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Eigenvalue Analysis for Transverse Vibration of Stepped Column with Lumped Mass at the Top by Finite Difference Approach

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***Abstract*—Differential eigenvalue problem of stepped column with lumped mass at the top was analysed by finite difference method. Exact differential governing equation and boundary conditions were discretized with central differences applied upon grid points along the column. Boundary value problem was transformed in an appropriate and compact structure of algebraic equations written in matrix form, suitable for development of computer routines. For a comparison purpose, a finite element method analysis was conducted in ANSYS. Presented model gave natural frequencies that were in a very good agreement with the results obtained from finite element simulation.**

***Keywords*—Stepped column, Transverse vibration, Euler–Bernoulli theory, Eigenvalue, Finite difference method, Finite element method.**

I. INTRODUCTION

Being frequently utilized as a part of many engineering structures, the beams with varying cross-section were extensively investigated during past decades. One group of non-uniform beams are the ones with continuously variable cross-section, i.e. tapered beams, whose free bending vibration was researched by many authors [1-5]. On the other hand, there is another type of non-uniform beams with step changes of cross-sections, such as multi step shaft parts, columns in civil engineering, etc. Published researches on free bending vibration of stepped beams with various types of supports are numerous. For example, S. Naguleswaran [6,7] used analytical method and continuity of deflection, slope, bending moment and shearing force to calculate the frequencies of beams with up to three step changes in cross-section. Li et al. [8] developed a finite element with generalized degrees of freedom for the dynamic analyses of beams and plates with cross-section varying in a continuous or discontinuous manner. Qiao et al. [9] presented analytical method for investigating free flexural vibrations of non-uniform multi-step Euler–Bernoulli beams with any kind of support configurations and carrying an arbitrary number of single-degree-of-freedom and two-degree-of-freedom spring–mass systems.

Koplow et al. [10] found an analytical solution for the dynamic response of a discontinuous beam with one-step change. Jaworski and Dowell [11] considered the accuracy and convergence of the Rayleigh–Ritz method, component modal analysis, and the finite element method in the free vibration analysis of a multiple-stepped cantilevered beam. Yet, the demand for shorter time for design force the mechanical and civil engineers to act promptly to the new challenges in structural design with some efficient but acceptably accurate solutions. In another words, real-life structural design tasks, with are becoming more complex, usually impose the use of some numerical approaches. Engineers mostly use finite element method (FEM) to solve everyday tasks, but it is mostly used for case studies. On the other hand, finite difference method (FDM) allows to obtain also approximate but wider solutions. Survey upon available literature revealed the fact that FDM was not utilized as often as it deserves to be. AL-Sadder and AL-Rawi [12] used FDM for static large-deflection analysis of non-prismatic cantilever beams subjected to different types of continuous and discontinuous loadings. Awrejcewicz et al. [13] studied regular and chaotic dynamics of the uniform Euler-Bernoulli beams and used FDM and finite element method (FEM) to verify the reliability of the obtained results. This paper presents a detailed workflow for development of the FDM scheme for the eigenvalue analysis of the stepped column with lumped mass at its top in transverse bending vibration. Presented approach gives a numerical scheme, which is suitable for development of computational algorithms. Such codes enable engineers to obtain natural frequencies of the stepped column for various design cases, i.e. input parameters, quickly but with acceptable accuracy. Presented approach can be extended to cases with any number of step changes in cross-section and with elastic support. Accuracy of the obtained results is verified by FEM analysis and the results are in a very good agreement.

II. MODEL DESCRIPTION AND GOVERNING EQUATION

The case being considered, a three-segment stepped column with lumped mass M at its top, is presented in Fig.1.

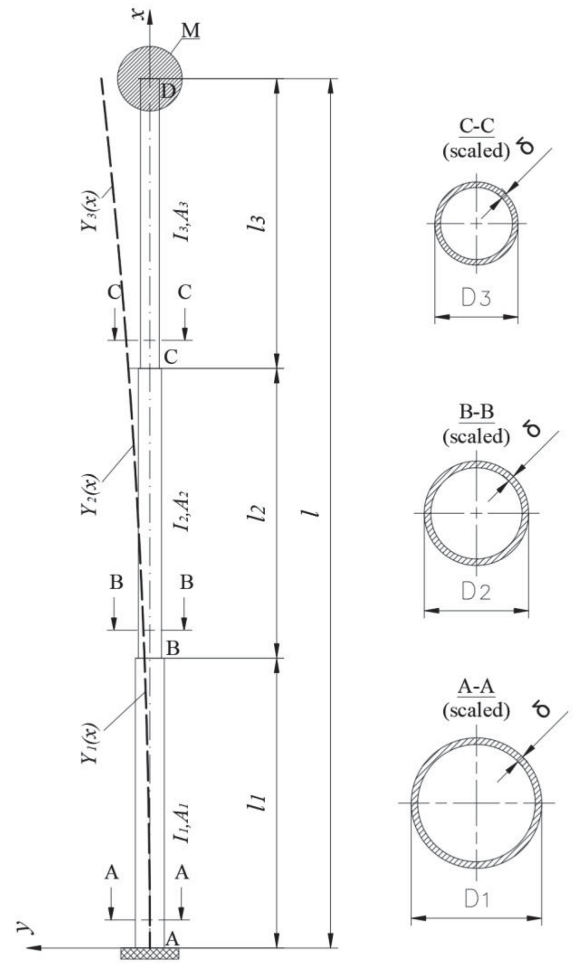

Fig. 1 The model of three-segment stepped column with lumped mass at the top and its cross-sections

Analysis is based on Euler-Bernoulli beam theory, i.e. on the assumption that the rotatory inertia of the differential element and shear effects are negligible. The partial differential equation for the free vibration of beams in bending, according to Euler-Bernoulli beam theory, is well known [14]

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},\ 0\le x\le l \quad (1)$$

Free vibration is harmonic, so the transverse displacement of the beam can be expressed in the form

$$y(x,t)=CY(x)\cos(\omega t-\varphi),\ 0\le x\le l \quad (2)$$

where C is amplitude, $Y(x)$ is a mode shape function, ω is circular natural frequency and φ is a phase angle. If the bending stiffness $I(x)$ and the unit mass $m(x)$ are constant within each segment, we obtain the differential eigenvalue problem for Euler-Bernoulli beam in bending

$$EI_i\frac{d^4Y_i(x_i)}{dx^4}=m_i\omega^2Y_i(x),\ i=1,2,3 \quad (3)$$

So, the whole column is considered as system consisted of three beams, where each segment has a mode shape governing equation and four boundary conditions, two conditions per end of segment. Boundary conditions at the clamped support are related to deflection and slope as follows:

$$Y_1(x=0)=0 \quad (4)$$

$$Y_1'(x=0)=0 \quad (5)$$

Boundary conditions at the junction of segments 1 and 2 (point B) are continuity conditions of deflection, slope, bending moment and shear force:

$$Y_1(x=l_1)=Y_2(x=l_1) \quad (6)$$

$$Y_1'(x=l_1)=Y_2'(x=l_1) \quad (7)$$

$$EI_1Y_1''(x=l_1)=EI_2Y_2''(x=l_1) \quad (8)$$

$$-EI_1Y_1'''(x=l_1)=-EI_2Y_2'''(x=l_1) \quad (9)$$

Similarly, boundary conditions at the junction of segments 2 and 3 (point C) are continuity conditions of deflection, slope, bending moment and shear force:

$$Y_2(x=l_1+l_2)=Y_2(x=l_1+l_2) \quad (10)$$

$$Y_2'(x=l_1+l_2)=Y_3'(x=l_1+l_2) \quad (11)$$

$$EI_2Y_2''(x=l_1+l_2)=EI_3Y_3''(x=l_1+l_2) \quad (12)$$

$$-EI_2Y_2'''(x=l_1+l_2)=-EI_3Y_3'''(x=l_1+l_2) \quad (13)$$

Boundary conditions for the column top, with fixed lumped mass M (point D), are related to bending moment and shear force:

$$Y_3''(x=l_1+l_2+l_3)=0 \quad (14)$$

$$[Y_3'''(x=l)+\frac{M\omega^2}{EI_3}Y_3(x=l)]=0 \quad (15)$$

III. Discretization of Boundary Value Problem by FDM

Fig. 2a shows central finite difference grid scheme where the total length of the column L is equally divided by grid points into N segments with length $s=l/N$. To apply the method, there are three fictitious grid points added to the scheme, one before the root grid point and two after free-end grid point. Also, there are additional fictitious displacements (superscript f) at the junction of each pair of adjacent segments as their shape functions have different domains (Fig. 2b).

Central finite difference approximations for derivatives of shape function $Y(x)$ for grid point n are as follows

$$\left(\frac{dY}{dx}\right)_n=Y_n'\approx\frac{-Y_{n-1}+Y_{n+1}}{2s} \quad (16)$$

$$\left(\frac{d^2Y}{dx^2}\right)_n=Y_n''\approx\frac{Y_{n-1}-2Y_n+Y_{n+1}}{s^2} \quad (17)$$

$$\left(\frac{d^3Y}{dx^3}\right)_n=Y_n'''\approx\frac{-Y_{n-2}+2Y_{n-1}-2Y_{n+1}+Y_{n+2}}{2s^3} \quad (18)$$

$$\left(\frac{d^4Y}{dx^4}\right)_n = Y_n^{IV} \approx \frac{Y_{n-2} - 4Y_{n-1} + 6Y_n - 4Y_{n+1} + Y_{n+2}}{s^4} \quad (19)$$

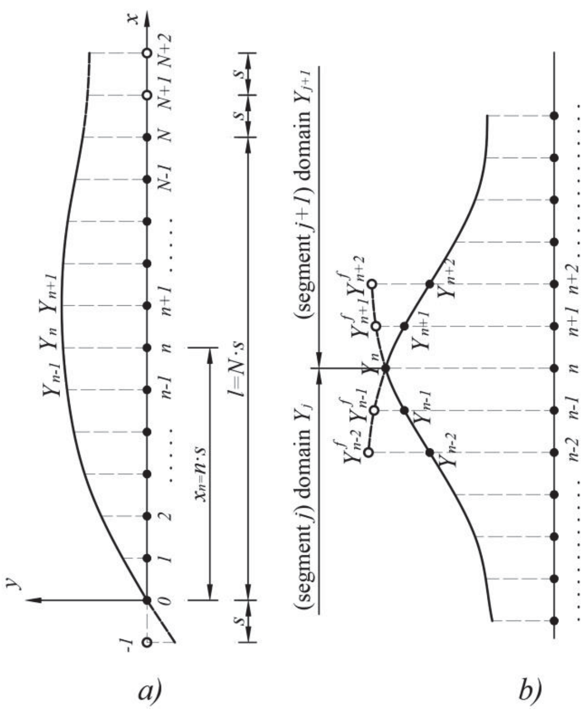

Fig. 2 a) Central finite difference grid scheme used for discretization of boundary value problem; b) additional fictitious displacements at the junction of two adjacent segments.

After conversion of all constituent equations through discretization, we combine them and form the algebraic eigenvalue problem. In fact, by insertion of discretized boundary conditions into discretized governing equation, all external fictitious displacements Y^f_{-1}, Y^f_{N+1}, Y^f_{N+2} and internal fictitious displacements Y^f_{n-2}, Y^f_{n-1}, Y^f_{n+1}, Y^f_{n+2} at segments' junctions are being eliminated, which yields a set of N algebraic equations with Y_n, $n=1,\ldots,N$ as unknowns and $\lambda = \omega^2 s^4$ as a parameter. Discretized governing equation for mode shape function was written for each grid point, where only expressions for junction grid points and their adjacent ones differ from a standard form of discretized governing equation. The system of equations is given as follows.

$$n=1: B_1\left[7Y_1 - 4Y_2 + Y_3\right] = \lambda Y_1 \quad (20)$$

$$n=2: B_1\left[-4Y_1 + 6Y_2 - 4Y_3 + Y_4\right] = \lambda Y_2 \quad (21)$$

For grid points within interval $n=3 \div B\text{-}2$ it reads:

$$B_1\left[Y_{n-2} - 4Y_{n-1} + 6Y_n - 4Y_{n+1} + Y_{n+2}\right] = \lambda Y_n \quad (22)$$

For grid point $n=B\text{-}1$ it reads:

$$B_1\left[Y_{B-3} - 4Y_{B-2} + P_1Y_{B-1} - P_2Y_B + P_3Y_{B+1}\right] = \lambda Y_{B-1} \quad (23)$$

For grid point $n=B$ it reads:

$$B_1\left[P_4Y_{B-2} - P_5Y_{B-1} + P_6Y_B - P_7Y_{B+1} + P_8Y_{B+2}\right] = \lambda Y_B \quad (24)$$

For grid point $n=B+1$ it reads:

$$B_2\left[P_9Y_{B-1} - P_{10}Y_B + P_{11}Y_{B+1} - 4Y_{B+2} + Y_{B+3}\right] = \lambda Y_{B+1} \quad (25)$$

For grid points within interval $n=B+1 \div C\text{-}2$ it reads:

$$B_2\left[Y_{n-2} - 4Y_{n-1} + 6Y_n - 4Y_{n+1} + Y_{n+2}\right] = \lambda Y_n \quad (26)$$

For grid point $n=C\text{-}1$ it reads:

$$B_2\left[Y_{C-3} - 4Y_{C-2} + Q_1Y_{C-1} - Q_2Y_C + Q_3Y_{C+1}\right] = \lambda Y_{C-1} \quad (27)$$

For grid point $n=C$ it reads:

$$B_2\left[Q_4Y_{C-2} - Q_5Y_{C-1} + Q_6Y_C - Q_7Y_{C+1} + Q_8Y_{C+2}\right] = \lambda Y_C \quad (28)$$

For grid point $n=C+1$ it reads:

$$B_3\left[Q_9Y_{C-1} - Q_{10}Y_C + Q_{11}Y_{C+1} - 4Y_{C+2} + Y_{C+3}\right] = \lambda Y_{C+1} \quad (29)$$

For grid points within interval $n=B+1 \div C\text{-}2$ it reads:

$$B_3\left[Y_{n-2} - 4Y_{n-1} + 6Y_n - 4Y_{n+1} + Y_{n+2}\right] = \lambda Y_n \quad (30)$$

For grid point $n=D\text{-}1$ it reads:

$$B_3\left[Y_{D-3} - 4Y_{D-2} + 5Y_{D-1} - 2Y_D\right] = \lambda Y_{D-1} \quad (31)$$

Finally, discretised equation for end grid point at the top of the column is:

$$W\left[2Y_{D-2} - 4Y_{D-1} + 2Y_D\right] = \lambda Y_D \quad (32)$$

The list of condensed expressions from previous system of equations is given in Table1.

TABLE I LIST OF CONDENSED EXPRESSIONS

$B_1 = EI_1 / m_1$	$P_9 = 2a/(1+a)$
$B_2 = EI_2 / m_2$	$P_{10} = (6a+2)/(1+a)$
$B_3 = EI_3 / m_3$	$P_{11} = (7a+5)/(1+a)$
$a = I_1 / I_2$	$Q_1 = (5c+7)/(1+c)$
$b = m_2 / m_1$	$Q_2 = (6+2c)/(1+c)$
$c = I_2 / I_3$	$Q_3 = 2/(1+c)$
$d = m_3 / m_2$	$Q_4 = 2/(1+d)$
$P_1 = (5a+7)/(1+a)$	$Q_5 = \dfrac{4(3+c)}{(1+d)(1+c)}$
$P_2 = (6+2a)/(1+a)$	$Q_6 = \dfrac{2c^2+20c+2}{c(1+d)(1+c)}$
$P_3 = 2/(1+a)$	$Q_7 = \dfrac{12c+4}{c(1+d)(1+c)}$
$P_4 = 2/(1+b)$	$Q_8 = 2/(c(1+d))$
$P_5 = \dfrac{4(3+a)}{(1+b)(1+a)}$	$Q_9 = \dfrac{2c}{1+c}$
$P_6 = \dfrac{2a^2+20a+2}{a(1+b)(1+a)}$	$Q_{10} = \dfrac{6c+2}{1+c}$
$P_7 = \dfrac{12a+4}{a(1+b)(1+a)}$	$Q_{11} = \dfrac{7c+5}{1+c}$
$P_8 = \dfrac{2}{a(1+b)}$	$W = \dfrac{sEI_3}{sm_3+2M}$

IV. NUMERICAL EXAMPLE AND VERIFICATION

Based on the previous system of equations, appropriate MatLAB code was written and the characteristic equation was solved. In order to verify the applied approach, the FEM analysis is performed by using ANSYS software. Circular hollow sections with constant and equal wall thickness are taken for segments' cross-sections, therefore the bending stiffness variation is done by changing the diameters values. Wall thickness of pipes was δ=5[mm] and overall height of column was l=3000[mm]. In all cases, the diameter of first segment is D_1=100[mm], while the diameters of second and third segment D_2 and D_3 are varied. All segments have equal lengths. For all cases, the lumped mass was $M=50kg$.

Table 2 shows the results for the first two natural frequencies obtained by both FDM and FEM analysis and corresponding relative deviations. A comparison between the results obtained by presented FDM approach and FEM analysis reveals very good agreement.

TABLE III COMPARISON OF FDM AND FEM RESULTS

D_2 [cm]	D_3 [cm]	f_1 [s^{-1}]		δ_1 [%]	f_2 [s^{-1}]		δ_2 [%]
		FDM	FEA		FDM	FEA	
9.50	9.00	4.053	4.047	0.14%	47.095	45.830	2.76%
	8.50	4.036	4.041	-0.13%	45.857	44.892	2.15%
	8.00	4.014	4.029	-0.36%	44.405	43.728	1.55%
	7.50	3.987	4.009	-0.56%	42.711	42.338	0.88%
	7.00	3.951	3.978	-0.67%	40.756	40.710	0.11%
9.00	8.50	3.936	3.940	-0.10%	44.495	44.450	0.10%
	8.00	3.929	3.930	-0.05%	43.987	43.481	1.16%
	7.50	3.891	3.913	-0.57%	41.720	42.283	-1.33%
	7.00	3.857	3.886	-0.74%	39.962	40.843	-2.16%
	6.50	3.813	3.846	-0.84%	37.933	39.170	-3.16%
8.50	8.00	3.802	3.811	-0.22%	41.953	43.103	-2.67%
	7.50	3.779	3.796	-0.46%	40.641	42.102	-3.47%
	7.00	3.748	3.773	-0.66%	39.083	40.858	-4.34%
	6.50	3.707	3.737	-0.80%	37.256	39.368	-5.36%
	6.00	3.653	3.685	-0.88%	35.148	37.634	-6.61%
8.00	7.50	3.648	3.655	-0.20%	39.475	41.815	-5.60%
	7.00	3.620	3.637	-0.46%	38.116	40.776	-6.52%
	6.50	3.584	3.607	-0.66%	36.496	39.488	-7.58%
	6.00	3.534	3.561	-0.76%	34.590	37.927	-8.80%
	5.50	3.467	3.496	-0.83%	32.398	36.139	-9.35%

V. CONCLUSION

Differential eigenvalue problem of the stepped column with lumped mass at the top was efficiently solved by building a stable and compact structure of algebraic equations derived from FDM approach. For common engineering needs, the results for first two natural frequencies revealed very good agreement with the results obtained by FEM simulation in ANSYS.

Presented approach has great versatility and can be utilized to solve an eigenvalue problem for the beam with any other type of cross-sectional variation and boundary conditions, without any limitations.

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