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Some Inequalities for the Forgotten Topological Index

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Abstract

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. Let d_i be the degree of its vertex i and $d(e_i)$ the degree of its edge e_i. We consider the recently introduced degree–based graph invariants: the forgotten index $F = \sum_{i\in V} d_i^3$, the hyper–Zagreb index $HM = \sum_{i\sim j}(d_i + d_j)^2$, and the reformulated first Zagreb index $EM_1 = \sum_{e_i\in E} d(e_i)^2$. A number of lower and upper bounds for F, HM, and EM_1 are established, and the equality cases determined.

Key Words: degree (of vertex), degree (of edge), Zagreb index, Forgotten index, Hyper–Zagreb index, Reformulated Zagreb index

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1 Introduction

Let $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, $E = \{e_1, e_2, \ldots, e_m\}$, be a simple connected graph with n vertices and m edges. Denote by $d_1 \geq d_2 \geq \cdots \geq d_n > 0$ and $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m) > 0$ the sequences of vertex and edge degrees of G, respectively. In addition, we use the following notation: $\Delta = d_1$, $\delta = d_n$, $\Delta_e = d(e_1) + 2$, $\Delta_{e_2} = d(e_2) + 2$, $\delta_e = d(e_m) + 2$, $\delta_{e_2} = d(e_{m-1}) + 2$. If the vertices i and j are adjacent, then we write

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$i \sim j$. If the edges e_i and e_j are incident, then we write $e_i \sim e_j$. As usual, $L(G)$ denotes the line graph of G.

In the 1970s, two degree–based topological indices were introduces [10], nowadays referred to as the first and the second Zagreb index, M_1 and M_2. These are defined as

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2$$

and

$$M_2 = M_2(G) = \sum_{i \sim j} d_i \, d_j \, .$$

Note that the first Zagreb index satisfies the identities

$$M_1 = \sum_{i \sim j} (d_i + d_j) = \sum_{i=1}^{m} [d(e_i) + 2] \, .$$

Details of the mathematical theory of Zagreb indices can be found in [3, 7–9].

Recently [11], a graph invariant similar to M_1 came into the focus of attention, defined as

$$F = F(G) = \sum_{i=1}^{n} d_i^3$$

which for historical reasons [7] was named *forgotten* topological index. It satisfies the identities

$$F = \sum_{i \sim j} (d_i^2 + d_j^2) = \sum_{i=1}^{m} [d(e_i) + 2]^2 - 2M_2 \, . \tag{1}$$

A further degree–based graph invariant was introduced in [20], and named hyper–Zagreb index, HM. It is defined as

$$HM = HM(G) = \sum_{i \sim j} (d_i + d_j)^2$$

and satisfies

$$HM = \sum_{i=1}^{m} [d(e_i) + 2]^2 \, .$$

However, HM can hardly be recognized as a new invariant. Namely, according to (1), the following immediate equality is valid [2]

$$HM = F + 2M_2\,.$$

In analogy with the first Zagreb index, by replacing vertex degrees by edge degrees, a so-called "reformulated first Zagreb index" EM_1 has been conceived as [14]

$$EM_1 = EM_1(G) = \sum_{i=1}^{m} d(e_i)^2 = \sum_{e_i \sim e_j} [d(e_i) + d(e_j)]\,.$$

In this paper, we are concerned with bounds for forgotten index. Then, we use the results obtained to establish upper and lower bounds for the invariants EM_1 and HM.

2 Preliminaries

In this section we outline some results for the invariants F, EM_1, and HM that will be needed in our subsequent consideration.

In [23], Zhou and Trinajstić proved the following equality which establishes a connection between EM_1, F, M_2, and M_1:

$$EM_1 = F + 2M_2 - 4M_1 + 4m\,. \tag{2}$$

In [12], Ilić and Zhou proved that

$$F \geq \frac{nM_1}{m} \tag{3}$$

with equality if and only G is regular.

Two of the present authors [11] proved that the following inequalities are valid

$$F \geq \frac{M_1^2}{2m} \tag{4}$$

and

$$F \geq \frac{M_1^2}{m} - 2M_2 \tag{5}$$

with equality in (4) if and only if G is regular, and in (5) if and only if $L(G)$ is regular. Let us note that (5) was also proved in [6] but in the form

$$HM \geq \frac{M_1^2}{m}\,. \tag{6}$$

Based on the relations (2), (4), and (5), the following can be easily obtained

$$EM_1 \geq \frac{M_1^2}{2m} + 2M_2 - 4M_1 + 4m \tag{7}$$

and

$$EM_1 \geq \frac{M_1^2}{m} - 4M_1 + 4m \tag{8}$$

which were, respectively, proven in [15] and [4].

For the invariants F and EM_1, the following was proven in [12]:

$$F \leq (\Delta + \delta)M_1 - 2m\Delta\delta \tag{9}$$

and

$$EM_1 \leq (\Delta + \delta - 4)M_1 + 2M_2 - 2m\Delta\delta + 4m \tag{10}$$

with equalities if and only if G is a regular or biregular graph.

In [6], several inequalities for HM were proven. Some of them are

$$\delta M_1 + 2M_2 \leq HM \leq \Delta M_1 + 2M_2 \tag{11}$$

with equality if and only if G is regular,

$$HM \leq 2(\Delta + \delta)M_1 - 4m\delta\Delta \tag{12}$$

with equality if and only if G is regular, and

$$HM \leq \frac{(\delta + \Delta)^2}{4m\Delta\delta}\, M_1^2 \tag{13}$$

with equality if and only if G is regular. Let us note that inequality (13) is consequence of (12). Namely, it is obtained from (12) and the arithmetic–geometric mean inequality.

3 Main results

Theorem 3.1. Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then

$$F \geq \frac{M_1^2}{m} + \frac{1}{2}(\Delta_e - \delta_e)^2 - 2M_2 \tag{14}$$

with equality if and only if $L(G)$ is regular, or $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$ and $\Delta_e + \delta_e = 2\Delta_{e_2}$.

Proof: Let $a_1 \geq a_2 \geq \cdots \geq a_m$ be real numbers with the property $r \leq a_i \leq R$, $i = 1, 2, \ldots, m$. In [21] (see also [19]) the following was proven

$$m\sum_{i=1}^{m} a_i^2 - \left(\sum_{i=1}^{m} a_i\right)^2 \geq \frac{m}{2}(R - r)^2 \tag{15}$$

with equality if and only if $R = a_1 = \cdots = a_m = r$, or $a_2 = \cdots = a_{m-1}$ and $a_1 + a_m = r + R = 2a_2$. For $a_i = d(e_i) + 2$, $i = 1, 2, \ldots, m$, $r = \delta_e$, and $R = \Delta_e$, the inequality (15) transforms into

$$m\sum_{i=1}^{m}[d(e_i) + 2]^2 - \left(\sum_{i=1}^{m}(d(e_i) + 2)\right)^2 \geq \frac{m}{2}(\Delta_e - \delta_e)^2 .$$

Bearing in mind the identities (1), the above inequality becomes

$$m(F + 2M_2) - M_1^2 \geq \frac{m}{2}(\Delta_e - \delta_e)^2$$

wherefrom we obtain (14).

Since equality in (15) holds if and only if $R = a_1 = \cdots = a_m = r$, or $a_2 = \cdots = a_{m-1}$ and $a_1 + a_m = R + r = 2a_2$, it follows that equality in (14) holds if and only if $\Delta_e = d(e_1) + 2 = \cdots = d(e_m) + 2 = \delta_e$, or $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$ and $\Delta_e + \delta_e = 2\Delta_{e_2}$. ∎

It is not difficult to observe that (14) is stronger than (5), i.e., (6). However, lower bounds for F established by (5) and (14) are incomparable, since the bound (14) requires that Δ_e and δ_e are known.

Corollary 3.2. Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then

$$F \geq \delta_e M_1 + \frac{1}{2}(\Delta_e - \delta_e)^2 - 2M_2 \geq m\delta_e^2 + \frac{1}{2}(\Delta_e - \delta_e)^2 - 2M_2 \tag{16}$$

with equality if and only if $L(G)$ is regular.

Proof: The inequality (16) is obtained from (14) and

$$M_1^2 \geq m\delta_e M_1 \geq m^2\delta_e^2 \,.$$

∎

It is not difficult to observe that the first inequality in (16) is stronger than the left–hand side of (11).

Corollary 3.3. Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then

$$EM_1 \geq \frac{M_1^2}{m} - 4M_1 + \frac{1}{2}(\Delta_e - \delta_e)^2 + 4m \tag{17}$$

with equality if and only if $L(G)$ is regular, or $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$ and $\Delta_e + \delta_e = 2\Delta_{e_2}$.

Proof: The inequality (17) is obtained from (14) and (2). ∎

The inequality (17) is stronger than (8).

Corollary 3.4. Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then

$$EM_1 \geq \delta_e M_1 + \frac{1}{2}(\Delta_e - \delta_e)^2 - 4M_1 + 4m \geq m\delta_e^2 + \frac{1}{2}(\Delta_e - \delta_e)^2 - 4M_1 + 4m$$

with equality if and only if $L(G)$ is regular.

By a similar procedure as in the case of Theorem 3.1, the following can be proven:

Theorem 3.5. Let G be a simple connected graph with $n \geq 4$ vertices and m edges. Then

$$F \geq \frac{(M_1 - \Delta_e)^2}{m-1} + \frac{1}{2}(\Delta_{e_2} - \delta_e)^2 + \Delta_e^2 - 2M_2$$

with equality if and only if $d(e_2) = \cdots = d(e_m)$, or $d_{e_3} + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$ and $\Delta_{e_2} + \delta = 2\delta_2$.

Theorem 3.6. Let G be a simple connected graph with $n \geq 5$ vertices and m edges. Then

$$F \geq \Delta_e^2 + \delta_e^2 + \frac{(M_1 - \Delta_e - \delta_e)^2}{m-2} - 2M_2 + \frac{1}{2}(\Delta_{e_2} - \delta_{e_2})^2$$

with equality if and only if $d(e_2) = \cdots = d(e_{m-1})$, or $d(e_3) + 2 = \cdots = d(e_{m-2}) + 2$ and $\Delta_{e_2} + \delta_{e_2} = d(e_3) + 2$.

Corollary 3.7. Let G be a simple connected graph with $n \geq 4$ vertices and m edges. Then

$$EM_1 \geq \Delta_e^2 + \frac{(M_1 - \Delta_e)^2}{m-1} + \frac{1}{2}(\Delta_{e_2} - \delta_e)^2 - 4M_1 + 4m$$

with equality if and only if $d(e_2) = \cdots = d(e_m)$, or $d_{e_3} + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$ and $\Delta_{e_2} + \delta = 2\delta_2$.

Corollary 3.8. Let G be a simple connected graph with $n \geq 5$ vertices and m edges. Then

$$EM_1 \geq \Delta_e^2 + \delta_e^2 + \frac{(M_1 - \Delta_e - \delta_e)^2}{m-2} + \frac{1}{2}(\Delta_{e_2} - \delta_{e_2})^2 - 4M_1 + 4m$$

with equality if and only if $d(e_2) = \cdots = d(e_{m-1})$, or $d(e_3) + 2 = \cdots = d(e_{m-2}) + 2$ and $\Delta_{e_2} + \delta_{e_2} = d(e_3) + 2$.

Corollary 3.9. Let G be a simple connected graph with $n \geq 4$ vertices and m edges. Then

$$HM \geq \Delta_e^2 + \frac{(M_1 - \Delta_e)^2}{m-1} + \frac{1}{2}(\Delta_{e_2} - \delta_e)^2$$

with equality if and only if $d(e_2) = \cdots = d(e_m)$, or $d_{e_3} + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$ and $\Delta_{e_2} + \delta = 2\delta_2$.

Corollary 3.10. Let G be a simple connected graph with $n \geq 5$ vertices and m edges. Then

$$HM \geq \Delta_e^2 + \delta_e^2 + \frac{(M_1 - \Delta_e - \delta_e)^2}{m-2} + \frac{1}{2}(\Delta_{e_2} - \delta_{e_2})^2$$

with equality if and only if $d(e_2) = \cdots = d(e_{m-1})$, or $d(e_3) + 2 = \cdots = d(e_{m-2}) + 2$ and $\Delta_{e_2} + \delta_{e_2} = d(e_3) + 2$.

In the following theorem we establish an upper bound for the forgotten index.

Theorem 3.11. Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then

$$F \leq (\Delta_e + \delta_e)M_1 - m\Delta_e\delta_e - 2M_2 \tag{18}$$

with equality if and only if there exists an integer k, $1 \leq k \leq m$, such that $\Delta_e = d(e_1) + 2 = \cdots = d(e_k) + 2 \geq d(e_{k+1}) + 2 = \cdots = d(e_m) + 2 = \delta_e$.

Proof: Let $p_1, p_2, \ldots, p_m$ and $a_1 \geq a_2 \geq \cdots \geq a_m$ be positive real numbers with the property $p_1 + p_2 + \cdots + p_m = 1$ and $r \leq a_i \leq R$, $i = 1, 2, \ldots, m$. In [18] (see also [17]) the following was proven

$$\sum_{i=1}^{m} p_i\, a_i + rR \sum_{i=1}^{m} \frac{p_i}{a_i} \leq r + R \tag{19}$$

with equality if and only if there exists an integer k, $1 \leq k \leq m$, such that $R = a_1 = \cdots = a_k \geq a_{k+1} = \cdots = a_m = r$.

For

$$p_i = \frac{d(e_i) + 2}{\sum_{i=1}^{m}(d(e_i) + 2)} \qquad \text{and} \qquad a_i = d(e_i) + 2$$

for $i = 1, 2, \ldots, m$, as well as $r = \delta_e$ and $R = \Delta_e$, the inequality (19) becomes

$$\frac{\sum_{i=1}^{m}[d(e_i)+2]^2}{\sum_{i=1}^{m}[d(e_i)+2]} + \frac{m\Delta_e\delta_e}{\sum_{i=1}^{m}[d(e_i)+2]} \leq \Delta_e + \delta_e$$

i.e.,

$$F + 2M_2 + m\Delta_e\delta_e \leq (\Delta_e + \delta_e)M_1$$

wherefrom (18) is obtained. Equality in (18) holds if and only if there exists an integer k, $1 \leq k \leq m$, such that $\Delta_e = d(e_1)+2 = \cdots = d(e_k)+2 \geq d(e_{k+1})+2 = \cdots = d(e_m)+2 = \delta_e$. ∎

Corollary 3.12. Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then

$$EM_1 \leq (\Delta_e + \delta_e - 4)M_1 - m(\Delta_e\delta_e - 4)$$

with equality if and only if there exists an integer k, $1 \leq k \leq m$, such that $\Delta_e = d(e_1)+2 = \cdots = d(e_k)+2 \geq d(e_{k+1})+2 = \cdots = d(e_m)+2 = \delta_e$.

Corollary 3.13. Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then

$$HM \leq (\Delta_e + \delta_e)M_1 - m\Delta_e\delta_e \tag{20}$$

with equality if and only if there exists k, $1 \leq k \leq m$, such that $\Delta_e = d(e_1)+2 = \cdots = d(e_k)+2 \geq d(e_{k+1})+2 = \cdots = d(e_m)+2 = \delta_e$.

The inequality (20) is stronger than (12).

Corollary 3.14. Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then

$$F \leq \frac{M_1^2}{4m}\left(\sqrt{\frac{\Delta_e}{\delta_e}} + \sqrt{\frac{\delta_e}{\Delta_e}}\right)^2 - 2M_2 \tag{21}$$

with equality if and only if $L(G)$ is regular.

It is not difficult to conclude that (21) is stronger than (13).

Similarly, as in the case of Theorem 3.11, the following can be proven.

Theorem 3.15. Let G be a simple connected graph with $n \geq 4$ vertices and m edges. Then

$$F \leq \Delta_e^2 + (\Delta_{e_2} + \delta_e)(M_1 - \Delta_e) - 2M_2 - (m-1)\Delta_{e_2}\delta_e$$

with equality if and only if there exists k, $2 \leq k \leq m$, so that $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_k) + 2 \geq d(e_{k+1}) + 2 = \cdots = d(e_m) + 2 = \delta_e$.

Theorem 3.16. Let G be a simple connected graph with $n \geq 5$ vertices and m edges. Then

$$F \leq \Delta_e^2 + \delta_e^2 + (\Delta_{e_2} + \delta_{e_2})(M_1 - \Delta_e - \delta_e) - 2M_2 - (m-2)\Delta_{e_2}\delta_{e_2}$$

with equality if and only if there exists k, $2 \leq k \leq m-1$, so that $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_k) + 2 \geq d(e_{k+1}) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$.

Corollary 3.17. Let G be a simple connected graph with $n \geq 4$ vertices and m edges. Then

$$EM_1 \leq \Delta_e^2 + (\Delta_{e_2} + \delta_e)(M_1 - \Delta_e) - (m-1)\Delta_{e_2}\delta_e - 4M_1 + 4m$$

and

$$HM \leq \Delta_e^2 + (\Delta_{e_2} + \delta_e)(M_1 - \Delta_e) - (m-1)\Delta_{e_2}\delta_e$$

with equality if and only if there exists k, $2 \leq k \leq m$, so that $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_k) + 2 \geq d(e_{k+1}) + 2 = \cdots = d(e_m) + 2 = \delta_e$.

Corollary 3.18. Let G be a simple connected graph with $n \geq 5$ vertices and m edges. Then

$$EM_1 \leq \Delta_e^2 + \delta_e^2 + (\Delta_{e_2} + \delta_{e_2})(M_1 - \Delta_e - \delta_e) - (m-2)\Delta_{e_2}\delta_{e_2} - 4M_1 + 4m$$

and

$$HM \leq \Delta_e^2 + \delta_e^2 + (\Delta_{e_2} + \delta_{e_2})(M_1 - \Delta_e - \delta_e) - (m-2)\Delta_{e_2}\delta_{e_2}$$

with equality if and only if there exists k, $2 \le k \le m-1$, so that $\Delta_{e_2} = d(e_2)+2 = \cdots = d(e_k)+2 \ge d(e_{k+1})+2 = \cdots = d(e_{m-1})+2 = \delta_{e_2}$.

Theorem 3.19. Let G be a simple connected graph with $n \ge 3$ vertices and m edges. Then

$$F \le \frac{M_1^2}{m} + m\alpha(m)(\Delta_e - \delta_e)^2 - 2M_2 \tag{22}$$

where

$$\alpha(m) = \frac{1}{4}\left(1 - \frac{(-1)^{m+1}+1}{2m^2}\right).$$

Equality in (22) holds if and only if $L(G)$ is regular.

Proof: Let $p = (p_i)$, $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \ldots, m$, be sequences of non-negative real numbers with the property

$$0 < r_1 \le a_i \le R_1 < +\infty, , \qquad \text{and} \qquad 0 < r_2 \le b_i \le R_2 < +\infty\,.$$

Further, let S be a subset of $I_m = \{1, 2, \ldots, m\}$ for which the expression

$$\left| \sum_{i \in S} p_i - \frac{1}{2}\sum_{i=1}^{m} p_i \right|$$

is minimized. Under the given conditions, Andrica and Badea [1] proved that

$$\begin{aligned} &\left| \sum_{i=1}^{m} p_i \sum_{i=1}^{m} p_i\, a_i\, b_i - \sum_{i=1}^{m} p_i\, a_i \sum_{i=1}^{m} p_i\, b_i \right| \\ &\le (R_1 - r_1)(R_2 - r_2)\sum_{i \in S} p_i \left(\sum_{i=1}^{m} p_i - \sum_{i \in S} p_i \right). \end{aligned} \tag{23}$$

For $p_i = 1$, $a_i = b_i = d(e_i)+2$, $i = 1, 2, \ldots, m$, $R_1 = R_2 = \Delta_e$, and $r_1 = r_2 = \delta_e$, the inequality (23) becomes

$$m\sum_{i=1}^{m}[d(e_i)+2]^2 - \left(\sum_{i=1}^{m}[d(e_i)+2]\right)^2 \le (\Delta_e - \delta_e)^2 \left\lfloor \frac{m}{2} \right\rfloor \left(m - \left\lfloor \frac{m}{2} \right\rfloor \right)$$

i.e.,

$$m(F+2M_2) \leq M_1^2 + (\Delta_e - \delta_e)^2\, m^2\, \alpha(m)$$

wherefrom we obtain the required result. Since equality in (23) holds if and only if $R_1 = a_1 = \cdots = a_m = r_1$ or $R_2 = b_1 = \cdots = b_m = r_2$, then the equality in (22) holds if and only if $\Delta_e = d(e_1)+2 = d(e_2)+2 = \cdots = d(e_m)+2 = \delta_e$. ∎

By a similar procedure, the following can be proven.

Theorem 3.20. Let G be a simple connected graph with $n \geq 4$ vertices and m edges. Then

$$F \leq \Delta_e^2 + \frac{(M_1 - \Delta_e)^2}{m-1} + (m-1)\alpha(m-1)(\Delta_{e_2} - \delta_e)^2 - 2M_2$$

with equality if and only if $\Delta_{e_2} = d(e_2)+2 = \cdots = d(e_m)+2 = \delta_e$.

Theorem 3.21. Let G be a simple connected graph with $n \geq 5$ vertices and m edges. Then

$$F \leq \Delta_e^2 + \delta_e^2 + \frac{(M_1 - \Delta_e - \delta_e)^2}{m-2} + (m-2)\alpha(m-2)(\Delta_{e_2} - \delta_{e_2})^2 - 2M_2$$

with equality if and only if $\Delta_{e_2} = d(e_2)+2 = \cdots = d(e_{m-1})+2 = \delta_{e_2}$.

Corollary 3.22. Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then

$$EM_1 \leq \frac{M_1^2}{m} + m\alpha(m)(\Delta_e - \delta_e)^2 - 4M_1 + 4m$$

and

$$HM \leq \frac{M_1^2}{m} + m\alpha(m)(\Delta_e - \delta_e)^2$$

with equality if and only if $L(G)$ is regular.

Corollary 3.23. Let G be a simple connected graph with $n \geq 4$ vertices and m edges. Then

$$EM_1 \leq \Delta_e^2 + \frac{(M_1 - \Delta_e)^2}{m-1} + (m-1)\alpha(m-1)(\Delta_{e_2} - \delta_e)^2 - 4M_1 + 4m$$

and

$$HM \leq \Delta_e^2 + \frac{(M_1 - \Delta_e)^2}{m-1} + (m-1)\alpha(m-1)(\Delta_{e_2} - \delta_e)^2$$

with equality if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_m) + 2 = \delta_e$.

Corollary 3.24. Let G be a simple connected graph with $n \geq 5$ vertices and m edges. Then

$$EM_1 \leq \Delta_e^2 + \delta_e^2 + \frac{(M_1 - \Delta_e - \delta_e)^2}{m-2} + (m-2)\alpha(m-2)(\Delta_{e_2} - \delta_{e_2})^2 - 4M_1 + 4m$$

and

$$HM \leq \Delta_e^2 + \delta_e^2 + \frac{(M_1 - \Delta_e - \delta_e)^2}{m-2} + (m-2)\alpha(m-2)(\Delta_{e_2} - \delta_{e_2})^2$$

with equality if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$.

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