time’ for equation (4.8).

Bearing in mind that the algebra of the  $P_\alpha$  and  $\Pi^{(\alpha)}$  projectors is formally the same as for the  $P_m$  and  $\Pi^{(m)}$  projectors of §3b, it is now straightforward to prove, in analogy with lemma 3.5, that the subsystem's approximate map equation (4.8) is divisible for every  $t_\circ$ .

In regard of complete positivity, it is straightforward to repeat the proof of lemma 3.4 to obtain in analogy with equation (3.25) (we also use notational analogy):

$$\frac{1}{d} \sum_\alpha \left( \sum_{i=1}^{g_\alpha} p_i \right)^2 + \frac{1}{d} \sum_\alpha \left( 2 \sum_\beta (\text{tr}_E \Pi_\beta \rho_E(0)) \cos(\delta_{\alpha\beta} t_\circ) \right) \left( \sum_{i=1}^{g_\alpha} p_i \right) \left( \sum_{v_i=1}^{g^{(\alpha)}} p_{v_i} \right) \quad (4.10)$$

where  $d$ ,  $g_\alpha$  and  $g^{(\alpha)}$  refer to the open  $S$  system and hence take the much smaller values for the  $d$  and  $g$ 's in equation (4.10) than in equation (3.25). Noticing that the sum over  $\beta$ , which we denote  $\epsilon_\alpha(t_\circ)$ , in equation (4.10) is an almost periodic function, from equation (4.10) we obtain the estimate

$$\frac{1}{d} \sum_\alpha \left( \sum_{i=1}^{g_\alpha} p_i \right)^2 + 2\epsilon(t_\circ) g' \sum_\alpha \frac{\chi_\alpha}{g'd} \geq 0 \quad (4.11)$$

for sufficiently large  $t_\circ$ , in analogy with equation (3.26);  $\epsilon(t_\circ) = \max\{\epsilon_\alpha(t_\circ)\}$  and  $\sum_\alpha \chi_\alpha / g'd \leq 1$ . Now all the coefficients in equation (4.11) refer to the  $S$  system, not to the closed system of §3. The  $g'$  in equation (4.11) can be roughly estimated to be at least  $\min\{\text{tr} \Pi_\beta\}$ -times smaller than the  $g$  appearing in equation (3.25). Therefore, we conclude that the subsystem's approximate dynamical-map equation (4.8) dynamically acquires complete positivity *much faster* than the total (closed) system's approximate dynamical-map but with the time bound that is determined by  $g'$  and is therefore longer than the decoherence time in lemma 4.1 (for which, formally,  $g' = 1$ ).

Hence the dynamical emergence of the TCGM of the subsystem's approximate dynamical map with the time bound that is determined by the  $g'$  parameter.

## 5. Markovianity state-domains for closed systems: quantitative estimates

From §4 we can learn that open systems in a measurement-like (decoherence-like) interaction with their environments are Markovian. Therefore, in this section, we consider the closed many-particle systems.

Given  $\exp(-4) \approx 0.018 \ll 1$  and bearing in mind §2 from which  $\hbar\sqrt{\lambda} > \min\{2\Delta H/\pi, 2(\langle H \rangle - E_g)/\pi\}$ , the numerical condition

$$\frac{E_m - E_n}{\min\{2\Delta H/\pi, 2(\langle H \rangle - E_g)/\pi\}} \equiv \frac{\pi \delta_{mn}}{2 \min\{\Delta H, (\langle H \rangle - E_g)\}} > 4, \forall m \neq n \notin \{v_m\} \quad (5.1)$$

is necessary in order to satisfy equation (3.20).

From equation (5.1), we can recognize two domains of states that allow for Markovianity and one domain not allowing for Markovianity as follows.

Denoting  $E = E_{\max} - E_g$  and setting  $\Delta H = E/d$  and  $(\langle H \rangle - E_g) = E/r$ , equation (5.1) implies

$$\delta_{mn} \equiv \frac{E}{k(m)} > 2.55 \min \left\{ \frac{E}{d}, \frac{E}{r} \right\}, \quad \forall m \quad (5.2)$$

with the real  $k(m), d, r > 0$ . Hence, equation (5.2) reveals the coarsening dependence on the initial state via

$$k(m) < \frac{\max\{d, r\}}{2.55}, \quad \forall m, \quad (5.3)$$

which exhibits: increase of  $k(m)$  (less coarse-grained spectrum) implies the decrease of  $\Delta H$  or  $\langle H \rangle \rightarrow E_g$ . Also the estimates follow: in order to have the coarse-graining possible,  $k(m) > 1$ ,  $\forall m$ , is necessary and hence  $\Delta H < E/2.55k(m) < E/2.56 \approx 0.39E$  or  $\langle H \rangle = E_g + E/2.56 \approx E_g + 0.39E$  are *required*. For the initial states not fulfilling any of the constraints, equation (3.21) is not valid. However, Markovianity can be apparently restored for some large values of *both*  $\Delta H$  and  $(\langle H \rangle - E_g)$ , for which the exact state equation (3.3) may be practically indistinguishable from the

state equation (2.1), which pertains to the universal, global time. In between these ‘extremes’—small versus large values of  $\Delta H$  and  $(\langle H \rangle - E_g)$ —are the states for which Markovian dynamics equations (2.1) and (3.21) are not valid. For every  $k(m)$  the coarse graining interval  $[E_m, E_m + E/k(m)]$  uniquely determines the set of the energy eigenprojectors  $P_{v_m}$  and the parameters  $g^{(m)} = \sum_{v_m} \text{tr} P_{v_m}$ .

On the other hand, numerical values of  $k(m)$  determine the limits of validity of the condition  $\exp(-\delta_m^2/4\lambda\hbar^2) \approx 1$ , §3b, that is also necessary for equation (3.21) to be valid. Introduce the measurement error  $\delta E(m) = E/xk(m)$ , where  $x > 1$ . By definition,  $\delta_m = r\delta E_m$ ,  $r \gtrsim 1$ . Also by definition, and due to operational indistinguishability of the values from the interval  $[E_m, E_m + \delta E_m]$ ,  $\delta_m = (E/k(m) - (E_m + E/xk(m)))/s$ ,  $s \gtrsim 1$ . Then easily follows the constraint  $x = (rs + 1)/(1 - k(m)E_m/E)$ ; for simplicity, we omit dependence of  $x, r, s$  on  $m$ . For the positive non-zero  $E_m$ 's,  $k(m) < E/E_m$ , in order for  $x > 0$ .

To illustrate, consider a closed system of  $N \gg 1$  non-interacting spin- $\frac{1}{2}$  particles (that can act as a single-spin's environment [29]), with the Hamiltonian  $H = \hbar\omega_o \sum_{i=1}^N S_{iz}$  and the energy-scale  $C = \hbar\omega_o$ . Energy spectrum  $\pm(N - p)\hbar\omega_o/2$  with the degeneracy  $\binom{N}{p}$ ,  $p = 0, 1, 2, \dots, N$ , while  $E = N\hbar\omega_o$ . For the initial state satisfying  $\langle H \rangle = 0$ ,  $\langle H \rangle - E_g = -E_g = N\hbar\omega_o/2$ , i.e.  $r = 2$  (cf. equation (5.3)). The Markovianity constraint  $\Delta H < 0.39E$  can be satisfied already with  $d = 3$ , i.e. with  $\Delta H = N\hbar\omega_o/3$  since  $3/2.56 \approx 1.172 > k > 1$  satisfies equation (5.3). Then we can choose  $k = 1.71$  and  $\hbar\sqrt{\lambda} = 0.7E/\pi > (2/\pi) \min\{E/2, E/3\}$  and obtain for the Gaussian terms in equation (3.20) approximately to equal 0.18. The smallest value for  $x$  is for the ground energy  $E_g < 0$ , which gives rise to the largest measurement error  $\delta E = 1.08E/(rs + 1)$  and therefore the largest  $\exp(-\delta_m^2/4\lambda\hbar^2) = 0.798$  for  $r = 1$ ,  $s = 9$ . On the other hand, for  $\Delta H = 0.4E$  one obtains  $d = 2.5$  and hence  $k < 2.5/2.56 < 1$ , for which the coarse graining is not possible and therefore dynamics is non-Markovian. However, for sufficiently large  $\Delta H$  (small  $d$ ), we can obtain approximate validity of equation (2.1) as follows. For example, for the initial states  $|\psi_{\pm}\rangle = (|E_{\max}\rangle \pm |E_g\rangle)/\sqrt{2}$ ,  $d = r = 2$  and  $k < 2/2.56 < 1$ , which says that the coarse graining is excluded. However, equation (3.3) gives (with the choice  $\lambda = 1.1(E/\pi\hbar)^2$ ):

$$\begin{aligned} \sigma_{\pm}(t_o) \approx \frac{1}{2} \left[ |E_{\max}\rangle \langle E_{\max}| + |E_g\rangle \langle E_g| \pm 0.11 \exp\left(-\frac{tEt_o}{\hbar}\right) |E_{\max}\rangle \langle E_g| \right. \\ \left. \pm 0.11 \exp\left(\frac{tEt_o}{\hbar}\right) |E_g\rangle \langle E_{\max}| \right]. \end{aligned} \quad (5.4)$$

Then the fidelity [3]  $\mathcal{F} = +\sqrt{\langle \psi_{\pm}(t_o) | \sigma_{\pm}(t_o) | \psi_{\pm}(t_o) \rangle} \gtrsim 0.745, \forall t_o$ , which makes the two states interchangeable (almost indistinguishable) for certain practical purposes and thus apparently the unitary (Markovian) dynamics of the system.

Let us now consider a system of non-interacting harmonic oscillators (or uncoupled optical modes) with unique frequency  $\omega_o$  that is defined by the Hamiltonian  $H = \hbar\omega_o \sum_{k=1}^M (a_k^{\dagger} a_k + \frac{1}{2})$  and the characteristic energy-scale  $C = \hbar\omega_o$ , where appear the creation and annihilation Bose operators. The energy spectrum [in the physical units  $\hbar\omega_o = 1$ ]  $\nu + M/2 > 0$ , where  $\nu = 0, 1, 2, 3, \dots$  is an eigenvalue of the Hermitian number-operator  $N = \sum_{k=1}^M a_k^{\dagger} a_k$ ;  $N|\nu\rangle = \nu|\nu\rangle$ . Assuming a finite ‘cutoff’  $\nu_{\max}$ , the finite  $E = \nu_{\max}\hbar\omega_o$  and  $\langle H \rangle - E_g = \langle N \rangle \hbar\omega_o$ , while  $\Delta H = \Delta N \hbar\omega_o$ . For comparison with the spin system, we consider  $k = 1.71$  and  $\hbar\sqrt{\lambda} = 0.7E/\pi$ , which gives the same estimate for the Gaussians appearing in equation (3.20) as for the system of spins. However, due to the non-negative energies, for the system of oscillators, it is easy to detect the smallest  $x = rs + 1$  and the largest  $\delta E = 0.58/(rs + 1)$ . Therefore, the largest  $\exp(-\delta_m^2/4\lambda\hbar^2) = 0.936$  for  $r = 1, s = 9$ . For the initial state considered for the spins system, it is easy to obtain  $r = d = 2$ , equation (5.4) and the fidelity  $\mathcal{F} \gtrsim 0.745, \forall t_o$ .

Better satisfied the Markovianity conditions, equation (3.20) and  $\exp(-\delta_m^2/4\lambda\hbar^2) \approx 1$  pertain to the rather small  $\Delta H$  and  $\langle H \rangle - E_g$  that, in turn, refer to the rather small energy contents of the system. In such cases, the larger values for  $k(m)$  allow the smaller measurement errors as well as make equation (3.21) better satisfied.

Placing different ‘frequencies’  $\omega_i$  in the self-Hamiltonians, or introducing interactions in the system, lead to a rather dense energy spectrum for  $N \gg 1$ , thus leading to the continuous limit that is often used in the condensed-matter physics [30]. The analysis of Markovianity is essentially the same as presented above, depending on the (non)existence of the negative energy values.

Therefore, observation of (non)Markovian dynamics (e.g. via quantum process tomography [3]) of closed systems depends on the initial state, the structure of the energy spectrum and on the subtle relations between the above-introduced parameters,  $k, r, s$ . Every measurement that cannot resolve the system’s energies better than the error  $\delta E = \max\{\delta E_m\}$  may reveal non-Markovian dynamics. The only option for detecting non-Markovian dynamics of an open many-particle system is to perform measurements in the time intervals shorter than  $t_o$ , cf. lemma 4.1 and §4b.

While numerical details are model sensitive, the above-distinguished structure of the Markovianity state-domains is universal. Thus we learn about the novel, rich behaviour of closed many-particle systems: Depending on the system’s energy, an observer can in principle detect low-energy Markovian dynamics, the unitary-like (apparently unique time) Markovian behaviour for relatively high energies, and non-Markovian dynamics for the rest of quantum states. In classical terms, ‘high energy’ can be linked with high temperature, which is model-specific yet. Lemma 3.3 suggests that the observation requires interaction of the energy scale much larger than  $\hbar\omega_o$ , for both here regarded models.

## 6. Relation to the standard theory

Despite the apparently strange foundational character, the LTS dynamical map offers the following lessons regarding ensembles of many-particle systems.

First, in contrast to the standard wisdom [15–17], Markovian character of the dynamics is not unconditional. On the one hand, it appears in the coarse-graining description, §3b. *Per se* this is not entirely a new position—some kind of ‘coarse graining’ is often found a basis for quasi-classical behaviour of quantum systems [1,24,31,32]. To this end, the quantitative criteria (equations (3.20), (3.24) and (4.9)) set the new layer of considerations that could, at least in principle, be experimentally tested. On the other hand, complete positivity or divisibility and hence Markovianity of the map are dynamically established and ‘should ripen’. That is, lemmas 3.4 and 4.1 indicate that the *dynamical maps themselves are dynamic*.

In the standard theory, similar conclusions appear while bearing model dependence in the context of a narrower definition of Markovianity in a perturbative treatment (weak interaction) [15,33]. Non-validity of equation (3.7) in the standard theory implies that, for short times, there always exists the map inverse and therefore dynamics is Markovian [16,17]. However, results of §§3 and 4 are model independent, non-perturbative and point out *non-Markovianity for the arbitrarily short time interval* ( $0 < t < t_o$ ) that is unknown to the standard theory.

Second, the exact, non-Markovian dynamics equation (4.2) straightforwardly reproduces certain results known for a Markovian counterpart from the standard theory. To see this, consider an orthonormalized basis  $\{|m\rangle\}$  that is adapted to the orthogonal projectors  $P_\alpha$  and choose only a subset of states such that  $P_\alpha|m\rangle = |m\rangle$  implies  $P_\gamma|m\rangle = 0, \forall \gamma \neq \alpha$ . Then lemma 4.1 (which assumes strong interaction) implies decoherence:

$$\lim_{t_o \rightarrow \infty} \rho_{Smn}(t_o) = \lim_{t_o \rightarrow \infty} B_{mn}(t_o) \rho_{Smn}(0) = 0, \quad \forall m \neq n, \quad (6.1)$$

as a non-Markovian process.

On the other hand, in the standard theory of strong interaction (‘singular-coupling’ limit), equation (6.1) appears for *Markovian dynamics*. To see this, we refer to the Markovian master equation (3.159) of ref. [15] [ $\hbar = 1$ ]:

$$\dot{\rho}_S(t) = -i[H_S + H_{LS}, \rho_S(t)] + \sum_{\alpha, \beta} \gamma_{\alpha\beta} \left( A_\beta \rho_S(t) A_\alpha - \frac{1}{2} \{A_\alpha A_\beta, \rho_S(t)\} \right) \quad (6.2)$$

where the curly brackets denote the ‘anticommutator’.

In order to comply with our considerations, we neglect the commutator in equation (6.2) and due to the separable form of the ('pure decoherence') interaction equation (4.1), the commutators  $[A_\alpha, A_\beta] = 0, \forall \alpha, \beta$ . Choose a common eigenbasis,  $\{|m\rangle\}$ , for the set  $\{A_\alpha\}$ . Then equation (6.2) gives rise to the matrix elements,  $\rho_{Smn} \equiv \langle m|\rho_S|n\rangle$ , differential equation

$$\dot{\rho}_{Smn}(t) = -\frac{\Gamma_{mn}}{2} \rho_{Smn}(t), \quad (6.3)$$

where

$$\Gamma_{mn} = \sum_{\alpha, \beta} \gamma_{\alpha\beta} (a_{\alpha m} - a_{\alpha n})(a_{\beta m} - a_{\beta n}) \geq 0, \quad (6.4)$$

and  $a_{\alpha m}$  is the  $m$ th eigenvalue of  $A_\alpha$ . The proof of non-negativity of  $\Gamma_{mn}$ 's directly follows from the observation that  $\Gamma_{mn} = (\mathbf{v}_{mn}, \gamma \mathbf{v}_{mn})$ , where the real vector  $\mathbf{v}_{mn} = (a_{\alpha m} - a_{\alpha n})$  while  $\gamma = (\gamma_{\alpha\beta})$  is a positive semi-definite matrix and  $\Gamma_{mm} = 0$ . Hence from equation (6.3)

$$\rho_{Smn}(t) = \exp\left(-\frac{\Gamma_{mn}}{2} t\right) \rho_{Smn}(0), \quad (6.5)$$

in quantitative agreement with equation (6.1) for  $t_o \sim 2\Gamma_{mn}^{-1}$ .

Third, bearing in mind that the limit  $t_o \rightarrow \infty$ , lemma 4.1, typically regards the short time intervals of the 'decoherence time', equation (4.5) is not valid for arbitrarily long time intervals. Rather, there is recurrence of the initial values [18,27] and hence the repeating cycles of dynamical change of the map, which is presented by equation (4.5). For large number of constituent particles of the environment, this recurrence typically occurs for very long time interval and thus the repetition of the 'cycles' may be virtually unobservable.

Fourth, in the standard theory [15–17], existence and properties of steady state(s) of the reduced dynamics is a complicated topic and an active area of research. The averaged state (also known as 'ergodic average'), that is known to be a steady state for the CP semigroup (Markovian) dynamical maps [15–17], also appears to be steady for the both exact (*non-Markovian*) dynamics equation (3.3) and (4.2), of the form:

$$\bar{\sigma} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \sigma(t_o) dt_o = \sum_n P_n \rho(0) P_n. \quad (6.6)$$

While the proof of equation (6.6) is an elementary integration of equations (3.3) and (4.2), the steady state on the r.h.s. of equation (6.6) is *unique*.

Finally, equations (3.25), (4.7) and (4.11) reveal different time scales for the emergence of Markovianity. Equation (4.7) is well known from the decoherence theory and qualitatively defines 'decoherence time',  $\tau_{\text{dec}}$  [18,27]. On the other hand, equations (3.25) and (4.11) require much longer time intervals to reach the respective small values  $g^{-1}$  (which for equation (4.7) is of the order of 1). In the absence of the exact mathematical relation, we just note the chain of inequalities,  $1 \ll g' \ll g$ , that directly implies the chain of the respective time scales  $\tau_{\text{dec}} < \tau' < \tau$ . The inequality  $g' \ll g$ , emphasized in §3b, follows from

$$g = \max\{\text{tr } \Pi^{(m)}\} = \max \left\{ \sum'_{\alpha, \beta} \text{tr}_S P_\alpha \text{tr}_E \Pi_\beta \right\} \sim g' \max \left\{ \text{tr}_E \sum'_{\beta} \Pi_\beta \right\}, \quad (6.7)$$

(where the primed sums do not take all the possible values of the summation coefficients) while bearing in mind that  $\Pi_\beta$ 's are the environmental projectors, equation (4.1). From equation (6.7) directly follows: for a closed system of fixed size (unique value of  $\max\{\text{tr } \Pi^{(m)}\}$ ), smaller subsystem is monitored by larger environment and, therefore,  $g'$  decreases with the decrease in the size of the subsystem. Physically this means that, for the same total, closed system, the smaller the open system the faster is reached the steady state and the longer 'recurrence time'. Observations of the open system that are not limited by coarse-graining give rise to the fastest approach (with the rate  $\tau_{\text{dec}}^{-1}$ ) to the steady state.

## 7. Discussion

Dynamical map imposed by LTS is mathematically of a rather special kind that is here introduced for the first time. In certain points, LTS fits with, goes beyond or extends the results of the standard open systems theory. Thereby, §6 answers the first question on usefulness of LTS that is posed in §1. Within the scope of the new fundamental dynamical law equation (2.2)—which takes the place and the role of the Schrödinger law, equation (2.1), of the standard theory—the findings of §§3 and 4 are *generic* and *universal*. Here we specifically emphasize non-existence of the short-time Markovianity, the novel, rich behaviour of closed many-particle systems and that the scheme provides a long-sought answer to the problem of microscopic origin of the phenomenological rule, that the smaller open systems relax (in our case, attain the steady state) faster than the larger ones.

We conclude that dynamics imposed by LTS successfully though specifically tackles the so-far-investigated topics in the foundations of quantum theory. Nevertheless, there remains yet much to be done. We can detect at least the following four research directions. First, experimental tests of our findings are methodologically on the same ground as the tests of the occurrence of decoherence. To this end, a rather precise knowledge of the energy spectrum as well as sophisticated control of quantum states of many-particle systems are needed. Beyond these technical requirements, we do not see any obstacles to the more-or-less straightforward experimental tests of our results. Second, usefulness of LTS regarding the interpretational issues of quantum theory (e.g. of quantum measurement) should be separately investigated. In this regard, of particular interest is description of a *single* system dynamics (as distinct from the ensemble of systems considered so far). Third, implications for the physical nature of time allowed by LTS may contribute to the debate of whether time is fundamental or not and so whether or not the task of spacetime quantization makes any sense. Finally, in certain points of the foundational character, LTS concurs with a cosmological program that has recently been reviewed by Hartle [34]. Here we briefly emphasize certain essential points of the program that have clear counterparts within LTS while leaving details for the sequel.

The low entropy initial state of the Universe [34] is directly recognized within LTS: owing to the points (a–e) in appendix A, the early Universe is a single ‘local’ system that is subjected to unique local time and therefore described by equation (2.1) for pure, i.e. for the zero-entropy states. Subsequent dynamics of the Universe ‘brings today’s nearly isolated systems’ [34], which, according to LTS, are described by their own local times and mixed (non-zero entropy) states of the form of equation (2.2), i.e. of equation (3.1). Robustness to external noise is the characteristic trait of the decoherence-induced ‘quasiclassical’ domain that requires *sufficiently strong interaction* of a *many*-particle system with its environment in a *coarse-grained* description [34]. Sections 3 and 4 provide the basis for the requirements, which are above emphasized by italics. Finally, the conjecture ‘*A situation of local equilibrium will generally be reached before complete equilibrium is established, if it ever is.*’ [34] can be recognized to summarize lemmas 3.2 and 4.1 and the two final remarks in §6.

## 8. Conclusion

The so-far investigated mathematical aspects of LTS prove to be consistent and physically useful. The dynamical map imposed by the LTS is mathematically of a special kind that physically quantitatively fits with, goes beyond or extends the results of the standard open quantum systems theory and qualitatively tackles certain foundational issues in cosmology. The interpretational implications of the scheme, such as quantum measurement problem and the transition from quantum to classical, are yet to be investigated.

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## Appendix A

In LTS, the primitive is *dynamics*, which is simply a map between the system's states:

$$\rho_1 \rightarrow \rho_2 \rightarrow \rho_3 \rightarrow \dots \tag{A 1}$$

Along the dynamical chain equation (A 1), the system can be subjected to changes of its local time, which can be shared by some other systems that, as a whole, constitute an at least approximately closed system subjected to the unitary Schrödinger dynamics. For a single closed system, the states are pure ( $\rho_i^2 = \rho_i$ ) and the chain equation (A 1) is a 'history', which assumes unique local time [35]. Therefore, in a composite system, such as the Universe itself, there may be more than one such closed ('local') system described by its own local time. Hence a composite system is described solely by the composite system's Hamiltonian, which defines (dynamically changing) distribution of local systems and their local times without 'history' in the standard sense [35].

The rules for determining local time are as follows [1]: (a) systems with different Hamiltonians such as those with a different numbers of particles, different kinds of particles or different kinds of interactions between the particles are subject to different local times; (b) systems that mutually interact are subjected to the same time; (c) non-interacting systems need not have a common time; (d) non-identical many-body systems which do not interact and locally follow independent Schrödinger dynamics do not have a common time—which makes the universal time undefinable; (e) local time refer even to the mutually identical many-body systems, as long as they represent the mutually independent local systems.

These rules constitute the foundational basis for LTS. Everything one might need is 'inscribed' in the Hamiltonian since, along the dynamical chain equation (A 1), the average value of every term of the Hamiltonian and the energy-scale  $C$  are uniquely determined. For a closed system, unique Hamiltonian determines unique energy scale  $C$ . On the other hand, interaction may change the energy scale for the total system's Hamiltonian. If interaction for a pair of closed systems is weak, the systems are approximately independent and approximately subjected to independent local times. This should be sharply distinguished from the case of 'weakly interacting systems' that are subsystems of a local (approximately closed) system; such cases are considered in the context of Markovian dynamics in the standard open systems theory [15–18]. The other 'extreme' is the sufficiently strong interaction that defines the formation of a new composite system and the start ( $t = 0$ ) of the new local time ticking; such situations are considered in lemma 3.3 and §4. In between the two 'extremes' are the cases for which the interaction does not dominate, neither is sufficiently weak.

For a sub-chain  $\rho_i \rightarrow \rho_{i+1} \rightarrow \rho_{i+2} \rightarrow \dots$  of the dynamical chain equation (A 1), we say to be of *fixed energy scale* if the respective energy scales,  $C_i, C_{i+1}, C_{i+2}, \dots$ , are approximately equal, i.e. if  $C_i \approx C_{i+1} \approx C_{i+2} \dots = C$ .

The LTS does not provide quantitative *kinematical* criteria for 'weak' versus 'sufficiently strong' interaction. Therefore, the following question is in order: are there any realistic (at least approximately) closed many-particle systems and how can they be recognized in the realistic physical situations? This concern raised in §2 is, in our opinion, purely operational.

Non-existence of at least approximately closed systems as described above would be fatal for most of the physical theories as well as for interpreting certain experimental tests. For the whole of the decoherence and measurement theory, as well as for the foundations of the open systems theory, validity of the Schrödinger law equation (2.1) is essential and is implemented by 'object of measurement + apparatus (+ the apparatus' environment)' as well as by '[open]system + environment' [15–18]. Approximate isolation from the rest of the Universe is indispensable also for the appearance of the energy band-structure of electrons in crystals [30], for the so-called alpha-clustering in nuclear physics [36] (and references therein), 'entanglement renormalization'

[37], as well as for existence of quasiclassical realms in cosmology [34] etc. Therefore, we do not regard the above-posed question foundational but rather of the more-or-less practical, technical nature that should be separately answered for every concrete physical situation.

The second concern raised in §2 refers to the sharp *dynamical* transition from the ‘weak’ to ‘strong’ interaction of the initially independent local systems. For the unsharp transition, the initial instant  $t=0$  of the common local time for interacting systems *a priori*, i.e. independently of lemma 3.3 (which concerns the fixed energy scales), is not well defined. Nevertheless, this does not produce any problem as long as dynamics equation (A 1) can be ‘coarse grained’ in order to provide a ‘quick’ dynamical transition from the weak to the sufficiently strong interaction.

Consider a dynamical chain

$$\rho_i \rightarrow \rho_{i+1} \rightarrow \cdots \rho_{i+k} \rightarrow \cdots \rho_p \rightarrow \rho_{p+1} \rightarrow \cdots \rho_{p+q} \quad (\text{A } 2)$$

where the sub-chains  $\rho_i \rightarrow \rho_{i+1} \rightarrow \cdots \rho_{i+k}$  and  $\rho_p \rightarrow \rho_{p+1} \rightarrow \cdots \rho_{p+q}$  are with the fixed energy scales,  $c$  for the self-Hamiltonian and  $C$  for the interaction term, respectively. Then reducing the sub-chain  $\rho_{i+k} \rightarrow \cdots \rho_p$  to arbitrary *single state* of the sub-chain is the dynamical coarse graining. By definition, this coarse graining assumes *practical indistinguishability* of certain dynamically close states and is finer than and embedded in the dynamical coarse graining necessary for the dynamical appearance of Markovianity, §§3 and 4.

Needless to say, the coarse graining is equally relevant for the strongly interacting systems in the standard theory of global time: indeed, it is not *a priori* clear how quickly the interaction becomes sufficiently strong and whether or not the initial ensemble is really pure as assumed by equation (2.1). The plausible answer provided below lemma 3.3, that is  $U(\delta t) \approx I$ , is encompassed by the state coarse graining; one can find an attempt which goes beyond this approximation, e.g. in [38]. Therefore, the dynamical coarse graining generally appears to be as necessary as it is of the operational nature and thus not producing any problems in the foundations and applications of LTS.

## References

- Jeknić-Dugić J, Arsenijević M, Dugić M. 2014 A local-time-induced pointer basis. *Proc. R. Soc. A* **470**, 20140283. (doi:10.1098/rspa.2014.0283)
- Kitada H. 1994 Theory of local times. *Il Nuovo Cim.* **109B**, 281–302. (doi:10.1007/BF02727290)
- Nielsen MA, Chuang IL. 2000 *Quantum computation and quantum information*. Cambridge, UK: Cambridge University Press.
- Milburn GJ. 1997 *Schrodinger’s machines: the quantum technology reshaping everyday life*. New York: W. H. Freeman.
- Ramsden J. 2011 *Nanotechnology: an introduction (micro and nano technologies)*. Amsterdam, The Netherlands: Elsevier.
- Durr D, Goldstein S, Zanghi N. 2013 *Quantum physics without quantum philosophy*. Berlin, Germany: Springer.
- Wallace D. 2012 *Emergent multiverse: quantum theory according to the Everett interpretation*. Oxford, UK: Oxford University Press.
- Kastner RE. 2012 *The transactional interpretation of quantum mechanics: the reality of possibility*. New York, NY: Cambridge University Press.
- Fuchs CA, David Mermin N, Schack R. 2014 An introduction to QBism with an application to the locality of quantum mechanics. *Am. J. Phys.* **82**, 749–754. (doi:10.1119/1.4874855)
- Saunders S, Barrett J, Kent A, Wallace D (eds). 2010 *Many worlds: everett, quantum theory, and reality*. Oxford, UK: Oxford University Press.
- Brunner N, Gühne O, Huber M (eds). 2014 A special issue on 50 years of Bell’s theorem. *J. Phys. A Math. Theor.* **47**, 424009. (doi:10.1088/1751-8113/47/42/420301)
- First iWorkshop on the Meaning of the Wave Function. See <http://www.ijqf.org/groups-2/meaning-of-the-wave-function/forum/>.
- John Bell Workshop. 2014 See <http://www.ijqf.org/groups-2/bells-theorem/forum/>.
- Quantum Foundations Workshop. 2015 See <http://www.ijqf.org/groups-2/quantum-foundations-workshop-2015/forum/>.

15. Breuer H-P, Petruccione F. 2002 *The theory of open quantum systems*. Oxford, UK: Oxford University Press.
16. Rivas Á, Huelga SF. 2011 *Open quantum systems: an introduction*. Berlin, Germany: Springer.
17. Rivas Á, Huelga SF, Plenio MB. 2014 Quantum non-Markovianity: characterization, quantification and detection. *Rep. Prog. Phys.* **77**, 094001. (doi:10.1088/0034-4885/77/9/094001)
18. Joos E, Zeh H-D, Kiefer C, Giulini DJW, Kupsch J, Stamatescu I-O. 2003 *Decoherence and the appearance of a classical world in quantum theory*, 2nd edn. Berlin: Springer.
19. Petrat S, Tumulka R. 2014 Multi-time Schrödinger equations cannot contain interaction potentials. *J. Math. Phys.* **55**, 032302. (doi:10.1063/1.4867524)
20. Margolus N, Levitin LB. 1999 The maximum speed of dynamical evolution. *Phys. D* **120**, 188–195. (doi:10.1016/S0167-2789(98)00054-2)
21. Dugić M, Ćirković MM. 2002 Quantum information processing: the case of vanishing interaction energy. *Phys. Lett. A* **302**, 291–298. (doi:10.1016/S0375-9601(02)01198-2)
22. Audenaert KMR, Scheel S. 2008 On random unitary channels. *New J. Phys.* **10**, 023011. (doi:10.1088/1367-2630/10/2/023011)
23. Jamiolkowski A. 1972 Linear transformations which preserve trace and positive semidefiniteness of operators. *Rep. Math. Phys.* **3**, 275–278. (doi:10.1016/0034-4877(72)90011-0)
24. Jeong H, Lim Y, Kim MS. 2014 Coarsening measurement references and the quantum-to-classical transition. *Phys. Rev. Lett.* **112**, 010402. (doi:10.1103/PhysRevLett.112.010402)
25. Kac M. 1943 On the distribution of values of trigonometric sums with linearly independent frequencies. *Am. J. Math.* **65**, 609–615. (doi:10.2307/2371869)
26. Besicovitch AS. 1963 *Almost periodic functions*. Dover: Cambridge University Press.
27. Zurek WH. 1982 Environment-induced superselection rules. *Phys. Rev. D* **26**, 1862–1880. (doi:10.1103/PhysRevD.26.1862)
28. Dugić M. 1997 On diagonalization of a composite-system observable. Separability. *Phys. Scr.* **56**, 560–565. (doi:10.1088/0031-8949/56/6/003)
29. Zurek WH. 1981 Pointer basis of quantum apparatus: into what mixture does the wave packet collapse? *Phys. Rev. D* **24**, 1516–1525. (doi:10.1103/PhysRevD.24.1516)
30. Kittel C. 2004 *Introduction to solid state physics*, 8th edn. New York: Wiley.
31. Von Neumann J. 1955 *Mathematical foundations of quantum mechanics*. Princeton, NJ: Princeton University Press.
32. Omnés R. 1994 *The interpretation of quantum mechanics*. Princeton, NJ: Princeton University Press.
33. Breuer HP, Kappler B, Petruccione F. 2001 The time-convolutionless projection operator technique in the quantum theory of dissipation and decoherence. *Ann. Phys. (NY)* **291**, 36–70. (doi:10.1006/aphy.2001.6152)
34. Hartle J. 2010 Quasiclassical realms, In *Many Worlds? Everett, quantum theory, and reality* (ed S Saunders), p. 73. Oxford, UK: Oxford University Press.
35. Griffiths RB. 2002 *Consistent quantum theory*. Cambridge, UK: Cambridge University Press.
36. Kanada-En'yo Y. 2015 Analysis of delocalization of clusters in linear-chain alpha-cluster states with entanglement entropy. *Phys. Rev. C* **91**, 034303. (doi:10.1103/PhysRevC.91.034303)
37. Evenbly G, Vidal G. 2014 Real-space decoupling transformation for quantum many-body systems. *Phys. Rev. Lett.* **112**, 220502. (doi:10.1103/PhysRevLett.112.220502)
38. Bonifacio R, Olivares S, Tombesi P, Vitali D. 2000 A model independent approach to non-dissipative decoherence. *Phys. Rev. A* **61**, 053802. (doi:10.1103/PhysRevA.61.053802)