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MATHEMATICAL MODELLING OF A PLATE WEAKENED BY A CIRCULAR HOLE IN PURE BENDING CONDITION

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Abstract: *The task of this paper is mathematical modelling of a plate weakened by a circular hole in pure bending condition. The mathematical model is formed using the complex variable method, and allows complete analytical solution of stresses at every point of the plate, and especially around the hole where the well-known problem of stress concentration is present. The presented methodology can be applied for the solution of any plate weakened by a circular hole in pure bending condition. In the paper, methodology is applied for the solution of the one concrete example. The comparative analysis has shown a high accuracy of analytically obtained results with FEM results obtained by the calculation in ANSYS 12 software package. The application of the results of this paper is of great importance for quality of design and optimization of thin-walled structures of type plate weakened by a circular hole, which are very common in engineering practice.*

1. Introduction

Most of today's modern machinery and engineering constructions are composed of thin-walled elements of the type of plate or shell combined with elements of the beam. Many of them contain a large number of openings and holes of different shape. The openings and holes are present because of various design requirements, or to optimize and reduce the weight which is one of the most important tasks of engineering practice. Increasingly stringent requirements in terms of load capacity, cost and reliability, cause that design and construction must be based on very accurate analysis of stresses and strains.

In design phase, for determining the stresses and strains in most cases engineers use the equations of the theory of elasticity [1–3]. A solution of analysed problem can be obtained by integration of differential equations with satisfaction of the compatibility equations and the contour conditions. The theory of elasticity found a wide application and gave the answer to some problems posed by engineering practices, especially in the cases of the lower approximation that can be simplified. When the analysed problem is significantly different from the theoretical model, a lot of rough approximations must be introduced. The obtained results in that case are not enough accurate which may be a very serious problem. The special difficulties create places with the existence of openings or holes. In these places there is the formation and distribution of stress condition that is significantly different from that of the other, just remote parts of the loaded element. This occurrence is well-known as stress concentration. It plays a very important role in problems of theory of elasticity since practice has confirmed that it leads to very frequent crashes and serious accidents. That is why this issue is very important and extensive researches are carried out with this theme.

In the investigation of this phenomenon, a thin plate weakened by a hole and loaded by a certain load is analysed in the largest number of cases. In the paper [4] Troyani et al. deal with theoretical stress concentration factors for short rectangular plates with centred circular holes. The similar problem is treated in the paper [5]. Bizic et al. analysed in the paper [6] (theoretical and numerical) the effects of circular hole on the stress state of homogeneous isotropic uniaxial tensioned plate. Wang et al. in the paper [7] deal with the stress concentration factors for an eccentric circular hole in a finite-width strip or in a semi-infinite plate in tension. The papers [8, 9] deal with the stress and strain concentrations due to the bending of a thick plate with circular hole. Rubayi, Cao and Bell in the papers [10, 11] analysed thick circular plate with central hole loaded by transverse force or bending moment. The papers [12, 13] analysed the elastic stability of plates with holes subjected to axial compression and bending moment. Bakhshandeh et al. in the paper [14] investigate the stress concentration factor around the circular hole using three-dimensional finite element model. Similar analysis are performed for problems with multiple holes in the papers [15-18]. The problem of elastic disc weakened by an eccentric circular hole is analysed by Radi, Strozzi, Bizic, et al. in the papers [19, 20].

In all cases the problem is reduced to exact determination of stress at certain points of the loaded plate weakened by an opening or hole, where there occurs the phenomenon of stress concentration. One of directions of solving this problem is based on the application of Muskhelishvili's complex variable function [21, 22] and complex analysis, and it is called the complex variable method. By applying this method, it is possible, theoretically, to determine stress state of some observed elements which classical method it was impossible to solve [23]. Accordingly, in the paper [24] Gao exposed application the complex variable displacement method in plane isotropic elasticity. In the paper [25] Simha and Mohapatra using complex variable method for analysis of stress concentration around irregular holes. The papers [26, 27] deal with the boundary element method using complex variables for solving the plane and plate problems of elasticity.

In line with all these researches very interesting problem is mathematical modelling and exact identification of stresses in the plate weakened by a circular hole in pure bending condition, using the complex variable method. Problems of this type are very frequently in construction of railway and road vehicles, vessels, aircrafts, civil engineering machinery, mining and transportation machinery, cranes, tooling machines, steel structures and many others. The holes exist on them for various reasons such as construction requirements, optimization of the structure, reducing the self-weight, aesthetic reasons, and so on. This was motivation for research published in this paper where the mathematical model that allows analytical solution of stresses, is obtained by using the complex variable method. The aim was to find the analytical expressions which allow exact determination of stresses at any point of the plate and especially at circular hole. In order to verification of analytically obtained results the stresses are also determined by the finite element method (FEM).

2. Determination of Stress Using Complex Variable Method

2.1. Theoretical formulation

The problem of plate weakened by a circular hole in pure bending condition belongs to the group of problems of plane stress condition. Literature [1, 2] gives the four group of equations that describe the plane stress condition. It is always possible to find function *U(x,y),* so-called stress function, through which it is possible to express the normal and shear stresses as follows:

(1) $$\sigma_x = \frac{\partial^2 U(x,y)}{\partial y^2}; \qquad \sigma_y = \frac{\partial^2 U(x,y)}{\partial x^2}; \qquad \tau_{xy} = -\frac{\partial^2 U(x,y)}{\partial x \partial y}$$

Substituting these expressions into the compatibility equation, stress function must satisfy the condition:

(2) $$\frac{\partial^4 U(x,y)}{\partial x^4} + 2\frac{\partial^4 U(x,y)}{\partial x^2 \partial y^2} + \frac{\partial^4 U(x,y)}{\partial y^4} = 0$$

Since stresses σ_x, σ_y and τ_{xy} must be unique and continuous functions to the derivative of the second order, the stress function *U(x,y)* must have continuous derivatives of the fourth order, while these derivatives starting from the second must be unique functions in the whole area. Solution the previous equation in real form is very complicated, and in some cases even impossible. That is why this function is translated into a complex area. This is realized by selecting two analytical functions ϕ_z and ψ_z which are called complex potentials in which a complex function is $z=x+iy$, where i is imaginary unit. In this way the problem of resolving a function of two independent variables is reduced to the problem of determining the two complex functions of one independent variable. So the expressions for stresses are [21]:

(3) $$\begin{aligned} &\sigma_x + \sigma_y = 4\,\mathrm{Re}\,\phi'(z) \\ &\sigma_y - \sigma_x + 2i\tau_{xy} = 2\left[\bar{z}\phi''(z) + \psi'(z)\right] \end{aligned}$$

where: Re – the real part; $\bar{z}$ – conjugated complex number

In many cases it is more convenient to introduce polar coordinates, and it is therefore necessary to match the previous equations with the new coordinate system. Transition on the polar coordinates is using a following shift:

(4) $$z = x + iy = re^{i\theta}; \qquad x = r\cos\theta; \qquad y = r\sin\theta$$

where: r – radius of the circle, θ – angle between the x-axis and radius

If the stresses components in polar coordinates are marked with σ_ρ, σ_θ and $\tau_{\rho\theta}$, equations (3) take the following form:

(5) $$\begin{aligned} &\sigma_\rho + \sigma_\theta = 4\,\mathrm{Re}\,\phi'(z) \\ &\sigma_\theta - \sigma_\rho + 2i\tau_{\rho\theta} = 2\left[\bar{z}\phi''(z) + \psi'(z)\right]e^{2i\theta} \end{aligned}$$

Connection between stresses in Cartesian and polar coordinates is:

(6) $$\begin{aligned} &\sigma_\rho + \sigma_\theta = \sigma_x + \sigma_y \\ &\sigma_\theta - \sigma_\rho + 2i\tau_{\rho\theta} = e^{2i\alpha}\left(\sigma_y - \sigma_x + 2i\tau_{xy}\right) \end{aligned}$$

Functions ϕ_z and ψ_z are determined from the conditions defined on the contours. For solving equations (3) and (5) can be used one of the methods such as: conformal mapping, Cauchy's integrals, power series, and so on. In this paper the method of conformal mapping is used.

If $z=x+iy$ and $\zeta=\xi+i\eta$ are two complex variables that are linked with the relation $z=\omega(\zeta)$, where $\omega(\zeta)$ is unambiguous analytical function in the area of Σ in plane area of change ζ, then each mapping (Fig. 1) that is applied by using these functions, in which the sizes of the angles are preserved, is called a conformal mapping [21].

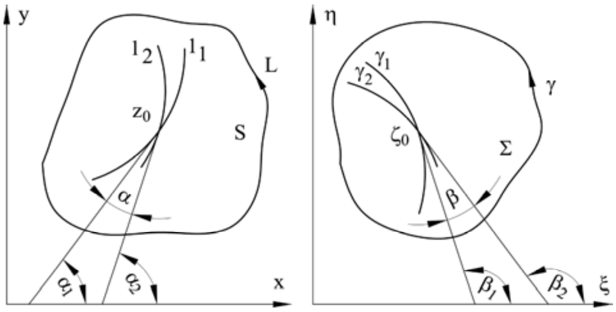

Fig. 1. Conformal mapping

Therefore, the task is to find a function for the mapping in which the angle between the two curves in the plane z will be copied without changes in the angle between corresponding curves in the plane ζ $(\alpha=\beta)$. There are advanced methods for the formation of conformal mapping function $\omega(\zeta)$. This is not primary task of this study and the ready-made function of conformal is used in this paper. The function of conformal mapping of the interior of the unit circle (radius of the surface varies in the limits of $\rho=0\div 1$) on the exterior of circular hole with radius R has the form [21]:

(7) $$z=\omega(\zeta)=\frac{R}{\zeta}$$

The complex potentials are:

(8) $$\phi(\zeta)=iA\left(\frac{1}{\zeta^2}-\zeta^2\right);\qquad \psi(\zeta)=iA\left(\zeta^2-2\zeta^4-\frac{1}{\zeta^2}\right)$$

The constant A depends on the geometry and load of the plate:

(9) $$A=\frac{MR^2}{8I_z}$$

where: M – bending moment; I_z – moment of inertia for the axis z

Expressions for the stresses in the polar coordinate system now have the following form:

$$\sigma_\theta=\frac{MR\rho^2}{2I_z}\left[\left(\frac{3}{2\rho^2}+\frac{1}{2\rho}\right)\sin\theta+\left(\frac{1}{2\rho}-2\rho^3-\frac{1}{2\rho^3}\right)\sin 3\theta\right]$$

(10) $$\sigma_\rho=\frac{MR\rho^2}{2I_z}\left[\left(\frac{1}{2\rho^3}+\frac{1}{2\rho}\right)\sin\theta+\left(2\rho^3+\frac{1}{2\rho^3}-\frac{5\rho}{2}\right)\sin 3\theta\right]$$

$$\tau_{\rho\theta}=-\frac{MR\rho^2}{4I_z}\left[\left(\frac{\rho}{2}-\frac{1}{2\rho^3}\right)\cos\theta+\left(\frac{3}{2}\rho-2\rho^3+\frac{1}{2\rho^3}\right)\cos 3\theta\right]$$

Stress at any point of contour of the hole is obtained for the unit of value of the radius $\rho=1$:

(11) $$\sigma_\theta=\frac{MR}{I_z}\left(\sin\theta-\sin 3\theta\right);\qquad \sigma_\rho=0;\qquad \tau_{\rho\theta}=0$$

When the contour of the hole is free from effects of external forces then the stresses σ_ρ and $\tau_{\rho\theta}$ are equal to zero, as the calculations has obtained.

2.2. Application on concrete example

It is very important to note that obtained mathematical relations is refer to the thin plate with infinite dimensions weakened by a circular hole of arbitrary radius. However, the most engineering problems is reduced to determination the stress state of a plate with finite dimensions. An example is the plate with length L, height b, thickness h, weakened by circular hole with radius R and loaded with bending moments M in plane of plate (Fig. 2).

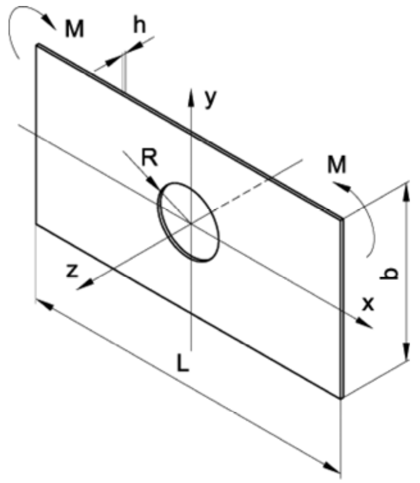

Fig. 2. The plate weakened by a circular hole in pure bending condition

Based on the given mathematical relations the software is developed and all values of stresses in the plane z are determined for the plate form Fig. 2. The unit values of radius hole R, bending moment M and moment of inertia I_z for the axis z are taken.

By substitution in the previous equations the values for ρ from 0.1 to 1.0 and angle θ from 0^o to 360^o, the values of stresses of each point of plate weakened by the circular hole are obtained. These results are shown on the following figures 3–4.

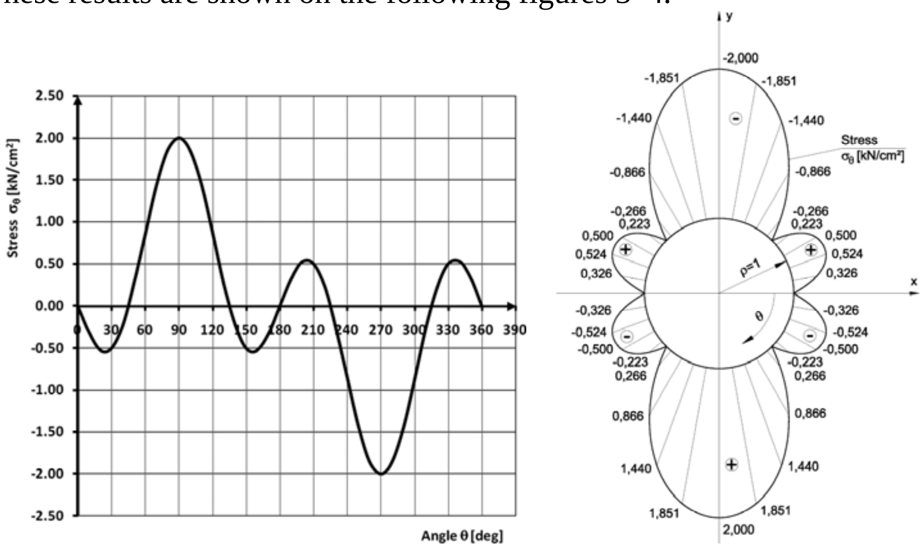

Fig. 3. The graphic illustration of stress σ_θ on the contour of the hole (left) and the diagram of stress σ_θ on the contour of the hole (right)

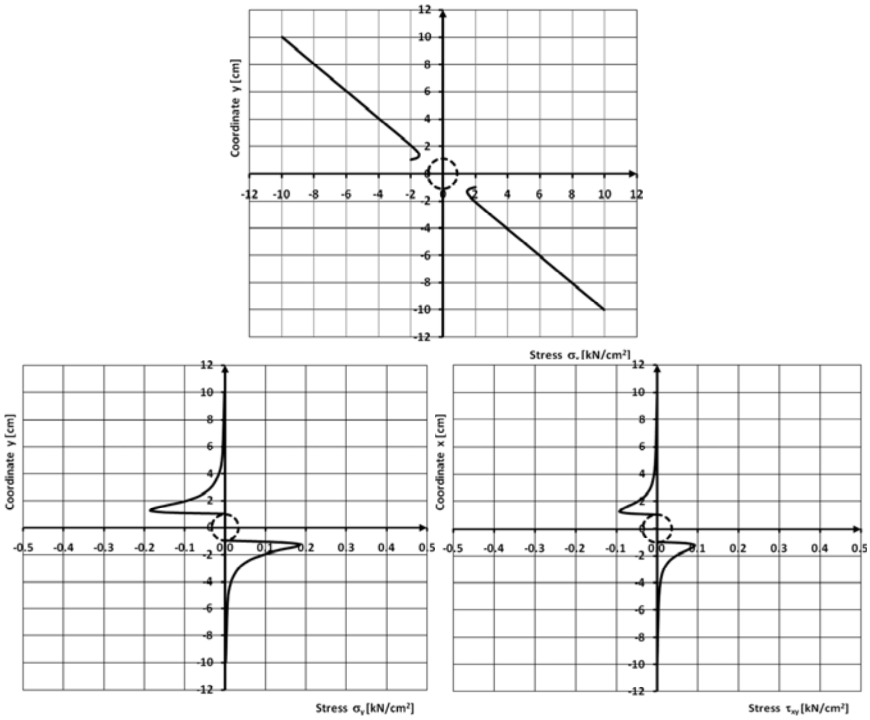

Fig. 4. The diagram of stresses σ_x for direction *x=0* (left), the diagram of stresses σ_y for direction *x=0* (middle), and the diagram of stress τ_{xy} for direction *y=0* (right)

3. Determination of Stress Using FEM

The numerical model of steel plate from the Fig. 2 with following dimensions is formed: L=100 mm; b=51 mm; h=0.9 mm; R=10 mm. These dimensions are equivalent to those in the concrete example taken in the Section 2.2. The homogeneous isotropic plate has been considered. The material is steel with modulus of elasticity E=21000 kN/cm^2, and Poisson's ratio υ=0.33. The calculation was carried out by using ANSYS 12 software package and the finite elements such as thin plates were applied. The FEM model consists of *12351* nodes and *1683* finite elements. The input data for the spatial discretization and mesh generation are not previously adjusted but was used a mesh that is generated automatically by the software. Analogous to the example from Section 2.2, the plate is loaded with unit bending moment M=1.0 kNcm in plane of plate. As a consequence, corresponding stresses are obtained as shown in Figs. 5–6.

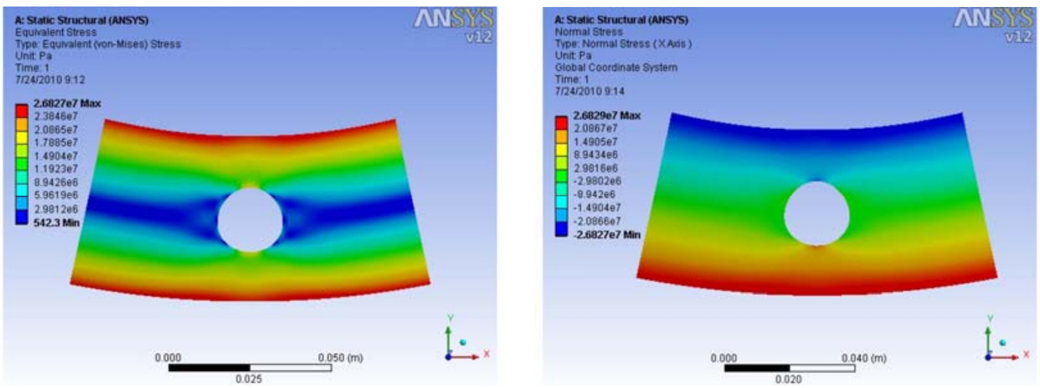

Fig. 5. The equivalent stress σ_e (left) and the normal stress σ_x (right)

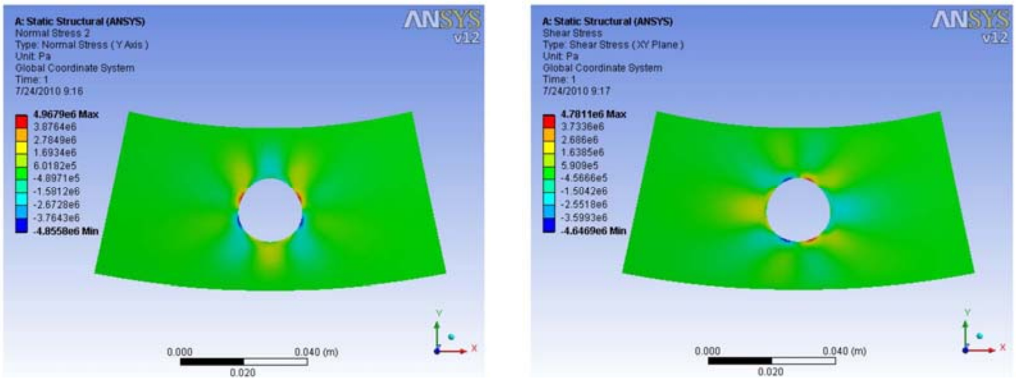

Fig. 6. The normal stress σ_y (left) and the tangential stress τ_{xy} (right)

In order to compare results with those from previous section, the numerical stresses values obtained by the FEM were read in specific points of the plate and contour of the hole. The diagrams and graphic illustration of stress σ_θ on the contour of the hole and diagrams of stresses σ_x, σ_y, and τ_{xy} were formed for specific values of coordinates x and y.

4. Comparison and Analysis of Obtained Results

The comparison of results obtained by the theoretical way using complex variable method and FEM are given in the following diagrams on Figs 7–8.

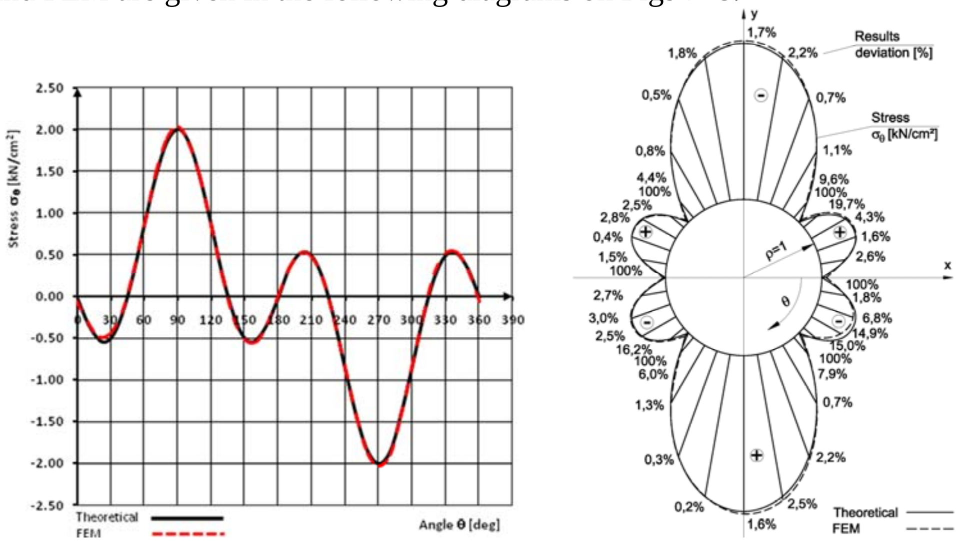

Fig. 7. The comparative diagram of stress σ_θ on contour of the hole (left) and the comparative graphic illustration with deviation in % of stress σ_θ on the contour of the hole (right)

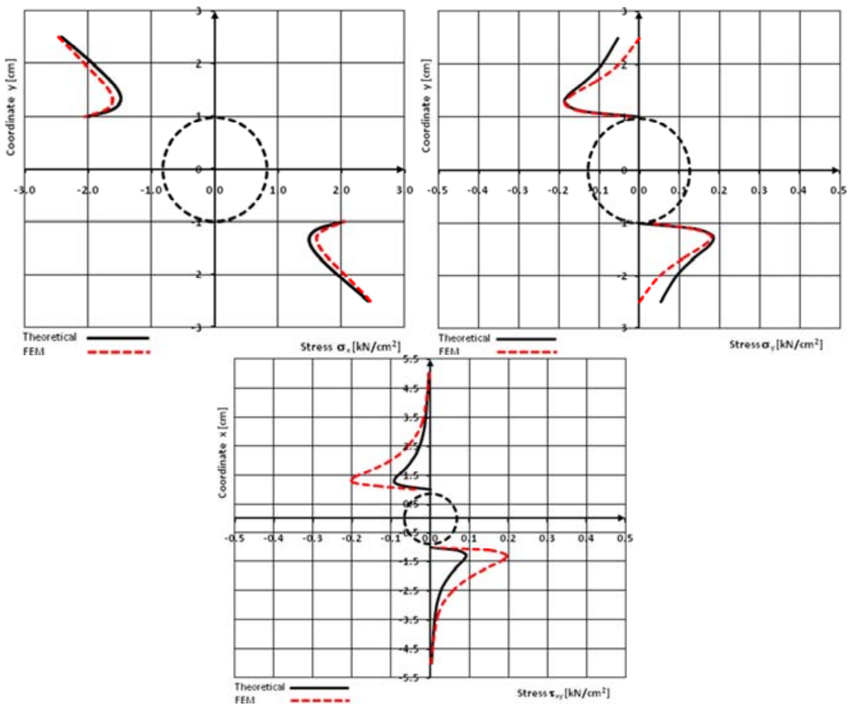

Fig. 8. The comparative diagrams of stresses σ_x for direction *x=0* (left), the comparative diagrams of stresses σ_y for direction *x=0* (middle), and the comparative diagram of stress τ_{xy} for direction *y=0* (right)

By analysing the results of stress σ_θ on the contour of the hole it can be noticed that the biggest values of stress are for the angle $\theta=\pm\pi/2$. In addition to this there are six singular points in which the stress is zero, which is consequence caused by sign changes when crossing from zone pressure to zone tension. The highest concentration of normal stress is present on the contour of the hole for the direction $x=0$. In contrast to the normal stress, the largest concentration of tangential stress is at the contour of the hole for the direction $y=0$.

From the comparative diagrams it can be noticed that the values and trend of stress distribution in both cases (theoretical and FEM) is approximately the same. There are very small deviations in the obtained values of stresses in the characteristic points of the plate and the contour of the hole. The deviations are less than 2.5% for largest values of stresses, while the deviations are higher for lower values of stresses. It should be taken into account that in the FEM, input data for the spatial discretization and mesh generation are not previously adjusted but was used a mesh that is generated automatically by the software. Proper choice of these parameters will certainly lead to the lower deviations.

5. Conclusion

The paper examines the mathematical modelling of a plate weakened by a circular hole in pure bending condition. The complex variable method was applied which is based on the application of Muskhelishvili's complex variable function technique. The formed mathematical model allows complete analytical solution of stresses at every point of the plate, and especially around the hole where the well-known problem of stress concentration is present. The technique was applied to the specific example of the plate for which the concrete numerical values of stresses were determined. Verification of obtained results is carried out by the finite element method (FEM) calculation using the software package ANSYS 12. Comparative analysis has shown that stresses obtained by the complex variable method and FEM are very similar in values and trend of distribution, which confirms the correctness of the established mathematical model. There is a tendency that the deviations are minimal and less than *2.5%* with the largest values of stress. It was also established that more significant stress difference are present with stresses whose values are close to zero. The application of the results of this paper is of great importance for quality of design and optimization of thin-walled structures of type plate weakened by a circular hole, which are very common in engineering practice.

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МАТЕМАТИЧЕСКО МОДЕЛИРАНЕ НА ОТСЛАБЕНА ПЛАСТИНА В РЕЗУЛТАТ НА КРЪГЪЛ ОТВОР ПРИ УСЛОВИЕ НА ЧИСТО ОГЪВАНЕ

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Ключови думи: *Моделиране, прастина, кръгъл отвор, чисто огъвана*

Резюме: *Задачата на тази статия е математическо моделиране на отслабена плоча от кръгъл отвор в състояние на чисто огъване. Математическият модел се формира с помощта на метод комплексна променлива, и дава възможност за пълно аналитично определяне на напреженията във всяка точка на пластината, и особено около отвора, когато е налице добре известния проблем на концентрация на напреженията. Представената методика може да се прилага за изчисление на всяка пластина отслабена от кръгъл отвор подложена на чисто огъване. Представената в статията методология се прилага за решаването на един конкретен пример. Сравнителният анализ показва висока точност на получените аналитични резултати по представената методология с резултатите, получени от FEM изчисление със софтуерен пакет ANSYS 12. Прилагането на резултатите от този документ е от голямо значение за качеството на проектиране и оптимизиране на тънкостенни конструкции от тип пластина отслабени от кръгъл отвор, които са много чести в инженерната практика.*