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ADVANCES IN MATHEMATICAL ANALYSIS AND ITS APPLICATIONS

Contents

Chapter 1

On Theorems of Galambos-Bojanić-Seneta Type

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1.1 Introduction

Working on Tauberian theory in the 1930's, J. Karamata initiated investigation in asymptotic analysis of divergent processes, nowadays known as *Karamata's theory of regular variation* (see [26, 27, 28, 29, 30], and also [4, 35]).

In 1970, de Haan [25] defined and investigated rapid variation and so initiated further development in asymptotic analysis.

The theory of regular and rapid variability has many applications in different mathematical disciplines: differential and difference equations, in particular in description of asymptotic properties of solutions of these equations, dynamic equations, number theory, probability theory, time scales theory, selection principles theory, game theory and so on (see, for example, [31] and references therein and also [19, 21, 24, 34, 36, 37]).

In this chapter we consider only measurable functions $\varphi : [a, \infty) \to (0, \infty)$, $a > 0$ (the set of such functions we denote by $\mathbb{F}$), and sequences of positive real numbers (the set of such sequences is denoted by $\mathbb{S}$). We use the notation $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, $\mathbf{y} = (y_n)_{n\in\mathbb{N}}$, and so on, for sequences from $\mathbb{S}$.

We will also need the following class $\mathbb{A}$ of functions:

$$\mathbb{A} = \{\varphi \in \mathbb{F} : \varphi \text{ is nondecreasing and unbounded}\}.$$

If $\varphi \in \mathbb{A}$, then the function $\varphi^{\leftarrow}$ defined by

$$\varphi^{\leftarrow}(t) := \inf\{u \geq a : \varphi(u) > t\},\ t \geq \varphi(a)$$

is the *generalized inverse of* φ [4].

Notice the following two facts:

(1) If $\varphi \in \mathbb{A}$ is continuous and increasing, then $\varphi^{\leftarrow}$ is the inverse function φ^{-1} of φ, i.e. $\varphi(t) = \varphi^{-1}(t)$, $t \geq \varphi(a)$;

(2) If $\varphi \in \mathbb{A}$, then $\varphi^{\leftarrow} \in \mathbb{A}$.

To each function $\varphi \in \mathbb{F}$ we assign the following three functions depending on $\lambda > 0$:

$$k_\varphi(\lambda) := \lim_{t\to\infty} \frac{\varphi(\lambda t)}{\varphi(t)|},\ \lambda > 0;$$

$$\overline{k}_\varphi(\lambda) := \limsup_{t\to\infty} \frac{\varphi(\lambda t)}{\varphi(t)},\ \lambda > 0;$$

$$\underline{k}_\varphi(\lambda) := \liminf_{t\to\infty} \frac{\varphi(\lambda t)}{\varphi(t)},\ \lambda > 0.$$

Similarly, to each sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ and each $\lambda > 0$ one assigns the following three functions:

$$k_{\mathbf{x}}(\lambda) := \lim_{n\to\infty} \frac{x_{[\lambda n]}}{x_n};$$

$$\overline{k}_{\mathbf{x}}(\lambda) := \limsup_{n\to\infty} \frac{x_{[\lambda n]}}{x_n};$$

$$\underline{k}_{\mathbf{x}}(\lambda) := \liminf_{n\to\infty} \frac{x_{[\lambda n]}}{x_n}.$$

The function $\overline{k}_\varphi(\lambda)$, $\lambda > 0$, is called the *index function of* φ, and the function $\underline{k}_\varphi(\lambda)$ is called the *auxiliary index function of* φ.

Karamata theory of regular variation has two basic lines of investigation: functional and sequential.

Definition 1 (1) A function $\varphi \in \mathbb{F}$, is said to be *regularly varying* if for each $\lambda > 0$ it satisfies the condition

$$k_\varphi(\lambda) < \infty.$$

The class of these functions is denoted by $\mathsf{RV_f}$. When $k_\varphi(\lambda) = 1$, one obtains the class $\mathsf{SV_f}$ of *slowly varying functions*.

(2) A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is said to be *regularly varying* if for each $\lambda > 0$ it satisfies the condition

$$k_{\mathbf{x}}(\lambda) < \infty.$$

The class of these sequences is denoted by $\mathsf{RV_s}$.

In the case $k_{\mathbf{x}}(\lambda) = 1$ we have the class $\mathsf{SV_s}$ of *slowly varying sequences.*

These two types of research developed independently of each other until the results by Galambos-Seneta [23] and Bojanić-Seneta [5].

In these two papers the authors proved the following result giving a natural connection between the two theories and nowadays these results are called Galambos-Bojanić-Seneta type theorems.

Theorem GBS. For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ the following are equivalent:

(a) $\mathbf{x}$ is slowly varying (respectively, regularly varying);

(b) The function $\varphi_{\mathbf{x}}$, $\varphi_{\mathbf{x}}(t) = x_{[t]}$, $t \geq 1$, is slowly varying (respectively, regularly varying).

During the years Karamata's theory of regular variation was extended and modified in different directions (see [4, 31]) and a natural question arose: what about Galambos-Bojanić-Seneta result in the context of those modifications? Several such results, showing that similar assertions are true for the modifications of regular variation, have been proved. We are going to review these known results obtained by several authors and to prove a new result of this type for one of modifications of regular variation. We also give (without proof) a new result for a subclass of the class of rapidly varying sequences.

Some known theorems will be proven in order to demonstrate the methods of proving results of Galambos-Bojanić-Seneta type. Let us mention that in all these proofs one uses the classical (topological) Baire category theorem [22].

1.2 Known results

1.2.1 Classes $\mathsf{ORV_s}$ and $\mathsf{ORV_f}$ and their subclasses

In this subsection we present results on the classes of $\mathcal{O}$-regularly varying sequences and functions and their important subclasses.

Definition 2 ([3]) A function $\varphi \in \mathbb{F}$ is said to be *$\mathcal{O}$-regularly varying* if for each $\lambda > 0$ satisfies the condition

$$\overline{k}_{\varphi}(\lambda) < \infty.$$

or, equivalently,

$$\underline{k}_{\varphi}(\lambda) > 0.$$

The class of all these functions is denoted by $\mathsf{ORV_f}$.

Definition 3 A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is said to be *$\mathcal{O}$-regularly varying* for each $\lambda > 0$ satisfies the condition

$$\overline{k}_{\mathbf{x}}(\lambda) < \infty.$$

The class of these sequences is denoted by $\mathsf{ORV_s}$.

The classes $\mathsf{ORV_f}$ and $\mathsf{ORV_s}$ have been studied in a number of papers (see, for instance, [1, 2, 8, 9]).

Observe that when $\overline{k}_\varphi(\lambda) = \underline{k}_\varphi(\lambda)$ (respectively, $\overline{k}_{\mathbf{x}}(\lambda) = \underline{k}_{\mathbf{x}}(\lambda)$) we have the class $\mathsf{RV_f}$ of regularly varying functions (respectively, the class of regularly varying sequences) in the sense of Karamata.

When the index function $\overline{k}_\varphi(\lambda)$, $\lambda > 0$, of a function $\varphi \in \mathsf{ORV_f}$ is continuous, we say that φ belongs to the class $\mathsf{CRV_f}$. Similarly, one defines the class $\mathsf{CRV_s}$ of sequences. Notice that a function φ belongs to the class $\mathsf{CRV_f}$ if and only if $\lim_{\lambda\to 1} \overline{k}_\varphi(\lambda) = 1$.

We begin with the Galambos-Bojanić-Seneta type theorem for the classes $\mathsf{ORV_s}$ and $\mathsf{ORV_f}$.

Theorem 1 ([9]) *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{S}$. Then the following assertions are equivalent:*

(a) $\mathbf{x} \in \mathsf{ORV_s}$;

(b) *The function $\varphi_{\mathbf{x}}(t) = x_{[t]}$, $t \geq 1$, belongs to* $\mathsf{ORV_f}$.

We also have a similar result for the classes $\mathsf{CRV_s}$ and $\mathsf{CRV_f}$.

Theorem 2 ([17]) *For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ the following assertions are equivalent:*

(a) $\mathbf{x} \in \mathsf{CRV_s}$;

(b) $\varphi_{\mathbf{x}}(t) = x_{[t]}$, $t \geq 1$, *belongs to* $\mathsf{CRV_f}$.

An important subclass of of the class $\mathsf{ORV_f}$ of $\mathcal{O}$-regularly varying functions is the class of Seneta functions defined as follows.

For a given $\beta \geq 1$, denote by $\mathsf{SO_f^\beta}$ the set of all functions $\varphi \in \mathsf{ORV_f}$ such that $\overline{k}_\varphi(\lambda) \leq \beta$ for all $\lambda > 0$. Put $\mathsf{SO_f} = \bigcup_{\beta\geq 1} \mathsf{SO_f^\beta}$. Functions in the class $\mathsf{SO_f}$ are called the *Seneta functions*.

Similarly, a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ belongs to the class $\mathsf{SO_s^\beta}$ for a given $\beta \geq 1$ if $\overline{k}_{\mathbf{x}}(\lambda) \leq \beta$ for each $\lambda > 0$. The set $\mathsf{SO_s} = \bigcup_{\beta\geq 1} \mathsf{SO_s^\beta}$ is called the class of *Seneta sequences*.

Theorem 3 ([18]) *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{S}$. Then the following are equivalent:*

(a) $\mathbf{x} \in \mathsf{SO_s}$;

(b) $\varphi_{\mathbf{x}}(t) = x_{[t]} \in \mathsf{SO_f}$ *on the interval* $[1, \infty)$.

In fact, we have the following more precise result whose consequence is the above theorem.

Theorem 4 ([18, Peroposition 1]) *The following hold:*

(*a*) *If a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *belongs to the class* $\mathsf{SO}_\mathsf{s}^\beta$, *then the function* $\varphi_{\mathbf{x}}(t) = x_{[t]}$, $t \geq 1$, *belongs to the class* $\mathsf{SO}_\mathsf{f}^{\beta^2}$*;*

(*b*) *If* $\varphi_{\mathbf{x}}(t) = x_{[t]}$, $t \geq 1$, *belongs to the class* $\mathsf{SO}_\mathsf{f}^\beta$, *then* $\mathbf{x} \in \mathsf{SO}_\mathsf{s}^\beta$.

Remark 1 (1) From the above theorem one concludes that a sequence $\mathbf{x}$ belongs to SO_s^1 if and only if $\varphi_{\mathbf{x}} \in \mathsf{SO}_\mathsf{f}^1$. But, it is not so for $\beta > 1$ (see [18, Proposition 2]). For each $\beta > 1$ consider the function

$$g(t) = \exp\{2\ln(\beta)\sqrt{|\sin(\ln t)|}\},\ t > 0$$

which is O-regularly varying, hence, by Theorem 1, the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$, $x_n = g(n)$, is in the class $\mathsf{ORV_s}$. Moreover, $\mathbf{x} \in \mathsf{SO}_\mathsf{s}^{\beta^2} \setminus \mathsf{SO}_\mathsf{s}^\beta$. On the other hand, the function $\varphi_{\mathbf{x}}(t) = x_{[t]}$, $t \geq 1$, belongs to $\mathsf{SO}_\mathsf{f}^{\beta^2}$.

(2) Following [5], call a sequence $\mathbf{x}$ *embedable* in the function $g(t)$, $t \geq a, a > 0$, if $g(n) = x_n$ for each $n \geq [a] + 1$. A sequence $\mathbf{x}$ is embedable in a Seneta function $g(t)$, $t \geq a, a > 0$, if and only if it is a Seneta sequence.

We present now results on another subclasses of $\mathsf{ORV_f}$ and $\mathsf{ORV_s}$ introduced by Matuszewska.

Definition 4 ([32, 33]) A function $\varphi \in \mathbb{F}$ is said to be in the *class* $\mathsf{ERV_f}$ *of Matuszewska* if for each $\lambda \geq 1$ there are $c, d \in \mathbb{R}$, $c \leq d$, such that

$$\lambda^c \leq \underline{k}_\varphi(\lambda) \leq \overline{k}_\varphi(\lambda) \leq \lambda^d.$$

Similarly one defines the class $\mathsf{ERV_s}$ of sequences.

Definition 5 A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ belongs to the *class* $\mathsf{ERV_s}$ if

$$\lambda^c \leq \underline{k}_{\mathbf{x}}(\lambda) \leq \overline{k}_{\mathbf{x}}(\lambda) \leq \lambda^d$$

for each $\lambda \geq 1$ and some $c, d \in \mathbb{R}$ with $c \leq d$.

Notice that the following inclusions hold:

$$\mathsf{RV_f} \subsetneqq \mathsf{ERV_f} \subsetneqq \mathsf{CRV_f} \subsetneqq \mathsf{ORV_f} \supsetneqq \mathsf{SO_f}$$

and

$$\mathsf{RV_s} \subsetneqq \mathsf{ERV_s} \subsetneqq \mathsf{CRV_s} \subsetneqq \mathsf{ORV_s} \supsetneqq \mathsf{SO_s}.$$

The following result holds for the ERV classes of sequences and functions.

Theorem 5 ([17]) *For a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ *the following assertions are equivalent:*

(a) $\mathbf{x} \in \mathsf{ERV_s}$*;*

(b) *The function* $\varphi_{\mathbf{x}}(t) = x_{[t]}$, $t \geq 1$, *belongs* $\mathsf{ERV_f}$.

1.2.2 Rapid and related variations

In this section we quote Galambos-Bojanić-Seneta type results for rapidly varying sequences and functions and their variants.

Definition 6 ([25]) A function $\varphi \in \mathbb{F}$ is said to be *rapidly varying of index of variability* ∞ if it satisfies the asymptotic condition

$$k_\varphi(\lambda) = \infty, \ \lambda > 1.$$

The class of rapidly varying functions of index of variability ∞ we denote by $\mathsf{R}_{f,\infty}$.

Definition 7 A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is *rapidly varying* (of index of variability ∞) if the following asymptotic condition is satisfied:

$$k_{\mathbf{x}}(\lambda) = \infty, \ \lambda > 1.$$

or, equivalently.

$$\lim_{n\to\infty} \frac{x_{[\lambda n]}}{x_n} = 0, \ 0 < \lambda < 1.$$

$\mathsf{R}_{s,\infty}$ denotes the class of rapidly varying sequences of index of variability ∞.

Properties of the important class of rapidly varying sequences have been studied in [11] where a result of Galambos-Bojanić-Seneta type theorem was proved.

Theorem 6 *For a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ *the following are equivalent:*

- (a) $\mathbf{x}$ *belongs to the class* $\mathsf{R}_{\mathsf{s},\infty}$*;*
- (b) *The function* φ *defined by* $\varphi(t) = x_{[t]}$, $t \geq 1$, *is in the class* $\mathsf{R}_{\mathsf{f},\infty}$.

Proof $(a) \Rightarrow (b)$: Let $\lambda \in (0,1)$. Then for every $\alpha \in (\lambda,1)$ we have $\lim_{n\to\infty} \frac{x_{[\alpha n]}}{x_n} = 1$. We prove that for a given $\epsilon > 0$ there exist an interval $[A,B]$ which is a proper subset of $(\lambda,1)$ and $n_0 \in \mathbb{N}$ such that $\frac{x_{[\alpha n]}}{x_n} < \epsilon$ for each $n \geq n_0$ and each $\alpha \in [A,B]$. For an arbitrary and fixed $\alpha \in (\lambda,1)$ define $n_\alpha \in \mathbb{N}$ in the following way:

$$n_\alpha = \begin{cases} 1, & \text{if } \frac{x_{[\alpha n]}}{x_n} < \epsilon \text{ for each } n \in \mathbb{N}; \\ 1 + \max\{n \in \mathbb{N} : \frac{x_{[\alpha n]}}{x_n} \geq \epsilon\}, & \text{otherwise.} \end{cases}$$

It is easy to see that $1 \leq n_\alpha < \infty$ for each considered α. For each $k \in \mathbb{N}$ define

$$A_k = \{\alpha \in (\lambda,1) : n_\alpha > k\}.$$

Then $(A_k)_{k\in\mathbb{N}}$ is a non-increasing sequence of sets such that $\bigcap_{k\in\mathbb{N}} A_k = \emptyset$. We prove that not all sets A_k are dense in $(\lambda,1)$. If $k \in \mathbb{N}$ is fixed and $\alpha \in A_k$, then

$$\frac{x_{[\alpha(n_\alpha-1)]}}{x_{n_\alpha-1}} \geq \epsilon$$

and there is some $\delta_\alpha > 0$ such that for each $t \in [\alpha, \alpha+\delta_\alpha) \subset (\lambda, 1)$ we have

$$\frac{x_{[t(n_\alpha-1)]}}{x_{n_\alpha-1}} = \frac{x_{[\alpha(n_\alpha-1)]}}{x_{n_\alpha-1}} \geq \epsilon$$

This means that each $t \in (\alpha, \alpha+\delta_\alpha)$ belongs to the set A_k, since $n_t \geq (n_\alpha-1)+1 > k$. It follows that if $\alpha \in A_k$, then $(\alpha, \alpha+\delta_\alpha) \subset A_k$. If we assume that some of the sets A_k is dense in $(\lambda, 1)$, then the set $\mathrm{Int}(A_k)$ is also dense in $(\lambda, 1)$. If, on the other side, we suppose that all the sets A_k are dense in $(\lambda, 1)$, then $(\mathrm{Int}(A_k))_{k\in\mathbb{N}}$ is a sequence of open, dense subsets of the set $(\lambda, 1)$ of the second category. It follows that the set $\bigcap_{k\in\mathbb{N}} A_k$ is dense in $(\lambda, 1)$ and thus nonempty, and we have a contradiction. Therefore, there is $n_0 \in \mathbb{N}$ such that the set A_{n_0} is not dense in $(\lambda, 1)$. Consequently, there is an interval $[A, B]$, a proper subset of $(\lambda, 1)$, such that

$$[A, B] \subset (\lambda, 1) \setminus A_{n_0} = \{\alpha \in (\lambda, 1) : n_\alpha \leq n_0\}.$$

From here it follows that $n_\alpha \leq n_0$ for each $\alpha \in [A, B]$, and thus for each $n \geq n_0 \geq n_\alpha$ and each $\alpha \in [A, B]$ it holds $\frac{x_{[\alpha n]}}{x_n} < \epsilon$.

We conclude that for $\lambda \in (0, 1)$ and each $t \in [1, \infty)$ large enough, we have

$$\frac{x_{[\lambda t]}}{x_{[t]}} = \frac{x_{[u[\eta[t]]]}}{x_{[\eta[t]]}} \cdot \frac{x_{[\eta[t]]}}{x_{[t]}},$$

where $u = u(x) \in [A, B]$ and $\eta = \frac{2\lambda}{A+B}$.

Since $\eta \in (0, 1)$ we have

$$\limsup_{t\to\infty} \frac{x_{[\lambda t]}}{x_{[t]}} \leq \epsilon \cdot \limsup_{t\to\infty} \frac{x_{[\eta[t]]}}{x_{[t]}} = 0.$$

This means that the function φ defined by $\varphi(t) = x_{[t]}$, $t \geq 1$, belongs to the class $\mathsf{R}_{\mathsf{f},\infty}$.

$(b) \Rightarrow (a)$: It is trivial, because for an arbitrary and fixed $\lambda \in (0, 1)$ we have

$$\lim_{n\to\infty} \frac{x_{[\lambda n]}}{x_n} = \lim_{x\to\infty} \frac{x_{[\lambda t]}}{c_{[t]}} = 0.$$

There is another class of rapidly varying functions (see [25] and also [4]).

Definition 8 A function $\varphi \in \mathbb{F}$ is said to be *rapidly varying of index of variability* $-\infty$ if for each $\lambda > 1$ it satisfies

$$\lim_{t\to+\infty} \frac{\varphi(\lambda t)}{\varphi(t)} = 0.$$

$\mathsf{R}_{f,-\infty}$ denotes the class of rapidly varying functions of index $-\infty$.

Definition 9 A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is said to belong to the class $\mathsf{R}_{s,-\infty}$

of *rapidly varying sequences of index of variability* $-\infty$ if for each $\lambda > 1$ the following condition is satisfied:

$$\lim_{n\to\infty} \frac{x_{[\lambda n]}}{x_n} = 0.$$

The class of rapidly varying sequences of index of variability $-\infty$ is denoted b $\mathsf{R}_{s,-\infty}$.

We have the following result which is parallel to Theorem 6.

Theorem 7 ([12]) *For a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{S}$ *the following are equivalent:*

(*a*) $\mathbf{x}$ *belongs to the class* $\mathsf{R}_{\mathsf{s},-\infty}$*;*

(*b*) *The function* $\varphi_{\mathbf{x}}$ *defined by* $\varphi_{\mathbf{x}}(t) = x_{[t]}$, $t \geq 1$, *is in the class* $\mathsf{R}_{\mathsf{f},-\infty}$*;*

(*c*) $\lim_{n\to\infty} \frac{x_{[\lambda n]}}{x_n} = \infty,\ 0 < \lambda < 1.$

The following is one more kind of rapid variation.

Definition 10 A function $\varphi \in \mathbb{F}$ belongs to the class $\mathsf{Tr}(\mathsf{R}_{\mathsf{f},\infty})$ of *translationally rapidly varying functions* if for each $\lambda \geq 1$, the following condition holds:

$$\lim_{n\to\infty} \frac{\varphi(t+\lambda)}{\varphi(t)} = \infty.$$

Definition 11 A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is in the class $\mathsf{Tr}(\mathsf{R}_{\mathsf{s},\infty})$ of *translationally rapidly varying sequences* if for each $\lambda \geq 1$, the following condition holds:

$$\lim_{n\to\infty} \frac{x_{[n+\lambda]}}{x_n} = \infty.$$

Note that

$$\mathsf{Tr}(\mathsf{R}_{\mathsf{f},\infty}) \subsetneq \mathsf{R}_{\mathsf{f},\infty} \quad \text{and} \quad \mathsf{Tr}(\mathsf{R}_{\mathsf{s},\infty}) \subsetneq \mathsf{R}_{\mathsf{s},\infty}.$$

The class $\mathsf{Tr}(\mathsf{R}_{\mathsf{s},\infty})$ (and its subclasses) was studied in [13, 14, 15], in particular in connection with selection principles and game theory,

The following new result of Galambos-Bojanić-Seneta type we give without proof because it is similar to proof of Theorem 9.

Theorem 8 *For a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ *the following are equivalent:*

(1) $\mathbf{x} \in \mathsf{Tr}(\mathsf{R}_{\mathsf{s},\infty})$*;*

(2) *The function* $\varphi(t) = x_{[t]}$, $t \geq 1$, *belongs to the class* $\mathsf{Tr}(\mathsf{R}_{\mathsf{f},\infty})$.

We consider now an important subclass of the class $\mathsf{R}_{\mathsf{s},\infty}$, that we denote by $\mathsf{KR}_{\mathsf{s},\infty}$.

Definition 12 For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ the *lower Matuszewska index* $d(\mathbf{x})$ is defined as the supremum of all $d \in \mathbb{R}$ such that for each $\Lambda > 1$

$$\frac{x_{[\lambda n]}}{x_n} \geq \lambda^d(1 + o(1)) \quad (n \to \infty)$$

holds uniformly (with respect to λ) on the segment $[1, \Lambda]$. The sequence $\mathbf{x}$ belongs to the *class* $\mathsf{KR}_{\mathsf{s},\infty}$ if $d(\mathbf{x}) = \infty$.

The definition of lower Matuszewska index for functions can be found in [4, p. 68]. By a result from [4] we have $\mathsf{KR}_{\mathsf{f},\infty} \subsetneqq \mathsf{R}_{\mathsf{f},\infty}$.

Lemma 1 *For a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ *the following are equivalent:*

(1) $\mathbf{x} \in \mathsf{KR}_{\mathsf{s},\infty}$*;*

(2) *For each* $d \in \mathbb{R}$ *it holds* $\liminf_{n\to\infty} \inf_{\lambda\geq 1} \frac{x_{[\lambda n]}}{\lambda^d x_n} \geq 1$.

Proof (1) $\Rightarrow$ (2) From $d(\mathbf{x}) = \infty$, it follows that for every $d \in \mathbb{R}$, every $\Lambda > 1$, and sufficiently large n we have $\frac{x_{[\lambda n]}}{x_n} \geq \lambda^d(1 + o(1))$, where $\lambda \in [1, \Lambda]$ is an arbitrary fixed element. For the same d, λ, Λ, for sufficiently large n we have $\inf_{\lambda\in[1,\Lambda]} \frac{x_{[\lambda n]}}{\lambda^d x_n} \geq 1 + o(1)$. In other words, for each $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\inf_{\lambda\in[1,\Lambda]} \frac{x_{[\lambda n]}}{\lambda^d x_n} \geq 1 - \varepsilon$ for each $n \geq n_0$. Because the last inequality is true for each $\Lambda > 1$, it follows that (for the same d) for each $\lambda \geq 1$ we have $\inf_{\lambda\geq 1} \frac{x_{[\lambda n]}}{\lambda^d x_n} \geq 1 - \varepsilon$. As ε was arbitrary (2) follows.

(2) $\Rightarrow$ (1) Suppose that for an arbitrarily fixed $d \in \mathbb{R}$, $\liminf_{n\to\infty} \inf_{\lambda\geq 1} \frac{x_{[\lambda n]}}{\lambda^d x_n} \geq 1$ is satisfied. Then for the same d and each $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\inf_{\lambda\geq 1} \frac{x_{[\lambda n]}}{\lambda^d x_n} \geq 1 - \varepsilon$ for each $n \geq n_0$. In other words, for the same d, ε, n_0, and for each $\lambda \geq 1$, especially for $\lambda \in [1, \Lambda]$, $\Lambda > 1$ an arbitrary real number, it holds $\frac{x_{[\lambda n]}}{x_n} \geq \lambda^d(1 - \varepsilon)$ for each $n \geq n_0$. This means that for each $\Lambda > 1$ we have $\frac{x_{[\lambda n]}}{x_n} \geq \lambda^d(1 + o(1))$ uniformly with respect to $\lambda \in [1, \Lambda]$ for $n \to \infty$. Since d was arbitrary, (1) follows.

The next statement is a result of the Galambos-Bojanić-Seneta type.

Theorem 9 ([10]) *For a sequence* $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ *the following are equivalent:*

(1) $\mathbf{x} \in \mathsf{KR}_{\mathsf{s},\infty}$*;*

(2) *The function* $\varphi(t) = x_{[t]}$, $t \geq 1$, *belongs to the class* $\mathsf{KR}_{\mathsf{f},\infty}$

Proof (1) $\Rightarrow$ (2) Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathsf{KR}_{\mathsf{s},\infty}$. Then by Lemma 1 we have $\liminf_{n\to\infty} \inf_{\lambda\geq 1} \frac{x_{[\lambda n]}}{\lambda^d x_n} \geq 1$ for each $d \in \mathbb{R}$. This means that for the same d and each $\varepsilon > 0$ there is $n_0 = n_0(d, \varepsilon) \in \mathbb{N}$ such that $\inf_{\lambda\geq 1} \frac{x_{[\lambda[t]]}}{\lambda^d x[t]} \geq 1 - \varepsilon$ for each $t \geq n_0$ (≥ 1). Therefore, for the same d, ε, n_0 it is true

$$\inf_{\lambda\geq 1} \frac{x_{[\lambda t]}}{\lambda^d x_{[t]}} = \inf_{\lambda\geq 1} \frac{x_{[\frac{t}{[t]}\cdot[t]\cdot\lambda]}}{\lambda^d x_{[t]}} \geq \inf_{\lambda\geq 1} \frac{x_{[[t]\cdot\lambda]}}{\lambda^d x_{[t]}} \geq 1 - \varepsilon,$$

i.e. (for this d)

$$\liminf_{t\to\infty} \inf_{\lambda\geq 1} \frac{x_{[\lambda t]}}{\lambda^d x_{[t]}} \geq 1.$$

By [4, Proposition 2.4.3(ii)] it follows that the function $x_{[t]}$ belongs to the class $\mathsf{KR}_{\mathsf{f},\infty}$.

(2) $\Rightarrow$ (1) From $\liminf_{t\to\infty} \inf_{\lambda\geq 1} \frac{x_{[\lambda t]}}{\lambda^d x_{[t]}} \geq 1$ it follows that for this d and each $\varepsilon > 0$ there is $t_0 = t_0(d,\varepsilon) \geq 1$ such that $\inf_{\lambda\geq 1} \frac{x_{[\lambda t]}}{\lambda^d x_{[t]}} \geq 1-\varepsilon$ for all $t \geq t_0$. Since

$$\inf_{\lambda\geq 1} \frac{x_{[\lambda n]}}{\lambda^d x_{[n]}} \geq \inf_{\lambda\geq 1} \frac{x_{[\lambda t]}}{\lambda^d x_{[t]}} \quad \text{for } n \geq [t_0]+1,$$

one obtains

$$\liminf_{n\to\infty} \inf_{\lambda\geq 1} \frac{x_{[\lambda n]}}{\lambda^d x_{[n]}} \geq 1,$$

i.e., (1) is true.

Then following are the definitions of classes of functions and sequences containing the classes of rapidly varying functions and sequences of index of variability ∞ (see ([7] and also [12, 13]).

Definition 13 A function $\varphi \in \mathbb{F}$ is said to be in the *class* ARV_f if for each $\lambda > 1$ it satisfies

$$\underline{k}_\varphi(\lambda) > 1.$$

Definition 14 A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{F}$ belongs to the *class* ARV_s if for each $\lambda > 1$ it satisfies

$$\underline{k}_\mathbf{x}(\lambda) > 1.$$

The next theorem show the relationship between the classes CRV_f and ARV_f .

Theorem 10 ([20, Theorems 3 and 4]). *Let $\varphi \in \mathbb{A}$. Then:*

(*a*) $\varphi \in \mathsf{ARV}_\mathsf{f}$ *if and only if* $\varphi^{\leftarrow} \in \mathsf{CRV}_\mathsf{f}$*;*

(*b*) $\varphi \in \mathsf{CRV}_\mathsf{f}$ *if and only if* $\varphi^{\leftarrow} \in \mathsf{ARV}_\mathsf{f}$*.*

Proof (a) ($\Rightarrow$) Let $\varphi \in \mathbb{A} \cap \mathsf{ARV}_\mathsf{f}$. Then for all $\lambda > 1$ and all $t \geq t_0 = t_0(\lambda)$ we have $\varphi(t) \geq c(\lambda)\varphi(t)$, where $c(\lambda)$ is a function depending on φ, such that $c(\lambda) > 1$, $\lambda > 1$. It follows $\frac{\varphi(\lambda t)}{c(\lambda)} \geq \varphi(t)$ for $\lambda > 1$, hence $\frac{\varphi^{\leftarrow}(c(\lambda)t)}{\lambda} \leq \varphi^{\leftarrow}(t)$. Therefore, for all $\lambda > 1$ we have

$$\overline{k}_{\varphi^{\leftarrow}}(c(\lambda)) = \limsup_{t\to\infty} \frac{\varphi^{\leftarrow}(c(\lambda)t)}{\varphi^{\leftarrow}(t)} \leq \lambda.$$

As $\varphi^{\leftarrow}$ is nondecreasing, its index function $\overline{k}_{\varphi^{\leftarrow}}(c(\overline{\lambda}))$, $\overline{\lambda} > 0$, is also nondecreasing in $\overline{\mathbb{R}}$. The facts that $\overline{k}_{\varphi^{\leftarrow}}(c(\overline{\lambda}))$ is defined for $\overline{\lambda} \in (0, c(\lambda))$,and $c(\lambda) > 1$ imply $\varphi^{\leftarrow} \in \mathsf{ORV_f}$. It follows

$$1 \leq \liminf_{\lambda \to 1+} \underline{k}_{\varphi^{\leftarrow}}(c(\lambda)) \leq \limsup_{\lambda \to 1+} \overline{k}_{\varphi^{\leftarrow}}(c(\lambda)) \leq 1,$$

which gives $\lim_{\lambda \to 1+} \underline{k}_{\varphi^{\leftarrow}}(c(\lambda) = 1$. Let $A = \liminf_{\lambda \to 1+} c(\lambda)$ we have $A > 1$. There is a sequence $(\lambda_n)_{n \in \mathbb{N}}$ with $\lambda_n > 1$ for all n, $\lim_{n \to \infty} \lambda_n = 1+$ and $\lim_{n \to \infty} c(\lambda_n) = A$. Define $c_n = c(\lambda_n)$, $n \in \mathbb{N}$. Then $\lim_{n \to \infty} \overline{k}_{\varphi^{\leftarrow}}(c_n) = 1$. If $A = 1$, then $\lim_{\lambda \to 1+} \overline{k}_{\varphi^{\leftarrow}}(\lambda) = 1$ since $\overline{k}_{\varphi^{\leftarrow}}$ is nondecreasing. So, in this case $\varphi^{\leftarrow} \in \mathsf{CRV_f}$. If $A > 1$, then similarly $\lim_{\lambda \to A-} \overline{k}_{\varphi^{\leftarrow}}(\lambda) = 1$. So, if $\lambda \in [1, (A+1)/2]$, then $\overline{k}_{\varphi^{\leftarrow}}(\lambda) = 1$, and by [4], $\varphi^{\leftarrow} \in \mathsf{SV_f} \subset \mathsf{CRV_f}$.

($\Leftarrow$) Suppose now $\varphi \in \mathbb{A}$ and $\varphi^{\leftarrow} \in \mathsf{CRV}$. Then by [8] it holds

$$\lim_{t \to \infty, \lambda \to 1} \frac{\varphi^{\leftarrow}(\lambda t)}{\varphi^{\leftarrow}(t)} = 1.$$

Therefore, for each $\varepsilon > 1$ there are $t_0 = t_0(\varepsilon) > 0$ and $\delta_0 = \delta_0(\varepsilon) > 0$ such that

$$\frac{1}{\varepsilon} \leq \frac{\varphi^{\leftarrow}(\lambda t)}{\varphi^{\leftarrow}(t)} \leq \varepsilon$$

for each $t \geq t_0$ and each $\lambda \in [1 - \delta_0, 1 + \delta_0]$. Hence, for these λ and t we have

$$\frac{\varphi^{\leftarrow}(\lambda t)}{\varepsilon} \leq \varphi^{\leftarrow}(t) \text{ and } (\varphi(\varepsilon t)/\lambda)^{\leftarrow} \leq \varphi^{\leftarrow}(t)$$

so that

$$((\varphi(\varepsilon t)/\lambda)^{\leftarrow})^{\leftarrow} \geq (\varphi^{\leftarrow}(t))^{\leftarrow}.$$

Since for every function h in $\mathbb{A}$ and each $\beta > 1$ it holds $h(t) \leq ((h(t)^{\leftarrow}))^{\leftarrow} \leq h(\beta t)$, $t \geq a$,we obtain $\varphi(t) \leq \varphi(\varepsilon^2 t)/\lambda$, i.e., $\varphi(\varepsilon^2 t) \geq \lambda \varphi(t)$. So $\varphi(\varepsilon^2 t) \geq (1 + \delta_0(\varepsilon))\varphi(t)$ for $t \geq t_0$.

If $\alpha > 1$, take $\varepsilon = \sqrt{\alpha}$. So, we have

$$\varphi(\alpha t) \geq (1 + \delta_0(\sqrt{\alpha}))\varphi(t)$$

for $t \geq t_0(\sqrt{\alpha}) > 0$, which means $\varphi \in \mathsf{ARV_f}$.

(b) ($\Rightarrow$) Since $\varphi \in \mathbb{A} \cap \mathsf{CRV_f}$ we have

$$\lim_{t \to \infty, \lambda \to 1} \frac{\varphi(\lambda t)}{\varphi(t)} = 1.$$

It follows that for each $\varepsilon > 1$ there exist $t_0 = t_0(\varepsilon)$ and $\delta_0 = \delta_0(\varepsilon)$ such that $\frac{1}{\varepsilon} \leq \frac{\varphi(\lambda t)}{\varphi(t)} \leq \varepsilon$ for each $t \geq t_0$ and each $\lambda \in [1 - \delta_0, 1 + \delta_0]$, so that for these λ and t, $\varphi(\lambda t)/\varepsilon \leq \varphi(t)$, and consequently $\varphi^{\leftarrow}(\varepsilon t) \geq \lambda \varphi^{\leftarrow}(t)$. Therefore,

$$\varphi^{\leftarrow}(\varepsilon t) \geq (1 + \delta_0)\varphi^{\leftarrow}(t) \text{ for } t \geq t_0$$

which means that $\varphi\leftarrow \in \mathsf{ARV_f}$.

($\Leftarrow$) Let now $\varphi \in \mathbb{A}$ and $\varphi^{\leftarrow} \in \mathsf{ARV_f}$. Then for each $\lambda > 1$ and each $t \geq t_0 = t_0(\lambda)$, we have $\varphi^{\leftarrow}(\lambda t) \geq c(\lambda)\varphi^{\leftarrow}(t)$, where $c(\lambda) > 1$ depends on φ. So, for these λ and t, $\varphi^{\leftarrow}(\lambda t)/c(\lambda) \geq \varphi(t)$, and thus $(\varphi(c(\lambda)t)/\lambda)^{\leftarrow} \geq \varphi^{\leftarrow}(t)$. Similarly to the proof of the second part of the previous theorem we get

$$\frac{\varphi(c(\lambda)t)}{\lambda} \leq \varphi(t\sqrt{c(\lambda)}), \text{ i.e. } \frac{\varphi(c(\lambda)t)}{\varphi(t\sqrt{c(\lambda)})} \leq \lambda.$$

Therefore, for each $\lambda > 1$ we have $\varphi(u\sqrt{c(\lambda)})/\varphi(u) \leq \lambda$ for each $u \geq t_0\sqrt{c(\lambda)} = u_0(\lambda)$ which means that $\overline{k}_\varphi(\sqrt{c(\lambda)} \leq \lambda$ (for each $\lambda > 1$). This implies

$$1 \leq \liminf_{\lambda\to 1+} \overline{k}_\varphi(\sqrt{c(\lambda)}) \leq \limsup_{\lambda\to 1+} \overline{k}_\varphi(\sqrt{c(\lambda)}) \leq 1,$$

which gives

$$\lim_{\lambda\to 1+} \overline{k}_\varphi(\sqrt{c(\lambda)}) = 1.$$

Set $A = \liminf_{\lambda\to 1+} \sqrt{c(\lambda)} \geq 1$. There exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $\lambda_n > 1$ for all n, $\lim_{n\to\infty} \lambda_n = 1+$ and $\lim_{n\to\infty} \sqrt{c(\lambda_n)} = A$.

Define $c_n = \sqrt{c(\lambda_n)}$, $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \overline{k}_\varphi(c_n) = 1$. If $A = 1$, then $\lim_{\lambda\to 1+} \overline{k}_\varphi(\lambda) = 1$ since $\overline{k}_\varphi$ is nondecreasing. So, $\varphi \in \mathsf{CRV_f}$. If $A > 1$, then $\lim_{\lambda\to A-} \overline{k}_\varphi(\lambda) = 1$. So, if $\lambda \in [1, (A+1)/2]$, then $\overline{k}_\varphi(\lambda) = 1$, and thus $\varphi \in \mathsf{SV_f} \subsetneq \mathsf{CRV_f}$.

The following Galambos-Bojanić-Seneta type result is true.

Theorem 11 ([7]) *Let* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{S}$. *Then the following assertions are equivalent:*

(a) $\mathbf{x} \in \mathsf{ARV_s}$;

(b) $\varphi_{\mathbf{x}}(t) = x_{[t]}$, $t \geq 1$, *belongs to* $\mathsf{ARV_f}$.

1.3 New result

In this section we prove a new result of Galambos-Bojanić-Seneta type.

Definition 15 ([6, 7]) A function $\varphi \in \mathbb{F}$ is said to be in the *class* $\mathsf{PI_f^*}$ if there exists $\lambda_0 \geq 1$ such that for each $\lambda > \lambda_0$

$$\underline{k}_\varphi(\lambda) := \liminf_{x\to\infty} \frac{\varphi(\lambda x)}{\varphi(x)} > 1.$$

Definition 16 A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \mathbb{S}$ is said to belong to the *class* PI_s^* if there is $\lambda_0 \geq 1$ such that for each $\lambda > \lambda_0$ it holds

$$\underline{k}_x(\lambda) > 1.$$

Clearly, if in the previous two definitions $\lambda_0 = 1$ we have the classes $\mathsf{ARV_f}$ and $\mathsf{ARV_s}$, respectively, from the previous section.

Observe that the following holds:

$$\mathsf{R}_{\mathsf{s},\infty} \subsetneqq \mathsf{ARV_s} \subsetneqq \mathsf{PI}_\mathsf{s}^*.$$

The next theorem shows the importance of the class PI_f^* because it is conjugate (by the generalized inverse) with the very important class $\mathsf{ORV_f}$.

Theorem 12 ([16, Propositions 3 and 4]) *Let $\varphi \in \mathbb{A}$. Then:*

(*a*) $\varphi \in \mathsf{PI}_\mathsf{f}^*$ *if and only if* $\varphi^{\leftarrow} \in \mathsf{ORV_f}$*;*

(*b*) $\varphi \in \mathsf{ORV_f}$ *if and only if* $\varphi^{\leftarrow} \in \mathsf{PI}_\mathsf{f}^*$*.*

Proof (a) First assume $\varphi \in \mathbb{A} \cap \mathsf{PI}_\mathsf{f}^*$. Then for some $\lambda_0 \geq 1$ and for some $\lambda > \lambda_0$ it holds

$$\varphi(\lambda t) \geq c(\lambda)(t),\ t \geq t_0 = t_0(\lambda),$$

where $c(\lambda) = c_\varphi(\lambda) > 1$ for $\lambda > \lambda_0$. Therefore, for these λ and t we have $\varphi(\lambda t)/c(\lambda) \geq \varphi(t)$. It follows $\varphi^{\leftarrow}(c(\lambda)t)/\lambda \leq \varphi^{\leftarrow}(t)$. Hence, for this λ we have

$$\overline{k}_{\varphi^{\leftarrow}}(c(\lambda)) \leq \lambda < \infty$$

which means $\varphi^{\leftarrow} \in \mathsf{ORV_f}$.

Conversely, assume $\varphi^{\leftarrow} \in \mathsf{ORV_f} \cap \mathbb{A}$. Then by [1] we have

$$\limsup_{t\to\infty} \sup_{\lambda\in[1,2]} \frac{\varphi^{\leftarrow}(\lambda t)}{\varphi^{\leftarrow}(t)} = \limsup_{t\to\infty} \frac{\varphi^{\leftarrow}(2x)}{\varphi^{\leftarrow}(t)} = \overline{k}_{\varphi^{\leftarrow}}(2) \geq 1.$$

For each $\varepsilon > 0$ there is a $t_0 = t_0(\varepsilon) > 0$ such that

$$\sup_{\lambda\in[1,2]} \frac{\varphi^{\leftarrow}(\lambda t)}{\varphi^{\leftarrow}(t)} \leq \overline{k}_{\varphi^{\leftarrow}}(2) + \varepsilon = M(\varepsilon),\ t \geq t_0,$$

so that for each $t \geq t_0$ and each $\lambda \in [1, 2]$ we have

$$\frac{\varphi^{\leftarrow}(\lambda t)}{\varphi^{\leftarrow}(t)} \leq M(\varepsilon).$$

It follows

$$\begin{aligned}\frac{\varphi^{\leftarrow}(\lambda t)}{M(\varepsilon)} \leq \varphi^{\leftarrow}(t) \quad &\Rightarrow \quad \left(\left(\frac{f(M(\varepsilon)t)}{\lambda}\right)^{\leftarrow}\right)^{\leftarrow} \geq (\varphi^{\leftarrow}(t))^{\leftarrow} \\ &\Rightarrow \quad \varphi(t) \leq \frac{\varphi(M^2(\varepsilon)t)}{\lambda} \\ &\Rightarrow \quad \frac{\varphi(M^2(\varepsilon)t)}{\varphi(t)} \geq \lambda \\ &\Rightarrow \quad \frac{\varphi(M^2(\varepsilon)t)}{\varphi(t)} \geq 2 > 1 \\ &\Rightarrow \quad \liminf_{t\to\infty} \frac{\varphi(M^2(\varepsilon)t)}{\varphi(t)} = \underline{k}_{\varphi}(M^2(\varepsilon)) \geq 2 > 1.\end{aligned}$$

Since $\underline{k}_{\varphi}(u)$ is nondecreasing for $u > 0$, we find that $\underline{k}_{\varphi}(\lambda) > 1$, for $\lambda > M^2(\varepsilon) > 1$. Hence, $\varphi \in \mathsf{PI}^{\mathsf{n}}_{\mathsf{f}} * \cap \mathbb{A}$.

(b) First assume $\varphi \in \mathbb{A} \cap \mathsf{ORV}_{\mathsf{f}}$. By [1]

$$\limsup_{t\to\infty} \sup_{\lambda\in[1,2]} \frac{\varphi(\lambda t)}{\varphi(t)} = \limsup_{t\to\infty} \frac{\varphi(2t)}{\varphi(t)} = \overline{k}_{\varphi}(2) \geq 1.$$

For each $\varepsilon > 0$, there is $t_0 = t_0(\varepsilon) > 0$ such that

$$\sup_{\lambda\in[1,2]} \frac{\varphi(\lambda t)}{\varphi(t)} \leq \overline{k}_{\varphi}(2) + \varepsilon = m(\varepsilon), \quad \text{for all } t \geq t_0.$$

So, for the same t and for each $\lambda \in [1, 2]$ we have $\varphi(\lambda t)/\varphi(t) \leq m(\varepsilon)$. Therefore, $\varphi(\lambda t)/m(\varepsilon) \leq \varphi(t)$. It follows

$$\begin{aligned}\frac{\varphi^{\leftarrow}(m(\varepsilon)t)}{\lambda} \geq \varphi^{\leftarrow}(t) \quad &\Rightarrow \quad \varphi^{\leftarrow}(m(\varepsilon)t) \geq \lambda\varphi^{\leftarrow}(t) \\ &\Rightarrow \quad \varphi^{\leftarrow}(m(\varepsilon)t) \geq 2\varphi^{\leftarrow}(t) \\ &\Rightarrow \quad \frac{\varphi^{\leftarrow}(m(\varepsilon)t)}{\varphi^{\leftarrow}(t)} \geq 2 > 1 \\ &\Rightarrow \quad \liminf_{t\to\infty} \frac{\varphi^{\leftarrow}(m(\varepsilon)t)}{\varphi^{\leftarrow}(t)} \geq 2 > 1 \\ &\Rightarrow \quad \underline{k}_{\varphi^{\leftarrow}}(m(\varepsilon)) < 1.\end{aligned}$$

Hence, $\underline{k}_{\varphi^{\leftarrow}}(\lambda) > 1$ for $\lambda > m(\varepsilon) = \lambda_0 \geq 1$, which means $\varphi^{\leftarrow} \in \mathsf{PI}^*_{\mathsf{f}}$.

Conversely, assume $\varphi^{\leftarrow} \in \mathsf{PI}^*_{\mathsf{f}} \cap \mathbb{A}$. Then for some $\lambda_0 \geq 1$ and all $\lambda > \lambda_0$ we have $\varphi^{\leftarrow}(\lambda t) \geq c(\lambda)\varphi^{\leftarrow}(t)$, for all $t \geq t_0 = t_0(\lambda)$, where $c(\lambda) = c_{\varphi}(\lambda) > 1$, $\lambda > \lambda_0$. Hence, for those λ and t we have $\varphi^{\leftarrow}(\lambda t)/c(\lambda) \geq \varphi^{\leftarrow}(t)$, so that $\big(\varphi(c(\lambda)t)/\lambda)\big)^{\leftarrow} \geq \varphi^{\leftarrow}(t)$. As in the previous proof, we have $\varphi(c(\lambda)t)/\lambda \leq \varphi(\sqrt{c(\lambda)}t)$. Therefore, $\varphi(c(\lambda)t)/\varphi(\sqrt{c(\lambda)}t) \leq \lambda$, and consequently, for a fixed $\lambda > \lambda_0$, we obtain $\overline{k}_{\varphi}(\sqrt{c(\lambda)}) \leq \lambda < \infty$. In other words, $\varphi \in \mathsf{ORV}_{\mathsf{f}}$.

We prove now a new result of the Galambos-Bojanić-Seneta type for the classes PI^*.

Theorem 13 *Let* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{S}$*. Then the following are equivalent:*

(*a*) $\mathbf{x}$ *belongs to the class* $\mathsf{PI}_{\mathsf{s}}^*$*;*

(*b*) *The function* $\varphi(t) = x_{[t]}$*,* $t \geq 1$*, belongs to the class* $\mathsf{PI}_{\mathsf{f}}^*$*.*

Proof (a) ⇒ (b): Let the sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ belong to the class $\mathsf{PI}_{\mathsf{s}}^*$. Then

$$\liminf_{n\to\infty} \frac{x_{[\lambda n]}}{x_n} > 1 \text{ for some } \lambda_0 \geq 1 \text{ and each } \lambda > \lambda_0.$$

Consider the interval (λ_0, λ_0^2). Then $k_{\mathbf{x}}(\lambda) > 1$ on the interval (λ_0, λ_0^2). For each $\lambda \in (\lambda_0, \lambda_0^2)$ define $n_\lambda \in \mathbb{N}$ in the following way:

$$n_\lambda = \begin{cases} 1, & \text{if } \frac{x_{[\lambda n]}}{x_n} > 1 \text{ for each } n \in \mathbb{N}; \\ 1 + \max\{n \in \mathbb{N} : \frac{x_{[\lambda n]}}{x_n} \leq 1\}, & \text{otherwise.} \end{cases}$$

Evidently, $1 \leq n_\lambda < \infty$ for each λ. Then one defines the sequence $(A_k)_{k\in\mathbb{N}}$ by

$$A_k = \{\lambda \in (\lambda_0, \lambda_0^2) : n_\lambda > k\}$$

which is non-increasing and $\bigcap_{k\in\mathbb{N}} A_k = \emptyset$.

We prove that not all sets A_k are dense in (λ_0, λ_0^2). If $\lambda \in A_k$ for some k, then $\frac{x_{[\lambda(n_\lambda - 1)]}}{x_{n_\lambda - 1}} \leq 1$ and there is $\delta_\lambda > 0$ such that

$$\frac{x_{[t(n_\lambda - 1)]}}{x_{n_\lambda - 1}} \leq 1 \text{ for each } t \in [\lambda, \lambda + \delta_\lambda) \subsetneqq (\lambda_0, \lambda_0^2).$$

So, each $t \in (\lambda, \lambda + \delta_\lambda)$ belongs to A_k, hence $n_t \geq (n_\lambda - 1) + 1 > k$. We conclude that $(\lambda, \lambda + \delta_\lambda) \subset A_k$ whenever $\lambda \in A_k$. Suppose now that each set A_k is dense in (λ_0, λ_0^2). Then each set $\mathrm{Int}(A_k)$ is also dense, hence we have the sequence $(\mathrm{Int}(A_k))_{k\in\mathbb{N}}$ of open dense subsets of (λ_0, λ_0^2) which is a Baire second category set. By the Baire category theorem we have that the set $\bigcap_{k\in\mathbb{N}} \mathrm{Int}(A_k)$ dense in (λ_0, λ_0^2) and so the set $\bigcap_{k\in\mathbb{N}} A_k$ is nonempty. It is a contradiction.

Therefore, there $k_0 \in \mathbb{N}$ so that the set A_{k_0} is not dense in (λ_0, λ_0^2). There is the closed interval $[A, B] \subsetneqq (\lambda_0, \lambda_0^2)$ such that $[A, B] \subset (\lambda_0, \lambda_0^2) \setminus A_{k_0} ==$ $\{\lambda \in (\lambda_0, \lambda_0^2) : n_\lambda \leq k_0\}$. Therefore, for each $\lambda \in [A, B]$, $n_\lambda \leq k_0$. It follows that for each $\lambda \in [A, B]$ and each $k \geq k_0 \geq k_\lambda$, $\frac{x_{[\lambda k]}}{x_k} > 1$. Thus for each $\lambda > \lambda_0^3$ and each sufficiently large $t \geq t_0 \geq 1$ it holds

$$\frac{x_{[\lambda t]}}{x_t} = \frac{x_{z[\eta t]}}{x_{[\eta t]}} \cdot \frac{x_{[\eta t]}}{x_t} \geq k_{\mathbf{x}}(\eta) > 1.$$

This means that the function φ, $\varphi(t) = x_{[t]}$, $t \geq 1$, belongs to the class PI_f^*.

(b) ⇒ (a): It is evident.

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