UNIVERSITY OF EAST SARAJEVO

FACULTY OF MECHANICAL ENGINEERING

4th INTERNATIONAL SCIENTIFIC CONFERENCE

COMETa2018

„Conference on Mechanical Engineering Technologies and Applications“

PROCEEDINGS

27th-30th November
East Sarajevo-Jahorina, RS, B&H

COMETa 2018
4th INTERNATIONAL SCIENTIFIC CONFERENCE

27th - 30th November 2018
Jahorina, Republic of Srpska, B&H

University of East Sarajevo
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ZBORNIK RADOVA

PROCEEDINGS

Istočno Sarajevo – Jahorina, BiH, RS
27 - 30. Novembar 2018.

East Sarajevo – Jahorina, B&H, RS
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PREFACE

Faculty of Mechanical Engineering East Sarajevo is organizing the 4th International Scientific Conference COMETa 2018 - "Conference on Mechanical Engineering Technologies and Applications“. The aim of the conference is to contribute to the implementation of new technologies in production processes by achieving better cooperation between scientific research institutions and companies, and to enable practical application of research results presented in the proceedings.

The main objective of the conference is to bring together eminent domestic and international experts in the field of engineering and the application of new technologies and the development of mechanical systems, and to contribute increasing the competitiveness of the domestic economy through the exchange of experience and knowledge, public presentations of current research and new construction solutions.

The organization of previous conferences COMETa2012, COMETa2014 and COMETa2016, according to the assessments of participants, especially foreign colleagues, were successful.

The efforts were recognized by the Ministry of Science and Technology of the Republic of Srpska, since in May 2018 the COMETa conference was ranked among international scientific conferences of the first category.

The COMETa 2018 conference program consists of the following thematic areas:

- *Manufacturing technologies and advanced materials,*
- *Applied mechanics and mechatronics,*
- *Machine design and product development,*
- *Energy and environmental protection,*
- *Maintenance and technical diagnostic,*
- *Quality, management and organization.*

At this year's COMETa2018 conference, a record number of papers from the country and abroad have been submitted. In total 277 authors from 13 countries participates in the international conference COMETa2018, 112 papers were accepted, including 4 plenary papers. Within the COMETa2018 conference, it is planned to organize two working meetings that will focus on the current topics of the Conference.

With the desire to improve the organizational as well as the scientific effect of the Conferences, and appreciating the contributions made by the scientific community in this way, we want to emphasize that each of your suggestions is more than welcome and will be appreciated in connection with the above.

On behalf of the Organizing and Scientific Committee of the COMETa2018 conference, we would like to express our gratitude to all authors, reviewers, institutions, companies and individuals who contributed to the Conference.

Hoping that the results of our joint work will meet expectations, the organizer of the Conference, Faculty of Mechanical Engineering East Sarajevo, wants you active participation that will contribute to the development of modern ideas and solutions, in the spirit of technical and technological development of the modern world.

We wish you a pleasant stay in Jahorina. Welcome to the COMETa2018 conference.

East Sarajevo, November 21st, 2018.

President of the Scientific Committee
Full Professor Dušan Golubović, PhD

President of the Organizing Committee
Assistant Professor Milija Kraišnik, PhD

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COMETa 2018
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REŠAVANJE KONTAKTNIH PROBLEMA METODOM KONAČNIH ELEMENATA

Snežana Vulović[1], Miroslav Živković[2], Rodoljub Vujanac[3], Jelena Živković[4]

*Rezime**:** U ovom radu razmatran je kontakt između dva deformabilna tela u najopštijem slučaju. Kako konfiguracija tela koja ulaze u kontakt prethodno nije poznata, kontakt predstavlja nelinearni problem čak i kada su tela elastična. Analogija između trenja i plastičnosti je korišćena u numeričkoj implementaciji kontakta i prikazana je u ovom radu. Razvijeni model je implementiran u programski paket za analizu konstrukcija - PAK.*

Ključne riječi: Metod konačnih elemenata, kontakt, penalti metod

SOLUTION OF CONTACT PROBLEMS USING THE FINITE ELEMENT METHOD

Abstract: The contact between two deformable bodies in general case is presented in this paper. Even when the bodies are elastic the contact represents a non-linear problem as the configuration of two bodies coming into the contact is not known beforehand. An analogy between friction and classical elasto-plasticity was used in the numerical implementation of the contact and it is presented in this paper. Developed numerical model is implemented in the inhouse software package PAK.

Key words: Finite Elemente Method, Contact, Penalty method

1 INTRODUCTION

Contact mechanics has its application in many engineering problems, for example: the interaction between soil and foundations in civil engineering, general bearing problems as well as bolt and screw joints in mechanical engineering. Effective application of solvers for contact problems based on the finite element method demands a high degree of experience since the general robustness and stability cannot be guaranteed. For this reason the development of more efficient, fast and stable finite element contact discretization is still a very important topic, especially due to the fact that engineering applications are becoming more and more complex.

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[1] PhD, Snežana Vulović, Faculty of Engineering, University of Kragujevac, Serbia, vsneza@kg.ac.rs
[2] PhD, Miroslav Živković, Faculty of Engineering, University of Kragujevac, Serbia, zile@kg.ac.rs
[3] PhD, Rodoljub Vujanac, Faculty of Engineering, University of Kragujevac, Serbia, vujanac@kg.ac.rs
[4] MSc, Jelena Živković, Faculty of Engineering, University of Kragujevac, Serbia, jelena.zivkovic@kg.ac.rs

The aim of this paper is to provide the framework for contact problems with friction, based on the penalty [2-4,6] and the Lagrange multiplier method [4,6]. The Lagrange multiplier method provides exact solution but involves additional degrees of freedom. The penalty formulation is purely geometrically based and therefore no additional degrees of freedom must be added but the solution is dependent on the introduced penalty factor. Standard shape routines are used for the detection of contact between previously separated meshes and for the aplication of displacement constraints where the contact is identified. Both models are implemented into the inhouse software package for structural analysis, PAK [1], based on the finite element method. Numerical examples are shown to demonstrate a possibility of applying the developed method in the analysis of finite deformation problems.

2 FORMULATION OF THE MULTI-BODY FRICTIONAL CONTACT PROBLEM

A contact between two deformable bodies ($B^{(1)}$ and $B^{(2)}$) is considered, Figure 1. The contact surface is defined in the way that all material points where contact may occur at any time t are included. Using a standard notation in contact mechanics, each pair of contact surfaces involved in the problem will consist of slave ($\Gamma_C^{(1)}$) and master ($\Gamma_C^{(2)}$) surfaces. The condition that must be satisfied is that any part of slave surface cannot penetrate the master surface.

The projection point ($\bar{\mathbf{x}}$) of the current position of the slave node $\mathbf{x}^k$ onto the master surface, is defined as

$$\frac{\mathbf{x}^k - \bar{\mathbf{x}}(\bar{\xi}^1, \bar{\xi}^2)}{\left\| \mathbf{x}^k - \bar{\mathbf{x}}(\bar{\xi}^1, \bar{\xi}^2) \right\|} \cdot \bar{\mathbf{a}}_\alpha (\bar{\xi}^1, \bar{\xi}^2) = 0 \tag{1}$$

where $\alpha = 1,2$ and $\bar{\mathbf{a}}_\alpha(\bar{\xi}^1, \bar{\xi}^2)$ are the tangent covariant base vectors at the point. The definition of the projection point allows us to define the distance between any slave node and the master surface. The normal gap or the penetration g_N for slave node k is defined as the distance between the current position of this node and the master surface

$$g_N = (\mathbf{x}^k - \bar{\mathbf{x}}) \cdot \bar{\mathbf{n}} \tag{2}$$

Normal $\bar{\mathbf{n}}$ to the master surface $\Gamma_C^{(2)}$ at point $\bar{\mathbf{x}}$, can be defined using tangent vectors at the point $\bar{\mathbf{x}}$

$$\bar{\mathbf{n}} = \frac{\bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2}{\left\| \bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2 \right\|} \tag{3}$$

The gap (2) gives the non-penetration conditions as follows

$$g_N = 0 \text{ perfect contact; } g_N > 0 \text{ no contact; } g_N < 0 \text{ penetration} \tag{4}$$

The function (4) completely defines the contact kinematics for frictionless contact problem. If friction is modeled, tangential relative displacement must be introduced. In that case, the sliding path of the node $\mathbf{x}^k$ over the contact surface is described by total tangential relative displacement, in time interval from t_0 to t, as

$$g_T = \int_{t_0}^{t} \left\| \dot{\mathbf{g}}_T \right\| dt = \int_{t_0}^{t} \left\| \dot{\bar{\xi}}^{\alpha} \bar{\mathbf{a}}_{\alpha} \right\| dt = \int_{t_0}^{t} \sqrt{\dot{\bar{\xi}}^{\alpha} \dot{\bar{\xi}}^{\beta} a_{\alpha\beta}}\, dt \tag{5}$$

The time derivatives of parameter $\bar{\xi}^{\alpha}$ in equation (5) can be computed from the relation (1), [8]. In the geometrically linear case we obtain the following result

$$\bar{a}_{\alpha\beta} \dot{\bar{\xi}}^{\beta} = \left[\dot{\mathbf{x}}^{k} - \dot{\bar{\mathbf{x}}} \right] \cdot \bar{\mathbf{a}}_{\alpha} = \dot{g}_{T\alpha} \tag{6}$$

where $\bar{a}_{\alpha\beta} = \bar{\mathbf{a}}_{\alpha} \cdot \bar{\mathbf{a}}_{\beta}$ is the metric tensor in point $\bar{\mathbf{x}}$ of the master surface. From the equations (5) and (6) we can deduce the relative tangential velocity at the contact point

$$\dot{\mathbf{g}}_T = \dot{\bar{\xi}}^{\alpha} \bar{\mathbf{a}}_{\alpha} = \dot{g}_{T\alpha} \bar{\mathbf{a}}^{\alpha} \tag{7}$$

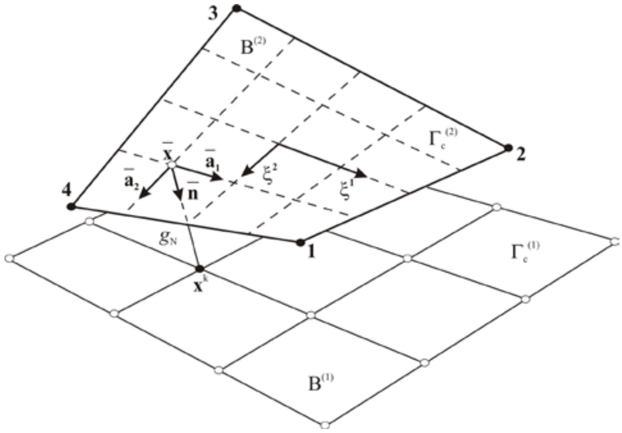

Figure 1. *Geometry of the 3D node-to-surface contact element*

For mathematical and computational modeling the surface characteristics have to be put into a constitutive interface constraint. A contact stress vector $\bar{\mathbf{t}}$ with respect to the current contact interface $\Gamma_C^{(2)}$ can be decomposed into a normal and tangential part

$$\bar{\mathbf{t}} = \bar{\mathbf{t}}_N + \bar{\mathbf{t}}_T = \bar{t}_N \bar{\mathbf{n}} + \bar{t}_{T\alpha} \bar{\mathbf{a}}^{\alpha} \tag{8}$$

where $\bar{\mathbf{a}}^{\alpha}$ is contravariant base vector. The stress acts on both surfaces obeying the action-reaction principle: $\bar{\mathbf{t}}(\bar{\xi}^1, \bar{\xi}^2) = -\mathbf{t}$ at the contact point $\bar{\mathbf{x}}$. The tangential stress $\bar{t}_{T\alpha}$ is zero in the case of frictionless contact. Condition $\bar{t}_N < 0$ indicates the existence of the contact. If there is no penetration between the bodies, then relations $g_N > 0$ and $\bar{t}_N = 0$ apply. This leads to

$$g_N \geq 0, \quad \bar{t}_N \leq 0, \quad \bar{t}_N g_N = 0 \tag{9}$$

which are known as Kuhn-Tucker conditions.

In tangential direction a distinction is made between stick and slip case. As long as no sliding between two bodies occurs, the relative tangential velocity is zero. If the velocity is zero, also the tangential relative displacement (5) is zero. This state is called stick case with the following restriction:

$$\dot{\mathbf{g}}_T = \mathbf{0} \;\Leftrightarrow\; \mathbf{g}_T = \mathbf{0} \tag{10}$$

A relative movement between two bodies occurs if the static friction resistance is overcome and the loading is large enough so that the sliding process can be kept. Therefore the relative sliding velocity, respectively the sliding displacement, shows in opposite direction to the friction force. With this the tangential stress vector is restricted as follows:

$$t_{T\alpha}^{sl} = -\mu \left| \mathbf{t}_N \right| \frac{\dot{g}_{T\alpha}^{sl}}{\left\| \dot{\mathbf{g}}_T^{sl} \right\|} \tag{11}$$

where μ is friction coefficient. In the simplest form of Coulomb's law (11), μ is constant and there is no distinction between static and sliding friction.

After the introduction of the stick and slip constraints, one needs an indicator to decide whether stick or slip actually take place. Therefore an indicator function

$$f = \left\| \mathbf{t}_T \right\| \text{-} \mu \left| t_N \right| \tag{12}$$

is evaluated, which respects the Coulomb's model for frictional interface law. In the equation (12) the first term is $\left\| \mathbf{t}_T \right\| = \sqrt{t_{T\alpha} \bar{a}^{\alpha\beta} t_{T\beta}}$. The following contact states can be distinguished:

$$f = \begin{cases} \left\| \mathbf{t}_T \right\| \text{-} \mu \left| t_N \right| \leq 0 & \rightarrow \text{ stick} \\ \left\| \mathbf{t}_T \right\| \text{-} \mu \left| t_N \right| > 0 & \rightarrow \text{ slip} \end{cases} \tag{13}$$

Using the penalty method for normal stress, constitutive equation can be formulated as

$$t_N = \varepsilon_N g_N \tag{14}$$

where ε_N is the normal penalty parameter. The tangential part is different for the stick and for the slip case. For the stick case a simple linear constitutive model can be used to describe the tangential stress

$$t_{T\alpha}^{stick} = \varepsilon_T g_{T\alpha} \tag{15}$$

where ε_T is the tangential penalty parameter. For the slip case the tangential stress is given by the constitutive law for frictional sliding (11). A backward Euler integration scheme and return mapping strategy are employed to integrate the friction equations (12). If the stick case is assumed, the trial values of the tangential contact pressure vector $t_{T\alpha}$, and the indicator function f at load step n+1 can be expressed in terms of their values at load step n as follows

$$t_{T\alpha\, n+1}^{trial} = t_{T\alpha\, n} + \varepsilon_T \Delta g_{T\alpha\, n+1} = t_{T\alpha\, n} + \varepsilon_T \bar{a}_{\alpha\beta} \Delta \xi_{n+1}^{\beta} \tag{16}$$

$$f_{Tn+1}^{trial} = \left\| \mathbf{t}_{Tn+1}^{trial} \right\| - \mu \left| t_{Nn+1} \right| \tag{17}$$

The return mapping is completed by

$$t_{T\alpha\ n+1} = \begin{cases} t_{T\alpha\ n+1}^{trial} & \text{if } f \le 0 \\ \mu \left| t_{Nn+1} \right| n_{T\alpha\ n+1}^{trial} & \text{if } f > 0 \end{cases} \tag{18}$$

with

$$n_{T\alpha\ n+1}^{trial} = \frac{t_{T\alpha\ n+1}^{trial}}{\left\| \mathbf{t}_{Tn+1}^{trial} \right\|} \tag{19}$$

The penalty method can be illustrated as a group of linear elastic springs that force the body back to the contact surface when overlapping or sliding occurs.

The principle of virtual works when two bodies at the time t are in contact can be written as (for a detailed legend of the symbols see [8])

$$\sum_{\alpha=1}^{2} \left(\int_{V^{(\alpha)}} \boldsymbol{\sigma}^{(\alpha)} : grad\,\delta\mathbf{u}^{(\alpha)} dV - \int_{V^{(\alpha)}} \rho^{(\alpha)} \left(\mathbf{b}^{(\alpha)} - \ddot{\mathbf{u}}^{(\alpha)} \right) \delta\mathbf{u}^{(\alpha)} dV - \int_{S_{\sigma}^{(\alpha)}} \boldsymbol{\sigma}^{(\alpha)} \cdot \mathbf{n} \cdot \delta\mathbf{u}^{(\alpha)} dA \right) - C_c = 0 \tag{20}$$

where C_c is “contact contribution”. Contact contribution is formulated for the Lagrange multiplier method in case of contact with friction for the stick case as

$$C_c = \int_{S_C} \left(\lambda_N \delta g_N + \boldsymbol{\lambda}_T \cdot \delta \mathbf{g}_T \right) dA \tag{21}$$

and for the case of sliding

$$C_c = \int_{S_C} \left(\lambda_N \delta g_N + \mathbf{t}_T \cdot \delta \mathbf{g}_T \right) dA \tag{22}$$

where δg_N and $\delta \mathbf{g}_T$ are variation of gap and tangential displacement; λ_N and $\boldsymbol{\lambda}_T$ are normal and tangential Lagrange multipliers and $\mathbf{t}_T$ is tangential stress vector which is determined from constitutive law for frictional slip. Note that the Lagrange multiplier λ_N can be identified as the contact stress t_N.

Contact contribution for the penalty method is formulated as

$$C_c = \int_{S_C} \left(\varepsilon_N g_N \delta g_N + \mathbf{t}_T \cdot \delta \mathbf{g}_T \right) dA \tag{23}$$

Detailed description of the finite element formulation for penalty method and Lagrange multiplier method can be found in [8].

3 NUMERICAL EXAMPLES

3.1 Sticking of elastic block

Numerical example given in [9] is used to verify whether the developed algorithm is able to represent stick/slip behavior correctly. The example consists of elastic block pressed against a rough rigid foundation. Simultaneously to the vertical loading the block is pulled at right side by an uniformly normal stress, as shown in Figure 2.

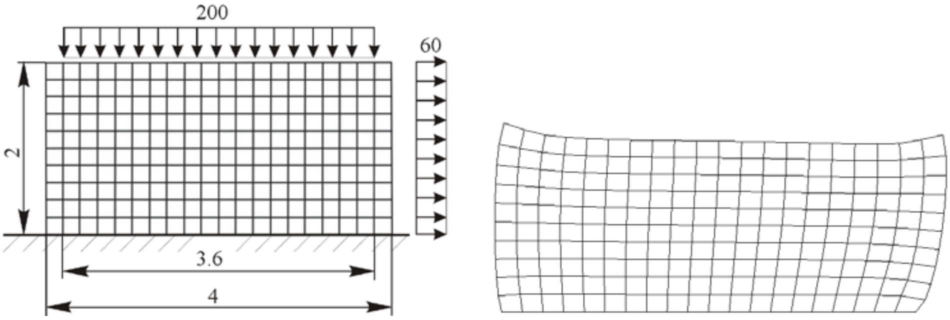

Figure 2. *Initial and deformed configuration*

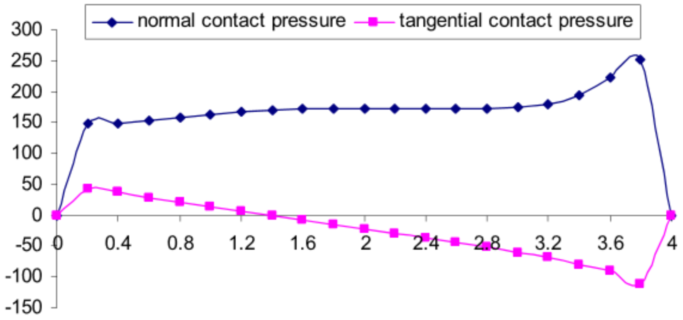

Figure 3. *Force-displacement repationship*

Material constants are: E = 1000MPa, v = 0.3. Properties of the contact surface have been chosen as $\varepsilon_N = 10^8$, $\varepsilon_T = 10^4$, friction coefficient $\mu = 0.5$. The block is discretized using 200 four-node isoparametric elements. It should be noted that in case of using the developed algorithm the total load can be applied in only one step. Calculated normal contact pressure and tangential contact stress are shown in Figure 3, and it can be seen that there is a good agreement between these solutions and the solutions given in [9].

3.2 Sliding of elastic-plastic block

Sliding of elastic-plastic block through rigid tool is considered, as shown in Figure 4. Material properties of the block are: $\mathrm{E} = 1000\mathrm{MPa}$, $\nu = 0.3$, $\sigma_y = 100\mathrm{MPa}$, $C_y = 10\mathrm{MPa}$, $n = 0$. Due to the symmetry, only one half of the problem is modeled. The block is discretized with 2D elements (the steady state of deformation). Displacement of the block of $79.2\mathrm{cm}$ is applied in 44 steps. Properties of the contact surface have been chosen as $\varepsilon_N = 1000$ и $\varepsilon_T = 500$, friction coefficient $\mu = 0.1$.

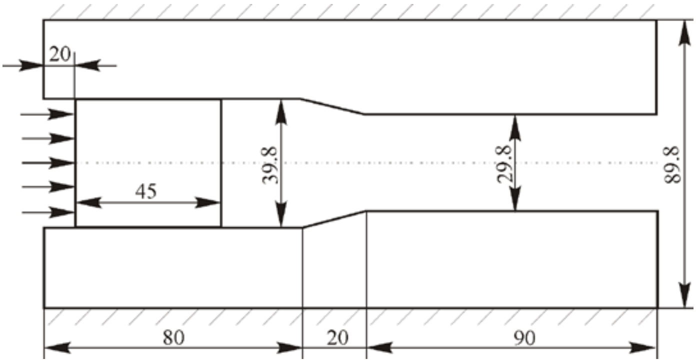

Figure 4. *Initial configuration*

Von Mises stress field on the considered block in 24th and 44th step is shown in Figure 5. Figure 6 shows relationship between contact force in x-direction and displacement.

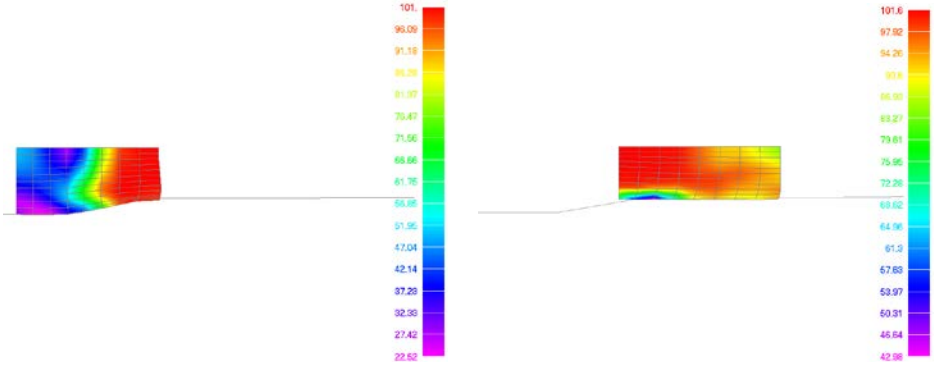

Figure 5. *Von Mises stress field in 24th step and 44th step*

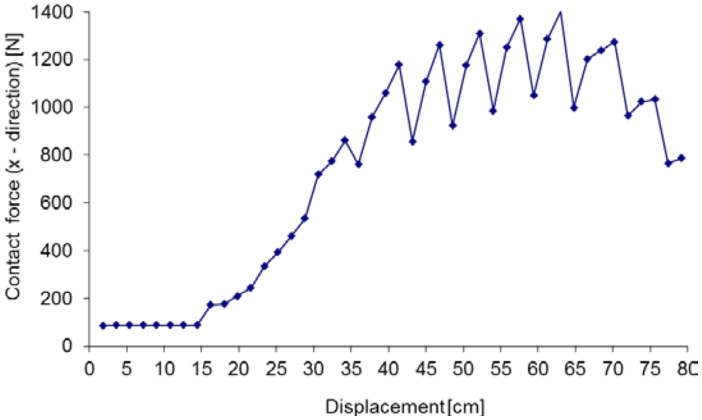

Figure 6. *Force-displacement relationship*

4 CONCLUSION

In this paper a model for three-dimensional contact problem with friction based on the penalty and Lagrange multiplier method was described. Using the penalty method calculation time is shorter but the results are strongly dependent on a chosen value of the penalty factor. The Lagrange multiplier method leads to the exact solution but with more iterations and significant extension of a number of degrees of freedom, as shown in [8]. Numerical examples shown in this paper and in [8] indicate a possibility of easy simultaneous use of both developed procedures in the analysis of finite deformation problems within one computer code.

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