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3 authors:

Milica Grbović
University of Kragujevac

Kazim Ilarslan
Kirikkale University

Emilija Nesovic
University of Kragujevac

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On null and pseudo null Mannheim curves in Minkowski 3-space

Milica Grbović, Kazım İlarslan and Emilija Nešović

Abstract. In this paper, we prove that there are no null Mannheim curves in Minkowski 3-space. We also prove that the only pseudo null Mannheim curves in Minkowski 3-space are pseudo null straight lines and pseudo null circles whose Mannheim partner curves are pseudo null straight lines. Finally, we give some examples of pseudo null Mannheim curves in E_1^3.

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1. Introduction

In the Euclidean space E^3 a regular smooth curve α is called *Mannheim curve*, if there exist another regular smooth curve α^* such that at the corresponding points of the curves, the principal normal lines of α coincide with the binormal lines of α^*, under bijection $\Phi : \alpha \to \alpha^*$ [2]. Then α^* is called a *Mannheim mate curve* (partner curve) of α and $\{\alpha, \alpha^*\}$ is called a *Mannheim pair* of curves. It is well-known that α is a Mannheim curve in E^3 if and only if its the first and the second curvature satisfy the equality [2]

$$\kappa_1 = a(\kappa_1^2 + \kappa_2^2),$$

for some positive constant a. In Euclidean spaces, Mannheim curves are characterized in [3,6,7] and [8]. In Minkowski spaces, non-null Mannheim partner curves with non-null principal normals are studied in [1] and [6]. Null Mannheim curves in Minkowski 3-space are characterized in [5] and [9].

In this paper, we prove that there are no null Mannheim curves in Minkowski 3-space. Hence characterizations of null Mannheim curves given in [5] and [9] are not valid. We also prove that the only pseudo null Mannheim curves in Minkowski 3-space are pseudo null straight lines and pseudo null circles whose

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Mannheim partner curves are pseudo null straight lines. Finally, we give some examples of pseudo null Mannheim curves in E_1^3.

2. Preliminaries

The Minkowski 3-space E_1^3 is the Euclidean 3-space equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Recall that a vector $v \neq 0$ in E_1^3 can be *spacelike* if $g(v, v) > 0$, *timelike* if $g(v, v) < 0$ and *null (lightlike)* if $g(v, v) = 0$. In particular, the vector $v = 0$ is a spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v, v)|}$. Two vectors v and w are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in E_1^3, can locally be *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null [4]. Spacelike curve in E_1^3 is called *pseudo null curve* if its principal normal vector N is null [10]. A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$. A non-null curve α is parameterized by arc-length if $g(\alpha'(s), \alpha'(s)) = \pm 1$. A *circle* in E_1^3 is a planar curve with non-zero constant first curvature. Let $\{T, N, B\}$ be the moving Frenet frame along a curve α in E_1^3, consisting of the tangent, the principal normal and the binormal vector field respectively. Depending on the causal character of α, the Frenet equations have the following forms.

Case I. If α is a null curve, the Frenet equations are given by [10]

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ \kappa_2 & 0 & -\kappa_1 \\ 0 & -\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.1}$$

where the first curvature $\kappa_1 = 0$ if α is straight line, or $\kappa_1 = 1$ in all other cases. In particular, the following conditions hold:

$$g(T, T) = g(B, B) = g(T, N) = g(N, B) = 0,\ g(N, N) = g(T, B) = 1. \tag{2.2}$$

Case II. If α is pseudo null curve, the Frenet formulas have the form [10]

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ 0 & \kappa_2 & 0 \\ -\kappa_1 & 0 & -\kappa_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.3}$$

where the first curvature $\kappa_1 = 0$ if α is straight line, or $\kappa_1 = 1$ in all other cases. In this case, the following conditions are satisfied:

$$g(T, T) = g(N, B) = 1,\ g(N, N) = g(B, B) = g(T, N) = g(T, B) = 0. \tag{2.4}$$

Case III. If α is a non-null curve with spacelike binormal vector field B, the Frenet equations read [4]

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & -\epsilon_0\kappa_1 & 0 \\ -\epsilon_0\kappa_1 & 0 & \kappa_2 \\ 0 & \epsilon_0\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.5}$$

where κ_1 and κ_2 are the first and the second curvature of the curve respectively. Moreover, the following conditions hold

$$g(T,T) = -g(N,N) = \epsilon_0 = \pm 1, \quad g(B,B) = 1, \tag{2.6}$$

$$g(T,N) = g(T,B) = g(N,B) = 0. \tag{2.7}$$

3. Null and pseudo null Mannheim curves in $\mathbf{E}_1^3$

In this section we prove that there are no null Mannheim curves in Minkowski 3-space. We also prove that the only pseudo null Mannheim curves in E_1^3 are pseudo null straight lines and pseudo null circles whose Mannheim partner curves are pseudo null straight lines. In Minkowski 3-space, Mannheim curves are defined analogously as in the Euclidean 3-space. Throughout this section, we will use "dot" to denote the derivative with respect to the arc-length or pseudo-arc parameter of the curve.

Theorem 3.1. *There are no null Mannheim curves in Minkowski space E_1^3.*

Proof. Let α be a null curve in E_1^3. Assume that there exists another curve α^* in E_1^3 such that at the corresponding points of the curves the principal normal vector N of α is collinear with binormal vector B^* of α^*. Then B^* is spacelike vector and hence α^* is a timelike or a spacelike curve whose Frenet frame satisfy Frenet equations (2.5). Since $\{\alpha, \alpha^*\}$ is a Mannheim pair of curves, it follows that

$$\alpha(s^*) = \alpha^*(s^*) + \lambda(s^*)B^*(s^*), \tag{3.1}$$

where s^* is arc-length parameter of α^* and $\lambda \neq 0$ is some differentiable function. Denote by s pseudo-arc parameter of α. We may distinguish two cases: (A) $\kappa_2^*(s^*) = 0$ and (B) $\kappa_2^*(s^*) \neq 0$.

(A) If $\kappa_2^*(s^*) = 0$, differentiating relation (3.1) with respect to s^* and applying (2.5) we find $T\frac{ds}{ds^*} = T^* + \dot{\lambda}B^*$. By taking the scalar product of the last equation with N and using (2.2), (2.6) and (2.7) we get $\dot{\lambda} = 0$. This implies that null vector T is collinear with non-null vector T^*, which is impossible.

(B) If $\kappa_2^*(s^*) \neq 0$, differentiating relation (3.1) with respect to s^* and applying (2.5), we obtain

$$T\frac{ds}{ds^*} = T^* + \dot{\lambda}B^* + \lambda\epsilon_0\kappa_2^* N^*. \tag{3.2}$$

By taking the scalar product of the last equation with N and using (2.2),(2.6) and (2.7) we get

$$\dot{\lambda} = 0, \tag{3.3}$$

which means that $\lambda = \text{constant} \neq 0$. Substituting (3.3) in (3.2) we find

$$T\frac{ds}{ds^*} = T^* + \lambda\epsilon_0\kappa_2^* N^*. \tag{3.4}$$

By using (3.4) and the conditions (2.2), (2.6), (2.7) it follows that

$$g\left(T\frac{ds}{ds^*},T\frac{ds}{ds^*}\right)=g(T^*,T^*)-\lambda^2\kappa_2^{*2}g(N^*,N^*)=\epsilon_0-\epsilon_0\lambda^2\kappa_2^{*2}=0.$$

The last relation implies

$$\kappa_2^*=\pm\frac{1}{\lambda}=constant\neq 0. \tag{3.5}$$

Since α is a null curve, we may distinguish two cases: (B.1) $\kappa_1(s)=0$ and (B.2) $\kappa_1(s)=1$.

(B.1) If $\kappa_1(s)=0$ then $T(s)$ =constant. By taking the derivative of the relation (3.4) with respect to s^* and using (2.5) and (3.5) we find

$$T\frac{d^2s}{ds^{*2}}=-\epsilon_0\kappa_1^*N^*\mp\kappa_1^*T^*\pm\epsilon_0\kappa_2^*B^*.$$

By using the last relation, (2.2),(2.6) and (2.7) we get

$$g\left(T\frac{d^2s}{ds^{*2}},T\frac{d^2s}{ds^{*2}}\right)=-\epsilon_0\kappa_1^{*2}+\epsilon_0\kappa_1^{*2}+\kappa_2^{*2}=0.$$

Hence $\kappa_2^*=0$ which is a contradiction with (3.5).

(B.2) If $\kappa_1(s)=1$, substituting (3.5) in (3.4) we obtain

$$T\frac{ds}{ds^*}=T^*\pm\epsilon_0N^*.$$

Differentiating the last equation with respect to s^* and applying (2.1) and (2.5), we find

$$N\left(\frac{ds}{ds^*}\right)^2+T\frac{d^2s}{ds^{*2}}=-\epsilon_0\kappa_1^*N^*\mp\kappa_1^*T^*\pm\epsilon_0\kappa_2^*B^*. \tag{3.6}$$

Consequently, relations (2.2),(2.6),(3.5) and (3.6) imply

$$g\left(N\left(\frac{ds}{ds^*}\right)^2+T\frac{d^2s}{ds^{*2}},N\left(\frac{ds}{ds^*}\right)^2+T\frac{d^2s}{ds^{*2}}\right)$$
$$=\left(\frac{ds}{ds^*}\right)^4=\kappa_2^{*2}=constant\neq 0. \tag{3.7}$$

By taking the scalar product of (3.6) with N^* and using (2.6) (2.7) and (3.7), we get $\kappa_1^*=0$ which implies $\kappa_2^*=0$. This is a contradiction with (3.5), which completes the proof of the theorem. □

Theorem 3.2. *Let α be pseudo null curve and α^* arbitrary curve in E_1^3. If $\{\alpha,\alpha^*\}$ is a Mannheim pair of curves then one of the following statements hold:*

(i) α and α^* are two parallel pseudo null straight lines;
(ii) α is a pseudo null circle and α^* is a pseudo null straight line.

Proof. Let α be a pseudo null curve in E_1^3. Assume that there exists another curve α^* such that at the corresponding points of the curves, the principal normal vector N of α is collinear with binormal vector B^* of α^*. Then B^* is a null vector which implies that α^* is pseudo null curve or null curve. We consider these two cases separately.

(A) Assume that α^* is a pseudo null curve. Since $\{\alpha, \alpha^*\}$ is a Mannheim pair of curves, it follows that

$$\alpha^*(s) = \alpha(s) + \mu(s)N(s), \tag{3.8}$$

where s is arc-length parameter of α and $\mu \neq 0$ is some differentiable function. Differentiating relation (3.8) with respect to s and applying (2.3) we obtain

$$\dot{\alpha^*} = T + (\dot{\mu} + \kappa_2\mu)N. \tag{3.9}$$

By using the last relation, it follows that $g(\dot{\alpha^*}, \dot{\alpha^*}) = 1$. Hence $\alpha^* = T^*$ so relation (3.9) becomes

$$T^* = T + (\dot{\mu} + \kappa_2\mu)N. \tag{3.10}$$

Now we may distinguish two subcases: (A.1) $\kappa_2(s) = 0$ and (A.2) $\kappa_2(s) \neq 0$.

(A.1) If $\kappa_2(s) = 0$, relation (3.10) becomes $T^* = T + \dot{\mu}N$. By taking the scalar product of the last equation with T and using (2.4), we find

$$g(T, T^*) = 1. \tag{3.11}$$

Since α is a pseudo null curve, we may distinguish two subcases: (A.1.1) $\kappa_1(s) = 0$ and (A.1.2) $\kappa_1(s) = 1$.

(A.1.1) If $\kappa_1(s) = 0$, differentiating the equation $g(T, B^*) = 0$ with respect to s and applying (2.3), we obtain $g(T, -\kappa_1^* T^* - \kappa_2^* B^*) = 0$. By using (3.11) it follows that $\kappa_1^* = 0$, which means that α^* is a straight line. Since N^* and B^* are two linearly independent null vectors and $N = \pm B^*$, it follows that that $B = \pm N^*$. Therefore, $T = \pm T^*$ which means that straight lines α and α^* are parallel. This proves statement (i).

(A.1.2) If $\kappa_1(s) = 1$, the curve α is a pseudo null circle lying in the lightlike plane of E_1^3. By taking the derivative of the equation $g(T, B^*) = 0$ with respect to s and applying (2.3), we obtain

$$g(N, B^*) + g(T, -\kappa_1^* T^* - \kappa_2^* B^*) = 0. \tag{3.12}$$

By using (2.4) and (3.11) it follows that $\kappa_1^* = 0$, which means that α^* is a pseudo null straight line. This proves statement (ii).

(A.2) If $\kappa_2(s) \neq 0$, by taking the scalar product of (3.10) with T and using (2.4) we obtain that (3.11) holds. Differentiating relation $g(T, B^*) = 0$ with respect to s and using (2.4) and (3.11), we get $\kappa_1^* = 0$. Thus $B^* =$ constant and hence $\dot{N} = 0$. Then relation (2.3) implies $\kappa_2(s) = 0$, which is a contradiction.

(B) Assume that α^* is a null curve. Since $\{\alpha, \alpha^*\}$ is a Mannheim pair of curves, there holds

$$\alpha(s^*) = \alpha^*(s^*) + \mu(s^*)B^*(s^*), \tag{3.13}$$

where s^* is pseudo-arc parameter of α^* and $\mu \neq 0$ is some differentiable function. Denote by s arc-length parameter of α. Then we may distinguish two subcases: (B.1) $\kappa_1^*(s^*) = 0$ and (B.2) $\kappa_1^*(s^*) = 1$.

(B.1) If $\kappa_1^*(s^*) = 0$, differentiating relation (3.13) with respect to s^* and applying (2.1) we obtain $T\frac{ds}{ds^*} = T^* + \dot{\mu}B^*$. By taking the scalar product of the last equation with N and using (2.2) and (2.4), it follows that $g(T^*, B^*) = 0$, which is a contradiction with (2.2).

(B.2) If $\kappa_1^*(s^*) = 1$, differentiating relation (3.13) with respect to s^* and applying (2.1) we obtain $T\frac{ds}{ds^*} = T^* + \dot{\mu}B^* - \mu\kappa_2^*N^*$. By taking the scalar product of the last equation with N and using (2.2) and (2.4), we get $g(T^*, N) = g(T^*, B^*) = 0$, which is a contradiction with (2.2). This completes the proof of the theorem. □

Example 1. Consider two parallel pseudo null straight lines in E_1^3 with parameter equations $\alpha(s) = (1, 1, s)$, $\alpha^*(s) = (\frac{1}{2}, \frac{1}{2}, s)$ and Frenet frames $T = T^* = (0, 0, 1)$, $N = B^* = (1, 1, 0)$, $B = N^* = (-\frac{1}{2}, \frac{1}{2}, 0)$. It can be easily verified that $\alpha^* = \alpha - \frac{1}{2}N$. Consequently, $\{\alpha, \alpha^*\}$ is a Mannheim pair of curves.

Example 2. Consider pseudo null circle given by $\alpha(s) = (\frac{s^2}{2}, \frac{s^2}{2}, s)$ and pseudo null straight line with parameter equation $\alpha^*(s) = (c, c, s)$, $c \in R$ in E_1^3. The corresponding Frenet frames of α and α^* read $T = (s, s, 1)$, $T^* = (0, 0, 1)$, $N = B^* = (1, 1, 0)$, $B = N^* = (-\frac{1}{2}, \frac{1}{2}, 0)$. Moreover, $\alpha^* = \alpha + (c - \frac{s^2}{2})N$ which means that $\{\alpha, \alpha^*\}$ is a Mannheim pair of curves.

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Milica Grbović
Faculty of Science
Radoja Domanovića 12
34000 Kragujevac
Serbia
e-mail: milica_grbovic@yahoo.com

Kazım İlarslan
Department of mathematics
Faculty of Sciences and Arts
Kırıkkale
Turkey
e-mail: kilarslan@yahoo.com

Emilija Nešović
Faculty of Science
Radoja Domanovića 12
34000 Kragujevac
Serbia
e-mail: emines@ptt.rs

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