Research Article

Akbar Dehghan Nezhad, Marija Paunovic, Ana Savic*, and Stojan Radenović

Some aspects of the n-ary orthogonal and $b_{(\alpha_n,\beta_n)}$-best approximations of $b_{(\alpha_n,\beta_n)}$-hypermetric spaces over Banach algebras

https://doi.org/10.1515/dema-2022-0200
received February 5, 2022; accepted January 20, 2023

Abstract: In this article, a definition of a $b_{(\alpha_n,\beta_n)}$-best approximations of $b_{(\alpha_n,\beta_n)}$-hypermetric spaces over Banach algebras is given. Our objective is to prove the concept of extension of fixed point theorems in n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric spaces over Banach algebras.

Keywords: $b_{(\alpha_n,\beta_n)}$-best approximations, $b_{(\alpha_n,\beta_n)}$-hypermetric spaces, b_n-metric space, fixed point

MSC 2020: Primary 54H25, Secondary 47H10, 46B20

1 Introduction and preliminaries

In recent years, clustering problem has become an essential factor in computer science. Commonly, one of Minkowski distances (e.g., Euclidean distances) is used for the similarity measure of clusters. In general, the metric (or distance in k-means) is a mathematical term; it provides information for scientists to solve problems. Considering the ineffectiveness of existing meters for data classification, the main motivation is this new look at meters. This opinion is derived from the general understanding and application of a meter in physical reality. In this view, the distance between two points is not considered a value, but a distance with a reasonable error approximation is calculated. Article [8] is an example of using this type of idea for an application.

Generalization of the metric space structure by weakening the triangle inequality is given by Bourbaki [4] and Bakhtin [2] (see more in [5,16]). The aim of our research is the generalization of a cone n-metric spaces to the n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric spaces.

In this section, we recall some definitions, notations, and terminologies, which will be used to prove the main results [1–3,6,7,9,11,17,18,21,23] and basic definitions can be found in [12,15] and [22].

Theorem 1.1. (Brouwer's fixed point theorem) [24]. *The continuous operator $T : \mathcal{X} \to \mathcal{X}$ has a fixed point, provided $\mathcal{X}$ is a compact, convex, non-empty set in a finite-dimensional normed space over $\mathbb{K}$.*

Theorem 1.2. (Schauder's fixed point theorem) [24]. *The compact operator $T : \mathcal{X} \to \mathcal{X}$ has a fixed point, provided $\mathcal{X}$ is a bounded, convex, non-empty subset of a Banach space $\mathcal{A}$ over $\mathbb{K}$.*

*** Corresponding author: Ana Savic,** School of Electrical and Computer Engineering of Applied Studies, Academy of Technical and Art Applied Studies, Vojvode Stepe 283, 11000 Belgrade, Serbia, e-mail: ana.savic@viser.edu.rs
Akbar Dehghan Nezhad: School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, 16846-13114, Iran, e-mail: dehghannezhad@iust.ac.ir
Marija Paunovic: Faculty of Hotel Management and Tourism, University of Kragujevac, 36210 Vrnjacka Banja, Serbia; Faculty of Business and Law, University „MB", 11000 Belgrade, Serbia, e-mail: majap@rcub.bg.ac.rs
Stojan Radenović: Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd 35, Serbia, e-mail: radens@beotel.rs

The Schauder fixed-point theorem is an extension of the Brouwer fixed-point theorem to topological vector spaces, which may be of infinite dimension.

Proposition 1.3. *Let A be a Banach algebra. If* $\dim(A) < +\infty$, *then the Schauder theorem is equivalent to the Brouwer theorem.*

The point is that in a finite dimension, all continuous operators are compact, while in an infinite dimension, you can have continuous operators that are not compact, as the following example shows:

In $l_2(\mathbb{N})$, consider the operator $T(x) = (\sqrt{1 - \|x\|^2}, x_1, x_2, \ldots)$ defined for $\|x\| \leq 1$, where $x = (x_1, x_2, \ldots)$ and $\|x\|^2 = \sum_{i=1}^{+\infty} |x_i|^2$. The operator T is continuous. In fact, $\|T(x) - T(y)\|^2 = |\sqrt{1 - \|x\|^2} - \sqrt{1 - \|y\|^2}|^2 + \|x - y\|^2 \leq |\|x\|^2 - \|y\|^2| + \|x - y\|^2 \leq (\|x\| + \|y\|)\|x - y\| + \|x - y\|^2 \leq 2\|x - y\| + \|x - y\|^2$.

Moreover, T maps the closed unit ball to its boundary because $\|T(x)\|^2 = 1 - \|x\|^2 + \|x\|^2 = 1$. The operator T does not have fixed points; by contradiction, if $T(x) = x$, we would not only have $\|x\| = 1$ but also $(0, x_1, x_2, \ldots) = (x_1, x_2, \ldots)$, i.e., $x_i = 0$ for every i. In that case, we shall have $\|x\|^2 = 0 \neq 1$, which is a contradiction. In the finite-dimension state being compact for a set is nothing more than being bounded and closed, so to better understand the main difference.

For further details, the readers are referred to [12–14,16,19,20] and the references therein.

2 Main findings

In this part, we will describe our results of the $b_{(\alpha_n,\beta_n)}$-hypermetric spaces of dimension n.

2.1 The n-ary $b_{(\alpha_n,\beta_n)}$-hypermetric spaces

Some fixed point theorems in set-valued metric spaces over Banach algebra are presented below, as well as corresponding examples and applications of our findings. Our first result is Definition 2.1. Here and subsequently, for $n \geq 2$, let X^n denotes the n-times Cartesian product $\underbrace{X \times \ldots \times X}_{n\text{-times}}$. To simplify, we let $(x_i)_{i=1}^n$ and $(x)_1^n$ stands for $(x_1, \ldots, x_n)$ and $(x)_{i=1}^n$. Let T be a mapping, and we will use Tx instead of $T(x)$.

Let $\mathcal{A}$ be an ordered Banach algebra. Let $P^*(\mathcal{A})$ represent the family of all non-empty subsets of $\mathcal{A}$. The subset relation on power set is partial ordering. Therefore, we have that $(P^*(\mathcal{A}), \subseteq)$ is an ordered set. Furthermore, $\mathcal{A}^+$ will represent a set of non-negative elements of $\mathcal{A}$ i.e., $\mathcal{A}^+ = \{a \in \mathcal{A} : 0_{\mathcal{A}} \preccurlyeq a\}$.

Definition 2.1. Let X be a non-empty set and $\alpha_n, \beta_n : X^n \longrightarrow \mathcal{A}^+$ with $1 \preccurlyeq \alpha_n(x_i)_{i=1}^n$ and $1 \preccurlyeq \beta_n(x_i)_{i=1}^n$. Let $\Gamma_{(\alpha_n,\beta_n)} : X^n \longrightarrow P^*(\mathcal{A}^+)$ be a mapping satisfying, for all n-tuples, $(x_i)_{i=1}^n \in X^n$ the following conditions:

(G0) $\{0_{\mathcal{A}}\} \subseteq \Gamma_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n$,

(G1) $\Gamma_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n = \{0_{\mathcal{A}}\}$ if $x_1 = \ldots = x_n$,

(G2) $\Gamma_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n \supsetneq \{0_{\mathcal{A}}\}$ for all $(x_i)_{i=1}^n \in X^n$ with $x_i \neq x_j$ and for some $i, j \in \{1, \ldots, n\}$,

(G3) $\Gamma_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n = \Gamma_{(\alpha_n,\beta_n)}(x_{\pi_i})_{i=1}^n$ for every permutation $(\pi_{(1)}, \ldots, \pi_{(n)})$ of $(1, 2, \ldots, n)$,

(G4) $\Gamma_{(\alpha_n,\beta_n)}((x_i)_{i=1}^{n-1}, x_{n-1}) \subseteq \Gamma_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n$ for all $x_1, \ldots, x_n \in X$,

(G5) $\Gamma_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n \subseteq \alpha_n(x_i)_{i=1}^n \cdot \Gamma_{(\alpha_n,\beta_n)}(x_1, (a)_2^n) + \beta_n(x_i)_{i=1}^n \cdot \Gamma_{(\alpha_n,\beta_n)}(a, (x_i)_{i=2}^n)$ for all $x_1, \ldots, x_n, a \in X$.

Furthermore, A_i are subsets of X, $(i = 1, \ldots, n)$, for any $B, B' \in P^*(\mathcal{A}^+)$ and $\alpha \in \mathcal{A}^+$. Now, we can define

$$\Gamma_{(\alpha_n,\beta_n)}(A_i)_{i=1}^m = \bigcup\{\Gamma_n(x_i)_{i=1}^n | x_i \in A_i, \quad i = 1, \ldots, n\},$$

$$B + B' = \{b + b' | b \in B, b' \in B'\}, \quad \text{and} \quad \alpha \cdot B = \{\alpha \cdot b | b \in B, \alpha \in \mathcal{A}^+\}.$$

To summarize, here we say: the function Γ_n indicates an *ordered $b_{(\alpha_n,\beta_n)}$-hypermetric over Banach algebra $\mathcal{A}$* of dimension. The pair (X, Γ_n) is a $b_{(\alpha_n,\beta_n)}$-hypermetric space over $\mathcal{A}$.

Here, for simplicity, it is assumed that $\mathcal{A}^+ = \mathbb{R}_+^0 \coloneqq [0, +\infty)$. The next useful properties of a b_n-hypermetric are easily obtained from the axioms.

Remark 1. If $\alpha_2(x_i)_{i=1}^2 = \beta_2(x_i)_{i=1}^2 = c$ for $c \geq 1$ and $\Gamma_2(x, y) = \{d_b(x, y)\} \in P^*(\mathcal{R})$, then we obtain the definition of b-metric space (Czerwik [5]). Obviously for $c = 1$, this b-metric becomes an ordinary metric.

Example 1. We assume that $\mathcal{A}^+ = \mathbb{R}_+^0$. Let $X = [0, 1]$ and $\alpha_n, \beta_n : X^n \to [1, +\infty)$, with $\alpha_n(x_i)_{i=1}^n = 1 + \frac{1}{1 + x_1 + \cdots + x_n}$ and $\beta_n(x_i)_{i=1}^n = 1 + \frac{2}{1 + x_1 + \cdots + x_n}$. Define

$$\Omega_{\alpha_n,\beta_n} : X^n \to P^*(\mathbb{R}_+^0)$$

with

$$\Omega_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n = \begin{cases} \left[1, \dfrac{1}{x_1 \cdots x_n}\right) \cup \{0\}; & x_1, \ldots, x_n \in (0, 1], \ x_i \neq x_j \\ \{0\}; & x_1, \ldots, x_n \in [0, 1], \ x_1 = \cdots = x_n \\ \Omega_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n = \left[1, \dfrac{1}{x_j}\right) \cup \{0\}; & x_i = 0, x_j \in (0, 1] \\ \Omega_{(\alpha_n,\beta_n)}\left(x_{\pi_i}\right)_{i=1}^n, ; & \text{for every permutation } (\pi_{(1)}, \ldots, \pi_{(n)}) \text{of } (1, 2, \ldots, n), \end{cases}$$

and also assume $A + B = A \cup B$ for all $A, B \in P^*(\mathbb{R}_+^0)$. Then, $(X, \Omega_{(\alpha_n,\beta_n)})$ is a $b_{(\alpha_n,\beta_n)}$-hypermetric space.

Proposition 2.2. *Let $(X, \Gamma_{(\alpha_n,\beta_n)})$ be a $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$. Then, for any $x_1, \ldots, x_n, a \in X$, it follows that*
(1) if $\Gamma_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n = \{0_{\mathcal{A}}\}$, *then* $x_1 = \cdots = x_n$,
(2) $\Gamma_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n \subseteq \sum_{j=2}^n \Gamma_{(\alpha_n,\beta_n)}((x_1)_1^{n-1}, x_j)$,
(3) $\Gamma_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n \subseteq \sum_{j=1}^n \Gamma_{(\alpha_n,\beta_n)}(x_j, (a)_2^n)$,
(4) $\Gamma_{(\alpha_n,\beta_n)}(x_1, (x_2)_2^n) \subseteq (n-1)\Gamma_{(\alpha_n,\beta_n)}((x_1)_1^{n-2}, x_2)$.

Proposition 2.3. *Let $(X, \Gamma_{(\alpha_n,\beta_n)})$ be a $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$. Then, $\{0_{\mathcal{A}}\} \subseteq \Gamma_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n$ for every $x_1, \ldots, x_n \in X$.*

Proof. Using the condition $(G4)$ of the definition of $b_{(\alpha_n,\beta_n)}$-hypermetric space, we obtain

$$\{0_{\mathcal{A}}\} = \Gamma_{(\alpha_n,\beta_n)}(x_1)_1^n \subseteq \Gamma_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n. \qquad \square$$

Proposition 2.4. *Every $b_{(\alpha_n,\beta_n)}$-hypermetric space $(X, \Gamma_{(\alpha_n,\beta_n)})$ over Banach algebra $\mathcal{A}$ defines a $b_{(\alpha_2,\beta_2)}$-hypermetric space $(X, \Gamma_{(\alpha_2,\beta_2)})$ over Banach algebra $\mathcal{A}$ as follows:*

$$\Gamma_{(\alpha_2,\beta_2)}(x, y) = \Gamma_{(\alpha_n,\beta_n)}(x, (y)_2^n) + \Gamma_{(\alpha_n,\beta_n)}(y, (x)_2^n) \quad \text{for all } x, y \in X,$$

where $\alpha_2(x, y) = \max\{\alpha_n(x, (y)_2^n), \alpha_n(y, (x)_2^n)\}$ *and* $\beta_2(x, y) = \max\{\beta_n(x, (y)_2^n), \beta_n(y, (x)_2^n)\}$.

Proof. For $[(G0)], \ldots, [(G4)]$ is trivial. Let us show that $\Gamma_{(\alpha_2,\beta_2)}$ is satisfied in

$$\Gamma_{(\alpha_2,\beta_2)}(x, y) \subseteq \alpha_2(x, y) \cdot \Gamma_{(\alpha_2,\beta_2)}(x, z) + \beta_2(x, y) \cdot \Gamma_{(\alpha_2,\beta_2)}(z, y) \quad \text{for all } x, y, z \in X.$$

The proof is straightforward, by setting $\alpha_2(x, y) = \max\{\alpha_n(x, (y)_2^n), \alpha_n(y, (x)_2^n)\}$ and $\beta_2(x, y) = \max\{\beta_n(x, (y)_2^n), \beta_n(y, (x)_2^n)\}$ and the condition $(G5)$ of definition (2.1). $\square$

Proposition 2.5. *Let δ be an arbitrary positive real value number, and (X, d_n) be a n-metric space. Let $\alpha_n, \beta_n : X^n \longrightarrow \mathcal{A}^+$ with $1 \preccurlyeq \alpha_n(x_i)_{i=1}^n$ and $1 \preccurlyeq \beta_n(x_i)_{i=1}^n$. We define an induced $b_{(\alpha_n,\beta_n)}$-hypermetric over Banach algebra $\mathbb{R}$ as follows:*

$$\Gamma^{\delta}_{(\alpha_n,\beta_n)} : X^n \to P^*(\mathbb{R}^0_+)$$

$$\Gamma^{\delta}_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n = \begin{cases} (0, d_n(x_i)_{i=1}^n + \delta) \cup \{0\}; & 1 \le i, j \le n,\ x_i \ne x_j,\ d_n(x_i)_{i=1}^n \ne \delta \\ \{0\}; & 1 \le i, j \le n,\ x_i = x_j \quad \text{or } d_n(x_i)_{i=1}^n = \delta. \end{cases}$$

Then, $(X, \Gamma^{\delta}_{(\alpha_n,\beta_n)})$ is a $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathbb{R}$.

Proof. It is sufficient to show that $\Gamma^{\delta}_{(\alpha_n,\beta_n)}$ is satisfied in all properties $[(G0)], \dots, [(G5)]$. The proofs of $[(G0)], \dots, [(G4)]$ can be obtained immediately from the definition of $\Gamma^{\delta}_{(\alpha_n,\beta_n)}$. We only need to show that $\Gamma^{\delta}_{(\alpha_n,\beta_n)}$ is satisfied in $[(G5)]$.

$\Gamma^{\delta}_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n \subseteq \alpha_n(x_i)_{i=1}^n \cdot \Gamma^{\delta}_{(\alpha_n,\beta_n)}(x_1, (a)_2^n) + \beta_n(x_i)_{i=1}^n \cdot \Gamma^{\delta}_{(\alpha_n,\beta_n)}(a, (x_i)_{i=2}^n)$ for all $x_1, \dots, x_n, a \in X$.

In the case $1 \le i, j \le n$, $x_i = x_j$ or $d_n(x_i)_{i=1}^n = \delta$. We have $\Gamma^{\delta}_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n = \{0\}$. Therefore,

$\{0\} \subseteq \alpha_n(x_i)_{i=1}^n \cdot \Gamma^{\delta}_{(\alpha_n,\beta_n)}(x_1, (a)_2^n) + \beta_n(x_i)_{i=1}^n \cdot \Gamma^{\delta}_{(\alpha_n,\beta_n)}(a, (x_i)_{i=2}^n)$ for all $x_1, \dots, x_n, a \in X$.

In the case $1 \le i, j \le n$, $x_i \ne x_j$, and $d_n(x_i)_{i=1}^n \ne \delta$. Clearly, according to the last definition, we have $\Gamma^{\delta}_{(\alpha_n,\beta_n)}(x_i)_{i=1}^n = (0, d_n(x_i)_{i=1}^n + \delta) \cup \{0\} \subseteq \{(0, d_n(x_1, (a)_2^n) + \delta) + (0, d_n(a, (x_i)_{i=2}^n) + \delta)\} \cup \{0\} \subseteq \Gamma^{\delta}_{(\alpha_n,\beta_n)}(x_1, (a)_2^n) + \Gamma^{\delta}_{(\alpha_n,\beta_n)}(a, (x_i)_{i=2}^n) \subseteq \alpha_n(x_i)_{i=1}^n \cdot \Gamma^{\delta}_{(\alpha_n,\beta_n)}(x_1, (a)_2^n) + \beta_n(x_i)_{i=1}^n \cdot \Gamma^{\delta}_{(\alpha_n,\beta_n)}(a, (x_i)_{i=2}^n)$, for all $x_1, \dots, x_n, a \in X$.

This completes the proof. □

2.2 Generalized to $b_{(\alpha_n,\beta_n)}$-best approximations

The subject of approximation theory is intertwined with almost all branches of mathematics; therefore, it is of great importance in the applied sciences. On the other hand, fixed point theorems are used in many cases in the approximation theory. In the approximation theory, it is interesting to know whether the approximating function inherits some beneficial properties of the function being approximated [10].

The aim of this subsection is to introduce and discuss the conception of $b_{(\alpha_n,\beta_n)}$-best approximation and x_0-orthogonality in the theory of $b_{(\alpha_n,\beta_n)}$-hypermetric space $(\mathcal{X}, \Gamma_{(\alpha_n,\beta_n)})$ over $\mathcal{A}$.

Definition 2.6. Let $\mathcal{Y}$ be a subspace of a $b_{(\alpha_n,\beta_n)}$-hypermetric space $(\mathcal{X}, \Gamma_{(\alpha_n,\beta_n)})$ over Banach algebra $\mathcal{A}$ and $x \in \mathcal{X}$. A point x_0 is a $b_{(\alpha_n,\beta_n)}$-best approximation to x if the following conditions are satisfied:

(i) $x_0 \in \mathcal{Y}$,
(ii) $\Gamma_{(\alpha_n,\beta_n)}(x, (x_0)_1^{n-1}) \subseteq \Gamma_{(\alpha_n,\beta_n)}(x, (y)_1^{n-1})$ for every $y \in \mathcal{Y}$.

In general, if $\mathcal{X}$ is a $b_{(\alpha_n,\beta_n)}$-hypermetric space over $\mathcal{A}$ and $x_0, x, y \in \mathcal{X}$. Hence x is called x_0-orthogonal to y, which is denoted by $x \perp_{\Gamma}^{x_0} y$, if and only if $\Gamma_{(\alpha_n,\beta_n)}(x, (x_0)_1^{n-1}) \subseteq \Gamma_{(\alpha_n,\beta_n)}(x, (y)_1^{n-1})$.

Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be subsets of $\mathcal{X}$. The $\mathcal{Y}_1 \perp_{\Gamma}^{x_0} \mathcal{Y}_2$ if and only if, for all $y_1 \in \mathcal{Y}_1$ and $y_2 \in \mathcal{Y}_2$ one has $y_1 \perp_{\Gamma}^{x_0} y_2$. If $x \perp_{\Gamma}^{x_0} y$, then it is not necessary that $y \perp_{\Gamma}^{x_0} x$.

Let $\mathcal{X}$ be a $b_{(\alpha_n,\beta_n)}$-hypermetric space over $\mathcal{A}$ and let $\mathcal{Y}$ be a subset of $\mathcal{X}$. A point $y_0 \in \mathcal{Y}$ is called a Γ-best approximation for $x \in \mathcal{X}$ if $x \perp_{\Gamma}^{y_0} \mathcal{Y}$ or $\Gamma_{(\alpha_n,\beta_n)}(x, (y_0)_1^{n-1}) \subseteq \Gamma_{(\alpha_n,\beta_n)}(x, (y)_1^{n-1})$ for every $y \in \mathcal{Y}$. The set of all Γ-best approximations of x in $\mathcal{Y}$ is denoted by $P_{\Gamma}^{\mathcal{Y}}(x)$.

If $\mathcal{Y}$ is a subset of $\mathcal{X}$, then it is obvious that $P_{\Gamma}^{\mathcal{Y}}(x) = \{y_0 \in \mathcal{Y} | x \perp_{\Gamma}^{y_0} y, \ \text{for all } y \in \mathcal{Y}\}$.

Proposition 2.7. *Let $(\mathcal{X}, \Gamma_{(\alpha_n,\beta_n)})$ be an n-ary $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$. For $x, y, z, x_0 \in \mathcal{X}$, the following statements are true:*

(*i*) $x \perp_{\Gamma}^{x_0} x$ *if and only if* $x = x_0$,

(*ii*) $x \perp_{\Gamma}^{x_0} y$ *and* $y \perp_{\Gamma}^{x} z$, *then* $x \perp_{\Gamma}^{x_0} z$,

(*iii*) $x \perp_{\Gamma}^{x_0} x_0$ *and* $x_0 \perp_{\Gamma}^{x_0} x$.

Proof.

(i) If $x \perp_{\Gamma}^{x_0} x$, then $\Gamma_{(\alpha_n,\beta_n)}(x, (x_0)_1^{n-1}) \subseteq \Gamma_{(\alpha_n,\beta_n)}(x, (x)_1^{n-1}) = 0_{\mathcal{A}}$. Hence, $\Gamma_{(\alpha_n,\beta_n)}(x, (x_0)_1^{n-1}) = 0_{\mathcal{A}}$, i.e., $x = x_0$.

(ii) Since $x \perp_{\Gamma}^{x_0} y$ and $y \perp_{\Gamma}^{x} z$, we conclude that

$$\Gamma_{(\alpha_n,\beta_n)}(x, (x_0)_1^{n-1}) \subseteq \Gamma_{(\alpha_n,\beta_n)}(x, (y)_1^{n-1}) \quad \text{and} \quad \Gamma_{(\alpha_n,\beta_n)}(y, (x)_1^{n-1}) \preccurlyeq \Gamma_{(\alpha_n,\beta_n)}(y, (z)_1^{n-1}).$$

Therefore, $\Gamma_{(\alpha_n,\beta_n)}(x, (x_0)_1^{n-1}) \subseteq \Gamma_{(\alpha_n,\beta_n)}(y, (z)_1^{n-1})$, i.e., $x \perp_{\Gamma}^{x_0} z$.

(iii) This statement is obvious. □

Proposition 2.8. *Let $(\mathcal{X}, \Gamma_{(\alpha_n,\beta_n)})$ be an n-ary $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$. Then, the following statements are true:*

(*i*) *if* $x \in \mathcal{X}$, *then* $x \in P_{\Gamma}^{\mathcal{Y}}(x)$,

(*ii*) *if* $x \in \mathcal{Y}$, *then* $x \in P_{\Gamma}^{\mathcal{Y}}(x) = \{x\}$.

Proof.

(i) Since $\Gamma_{(\alpha_n,\beta_n)}(x)_1^n = 0_{\mathcal{A}}$, we have $x \in P_{\Gamma}^{\mathcal{Y}}(x)$.

(ii) If $x \in \mathcal{Y}$ and $y_0 \in P_{\Gamma}^{\mathcal{Y}}(x)$, then $\Gamma_{(\alpha_n,\beta_n)}(x, (y_0)_2^n) \subseteq 0_{\mathcal{A}} = \Gamma_{(\alpha_n,\beta_n)}(y, (x)_2^n)$. Therefore, $x = y_0$. □

Proposition 2.9. *Let $(\mathcal{X}, \Gamma_{(\alpha_n,\beta_n)})$ be an n-ary $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$ and $x_0 \in \mathcal{X}$. Then, the following statements are equivalent:*

(*m*1) $x \perp_{\Gamma}^{x_0} \mathcal{Y}$,

(*m*2) *there is a function* $f : \mathcal{X} \longrightarrow P^*([0, +\infty))$ *such that* $\Gamma_{(\alpha_n,\beta_n)}(x, (x_0)_2^n) \subseteq f(y) \preccurlyeq \Gamma_{(\alpha_n,\beta_n)}(x, (y)_2^n)$ *for all* $y \in \mathcal{Y}$.

Proof. The case: (m1) implies (m2).

Define a function $f : \mathcal{X} \longrightarrow P^*([0, +\infty))$ by the equality $f(z) = \Gamma_{(\alpha_n,\beta_n)}(x, (z)_2^n)$. Suppose that $y \in \mathcal{Y}$. Since $x \perp_{\Gamma}^{x_0} y$, we have $\Gamma_{(\alpha_n,\beta_n)}(x, (x_0)_2^n) \subseteq \Gamma_{(\alpha_n,\beta_n)}(x, (y)_2^n) = f(y)$.

The case: (m2) implies (m1).

By definition, we have $x \perp_{\Gamma}^{x_0} y$ for every $y \in \mathcal{Y}$. □

Proposition 2.10. *Let $(\mathcal{X}, \Gamma_{(\alpha_n,\beta_n)})$ be an n-ary $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$ and $y_0 \in \mathcal{Y}$. Then, the following statements are equivalent:*

(*n*1) $y_0 P_{\Gamma}^{\mathcal{Y}}(x)$,

(*n*2) *there is a function* $f : \mathcal{X} \longrightarrow P^*([0, +\infty))$ *such that* $\Gamma_{(\alpha_n,\beta_n)}(x, (y_0)_2^n) \subseteq f(y) \subseteq \Gamma_{(\alpha_n,\beta_n)}(x, (y)_2^n)$ *for all* $y \in \mathcal{Y}$.

Proposition 2.11. *Let $(\mathcal{X}, \Gamma_{(\alpha_n,\beta_n)})$ be an n-ary $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$. Let $\mathcal{Y}$ be a subset of $\mathcal{X}$, $\mathcal{Z}$ be a subset $\mathcal{Y}$, and $x, x_0 \in \mathcal{X}$. If there is a function $f : \mathcal{X} \longrightarrow P^*([0, \infty))$ such that $\Gamma_{(\alpha_n,\beta_n)}(x, (y_0)_2^n) \subseteq f(y) \subseteq \Gamma_{(\alpha_n,\beta_n)}(x, (y)_2^n)$ for all $y \in \mathcal{Z}$, then $\mathcal{Z} \subseteq P_{\Gamma}^{\mathcal{Y}}(x)$.*

2.3 The n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$

In this subsection, we will describe some properties and findings of the n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric spaces. In addition, we will give some fixed point theorems in set-valued n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric spaces over $\mathcal{A}$. In addition, examples with the application of our main findings will be presented.

An ordered n-tuple is a set of n objects with an order associated with them. If n objects are represented by $x_1, x_2, \ldots, x_n$, hence we write the ordered n-tuple as $(x_i)_{i=1}^n$. Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be n sets. Then, the set of all ordered n-tuples $(x_i)_{i=1}^n$, where $x_i \in X_i$ for all $i = 1, \ldots, n$, is called the Cartesian product of $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ and is denoted by $(\mathcal{X}_i)_{i=1}^n$.

Definition 2.12. An n-ary relation Υ over sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$ is an $(n+1)$-tuple $(\mathcal{X}_1, \ldots, \mathcal{X}_n, G)$, where G is a subset of the Cartesian product $(\mathcal{X}_i)_{i=1}^n$ called the graph of Γ.

An n-ary relation on sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$ is a set of ordered n-tuples $(x_i)_{i=1}^n$, where x_i is an element of $\mathcal{X}_i$ for all $i = 1, \ldots, n$. Thus, an n-ary relation on sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$ is a subset of the Cartesian product $(\mathcal{X}_i)_{i=1}^n$. The set $\mathcal{X}_i$ is called the i-th domain of Υ. The sets $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$ are called the domains of the relation, and n is its degree.

We begin with the following definition. For $n \geq 2$, the set $\mathcal{X}^n$ denotes the n-times Cartesian product $\underbrace{\mathcal{X} \times \ldots \times \mathcal{X}}_{n\text{-times}}$ and $\mathcal{A}$ be a Banach algebra. We denote $\mathcal{A}^+$ as a set of non-negative elements of $\mathcal{A}$. The set $\mathcal{A}^+ = \{a \in \mathcal{A} : 0_{\mathcal{A}} \preccurlyeq a\}$ and $P^*(\mathcal{A}^+)$ denotes the family of all non-empty subsets of $\mathcal{A}^+$. We denote

(1) $\Gamma_{\min}^{\Upsilon}(x_i)_{i=1}^n = \min\{\Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x_1)_1^n, \ldots, \Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x_n)_1^n\}$ and

(2) $\Gamma_{\max}^{\Upsilon}(x_i)_{i=1}^n = \max\{\Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x_1)_1^n, \ldots, \Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x_n)_1^n\}$.

Definition 2.13. Let $\mathcal{X}$ be a non-empty set and $\alpha_n, \beta_n : \mathcal{X}^n \longrightarrow \mathcal{A}^+$. Let $(\mathcal{X}, \Upsilon)$ be an n-ary orthogonal set. Let $\Gamma_{(\alpha_n,\beta_n)}^{\Upsilon} : \mathcal{X}^n \longrightarrow P^*(\mathcal{A}^+)$ be a mapping (the n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric over $\mathcal{A}$ on n-ary orthogonal set $(\mathcal{X}, \Upsilon)$) satisfying the following conditions for all n-tuples $(x_i)_{i=1}^n \in \mathcal{X}^n$:

(O1) $x = y$ if and only if $\Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x)_1^n = \Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x, (z_j)_{j=2}^{n-1}, y) = \Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(y)_1^n$, where $z_j = x$ or y,

(O2) $\Gamma_{\min}^{\Upsilon}(x_i)_{i=1}^n \subseteq \Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x_i)_{i=1}^n$ for all $x_1, \ldots, x_n \in \mathcal{X}$,

(O3) $\Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x_i)_{i=1}^n = \Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x_{\pi_i})_{i=1}^n$ for every permutation $(\pi_{(1)}, \ldots, \pi_{(n)})$ of $(1, 2, \ldots, n)$,

(O4) $\Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x_i)_{i=1}^n - \Gamma_{\min}^{\Upsilon}(x_i)_{i=1}^n \subseteq \alpha_n(x_i)_{i=1}^n \cdot \Omega_1 + \beta_n(x_i)_{i=1}^n \cdot \Omega_2 - \Gamma_{\min}^{\Upsilon}(a)_{i=1}^n$ for all $x_1, \ldots, x_n, a \in X$

where $\Omega_1 = \Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x_1, (a)_2^n) - \Gamma_{\min}^{\Upsilon}(x_1, (a)_2^n)$ and $\Omega_2 = \Gamma_{(\alpha_n,\beta_n)}(a, (x_i)_{i=2}^n) - \Gamma_{\min}^{\Upsilon}(a, (x_i)_{i=2}^n)$.

Let A_i subsets of $\mathcal{X}$ $(i = 1, \ldots, n)$. For any $B, B' \in P^*(\mathcal{A}^+)$ and $\alpha \in \mathcal{A}^+$, we define

$$\Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(A_i)_{i=1}^n = \bigcup\{\Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}(x_i)_{i=1}^n | x_i \in A_i, \quad i = 1, \ldots, n\},$$

$$B + B' = \{b + b' | \in B, b' \in B'\} \quad \text{and} \quad \alpha \cdot B = \{\alpha \cdot b | b \in B, \alpha \in \mathcal{A}^+\}.$$

The pair $(\mathcal{X}, \Gamma_{(\alpha_n,\beta_n)}^{\Upsilon})$ is called an n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$ on n-ary orthogonal set $(\mathcal{X}, \Upsilon)$.

For example, we can place $\mathcal{A}^+ = \mathbb{Z}_+^0$ or $\mathbb{R}_+^0$, where $\mathbb{Z}_+^0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ and $\mathbb{R}_+^0 := [0, +\infty)$. For simplicity, we assume that $\mathcal{A}^+ = \mathbb{R}_+^0$. The next helpful properties of a b_n-hypermetric are simple and derived from the axioms.

Remark 2. If $\alpha_n(x_i)_{i=1}^n = \beta_n(x_i)_{i=1}^n = c$ for $c \geq 1$ and $n = 1$, then we have the definition of b-metric space (Czerwik [5]). It is clear that for $c = 1$, this b-metric becomes a regular metric.

Definition 2.14. Let $(\mathcal{X}, \Gamma_{(\alpha_n,\beta_n)}^{\Upsilon})$ be an n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$ on n-ary orthogonal set $(\mathcal{X}, \Upsilon)$. Then,

(i) a sequence $\{x_i\}_{i=1}^{+\infty}$ in an n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric space $(\mathcal{X}, \Gamma_{(\alpha_n,\beta_n)}^{\Upsilon})$ converges to a point $x_0 \in \mathcal{X}$ if and only if $\lim_{k \longrightarrow +\infty}(\Gamma_{(\alpha_n,\beta_n)}^{\Upsilon}((x_{i+k})_{i=1}^{n-1}, x_0) - \Gamma_{\min}^{\Upsilon}((x_{i+k})_{i=1}^{n-1}, x_0)) = 0_{\mathcal{A}}$;

(ii) a sequence $\{x_i\}_{i=1}^{+\infty}$ in an n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric space $(\mathcal{X}, \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)})$ is called $b_{(\alpha_n,\beta_n)}$-Cauchy sequence if $\lim_{k\longrightarrow+\infty}(\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_{i+k})_{i=1}^{n} - \Gamma^{\Upsilon}_{\min}(x_{i+k})_{i=1}^{n})$ and $\lim_{k\longrightarrow+\infty}(\Gamma^{\Upsilon}_{\max}(x_{i+k})_{i=1}^{n} - \Gamma^{\Upsilon}_{\min}(x_{i+k})_{i=1}^{n})$ exist (and are finite);

(iii) an n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric space $(\mathcal{X}, \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)})$ is said to be orthogonally $b_{(\alpha_n,\beta_n)}$-complete if every n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-Cauchy sequence $\{x_i\}_{i=1}^{+\infty}$ converges to a point $x_0 \in \mathcal{X}$ such that $\lim_{k\longrightarrow+\infty}(\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}((x_k)_1^{n-1}, x_0) - \Gamma^{\Upsilon}_{\min}((x_k)_1^{n-1}, x_0) = 0_{\mathcal{A}}$ and $\lim_{k\longrightarrow+\infty}(\Gamma^{\Upsilon}_{\max}((x_k)_1^{n-1}, x_0) - \Gamma^{\Upsilon}_{\min}((x_k)_1^{n-1}, x_0) = 0_{\mathcal{A}}$.

Definition 2.15. Let $\mathcal{X}$ be a non-empty set and Υ be an n-ary relation on sets $\mathcal{X}_1, \dots, \mathcal{X}_n$. Then, $(\mathcal{X}, \Upsilon)$ is said to be an n-ary orthogonal set (in short, O_n-set) if there exists $x_0 \in \mathcal{X}$ such that for all $x_i \in \mathcal{X}, (1 \le i \le n)$, $(x_0, x_2, \dots, x_n) \in \Upsilon$ or $\cdots$ or $(x_1, \dots, x_{i-1}, x_0, x_{i+1}, \dots, x_n) \in \Upsilon$ or $\cdots$ or $(x_1, x_2, \dots, x_{n-1}, x_0) \in \Upsilon$. The element x_0 is called an n-ary orthogonal element.

Definition 2.16. Let $(\mathcal{X}, \Upsilon)$ be an n-ary orthogonal set. A sequence $\{x_j\}$ is called an O_n-set sequence if for all $j \in \mathbb{N}$, $(x_j, x_{j+1}, \dots, x_{j+n}) \in \Upsilon$ or $\cdots$ or $(x_{1+j}, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_{n+j}) \in \Upsilon$ or $\cdots$ or $(x_{j+1}, x_{2+j}, \dots, x_{n-1+j}, x_j) \in \Upsilon$.

Definition 2.17. Let $(\mathcal{X}, \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)})$ be an n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$ on n-ary orthogonal set $(\mathcal{X}, \Upsilon)$. A mapping $T : \mathcal{X} \longrightarrow \mathcal{X}$ is an n-ary orthogonal continuous (or Υ-continuous) in $x_0 \in \mathcal{X}$ if, for each O_n-set sequence, $\{x_j\}_{j\in\mathbb{N}} \in \mathcal{X}$ such that $x_j \longrightarrow x_0$, then $T(x_j) \longrightarrow T(x_0)$. Furthermore, T is said to be Υ-continuous on $\mathcal{X}$ if T is Υ-continuous at each $x_0 \in \mathcal{X}$.

Definition 2.18. Let $(\mathcal{X}, \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)})$ be an n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$ on n-ary orthogonal set $(\mathcal{X}, \Upsilon)$. Then, $\mathcal{X}$ is said to be n-ary orthogonally $b_{(\alpha_n,\beta_n)}$ -complete if every $b_{(\alpha_n,\beta_n)}$-Cauchy sequence is convergent.

Definition 2.19. Let $(\mathcal{X}, \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)})$ be an n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathcal{A}$ on n-ary orthogonal set $(\mathcal{X}, \Upsilon)$. A mapping $T : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be Υ-preserving if $T((x_i)_{i=1}^{n}) \in \Upsilon$ whenever $(x_i)_{i=1}^{n} \in \Upsilon$.

Theorem 2.20. *Let $(\mathcal{X}, \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)})$ be an n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric space and $T : \mathcal{X} \longrightarrow \mathcal{X}$ be Υ-preserving, Υ-continuous, and satisfy the following condition:*

$$\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(T((x_i))_{i=1}^{n}) \subseteq \lambda \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_i)_{i=1}^{n} \quad \text{for all } (x_i)_{i=1}^{n} \in \mathcal{X}^n \quad \text{with } (x_i)_{i=1}^{n} \in \Upsilon, \quad \text{where } 0_{\mathcal{A}} \preccurlyeq \lambda \preccurlyeq e_{\mathcal{A}}.$$

Then, T has a unique fixed point $x_0 \in \mathcal{X}$ and $\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0)_1^{n} = \{0_{\mathcal{A}}\}$.

Proof. From the definition of n-ary orthogonality, there exists $x_0 \in \mathcal{X}$ such that for all $x_i \in \mathcal{X}, (1 \le i \le n)$, $(x_0, x_2, \dots, x_n) \in \Upsilon$ or $\cdots$ or $(x_1, \dots, x_{i-1}, x_0, x_{i+1}, \dots, x_n) \in \Upsilon$ or $\cdots$ or $(x_1, x_2, \dots, x_{n-1}, x_0) \in \Upsilon$. Hence, $(x_0, (T(x_0))_2^{n}) \in \Upsilon$ or $\cdots$ or $(T(x_0))_1^{j-1}, x_0, (T(x_0)_{j+1}^{n}) \in \Upsilon$ or $\cdots$ or $((T(x_0))_1^{n-1}, x_0) \in \Upsilon$. Let $x_j = T(x_{j+1})$ for all $j \in \mathbb{N}$ since the operator T is Υ-preserving. Therefore, $(x_i(t))_{i=1}^{\infty}$ is O_n-set sequence. By assumption, we can see that

$$\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_j)_{j=i}^{i+n} = \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(T(x_j))_{j=i-1}^{i+n-1} \subseteq \lambda^n \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_j)_{j=1}^{n}$$

for all $i \in \mathbb{N}$. For any two positive integers $j > i$, we have

$$\begin{aligned}
\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_i, (x_j)_2^{n}) &\subseteq \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(T(x_i), T((x_{j-1}))_2^{n}) \subseteq \lambda \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_{i-1}, (x_{j-1})_2^{n}) \\
&\subseteq \lambda \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}\left(\lambda \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_{i-2}, (x_{j-2})_2^{n})\right) \\
&\subseteq \lambda^2 \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}((x_{i-2}, (x_{j-2})_2^{n})) \\
&\vdots \\
&\subseteq \lambda^i \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0, (x_{j-i})_2^{n}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_i,(x_j)_2^n)-\Gamma^{\Upsilon}_{\min}(x_i,(x_j)_2^n)\\
&\quad\subseteq\lambda^i\Big(\alpha_n(x_0,(x_{j-i})_2^n)\cdot\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(x_1)_2^n)+\Big(\beta_n(x_0,(x_{j-i})_2^n)\cdot\Gamma_{(\alpha_n,\beta_n)}(x_0,(x_i)_{i=2}^n)-\Gamma^{\Upsilon}_{\min}(x_0,(x_i)_{i=2}^n)\Big)-\Gamma^{\Upsilon}_{\min}(x_0)_1^n\Big)\\
&\quad\subseteq\lambda^i(\alpha_n(x_0,(x_{j-i})_2^n)\cdot\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(x_1)_2^n)+\Big(\beta_n(x_0,(x_{j-i})_2^n)\cdot\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_1,(x_j)_2^n)\Big)\\
&\quad\subseteq\lambda^i\Big[\alpha_n(x_0,(x_{j-i})_2^n)\cdot\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(x_1)_2^n)+\beta_n(x_0,(x_{j-i})_2^n)\cdot\Big(\alpha_n(x_1,(x_i)_2^n)\cdot\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_2,(x_j)_2^n)\\
&\qquad+\beta_n(x_1,(x_i)_2^n)\cdot\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_2,(x_j)_2^n)\Big)\Big]\\
&\qquad\vdots\\
&\quad\subseteq\Delta\lambda^i[1+\Delta\lambda+(\Delta\lambda)^2+\cdots+(\Delta\lambda)^{j-i-1}]\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(x_1)_2^n)=\Delta\lambda^i\frac{(\Delta\lambda)^{j-i-1}}{1-\Delta\lambda}\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(x_1)_2^n),
\end{aligned}$$

where $\Delta=\sup_j\{\alpha_n(x_0,(x_{j-i})_2^n),\beta_n(x_0,(x_{j-i})_2^n)\cdot\alpha_n(x_1,(x_i)_2^n),\beta_n(x_0,(x_{j-i})_2^n)\cdot\beta_n(x_1),(x_i)_2^n,\ldots\}$. Since $0_{\mathcal{A}}\preccurlyeq\lambda\preccurlyeq e_{\mathcal{A}}$ and $\Delta>0$, it follows from the above inequality that $\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_i,(x_j)_2^n)-\Gamma^{\Upsilon}_{\min}(x_i,(x_j)_2^n)\longrightarrow 0_{\mathcal{A}}$ as $i,j\longrightarrow+\infty$.

A passage similar to the above implies that (similarly) $\Gamma^{\Upsilon}_{\max}(x_i,(x_j)_2^n)-\Gamma^{\Upsilon}_{\min}(x_i,(x_j)_2^n)\longrightarrow 0_{\mathcal{A}}$ as $i,j\longrightarrow+\infty$ and so $\Gamma^{\Upsilon}_{\min}(x_i,(x_j)_2^n)\longrightarrow 0_{\mathcal{A}}$ as $i\longrightarrow+\infty$. Therefore, we have $\Gamma^{\Upsilon}_{\min}(x_i,(x_j)_2^n)\longrightarrow\{0_{\mathcal{A}}\}$ as $i\longrightarrow+\infty$. Thus, $\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_i,(x_j)_2^n)=0_{\mathcal{A}}=\Gamma^{\Upsilon}_{\min}(x_i,(x_j)_2^n)$.

Consequently, $\{x_i\}_{i=1}^{\infty}$ is a $b_{(\alpha_n,\beta_n)}$-Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is n-ary orthogonally $b_{(\alpha_n,\beta_n)}$-complete, there exists $x_0\in\mathcal{X}$ such that $\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_i,(x_j)_2^n)-\Gamma^{\Upsilon}_{\min}(x_i,(x_j)_2^n)\longrightarrow\{0_{\mathcal{A}}\}$ as $i\longrightarrow+\infty$.

Now, we present that $\mathcal{X}$ is a fixed point of T in $\mathcal{X}$.

$$\begin{aligned}
\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(T(x_0))_2^n)&\subseteq\lim_{j\longrightarrow+\infty}\sup\Big(\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(x_j)_2^n)\Big)+\lim_{j\longrightarrow+\infty}\sup\Big(\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_j,(T(x_0))_2^n)\Big)\\
&\subseteq\lim_{j\longrightarrow+\infty}\sup\Big(\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_j,(T(x_0))_2^n)\Big)\\
&\subseteq\lim_{j\longrightarrow+\infty}\sup\Big(\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(T(x_{j-1}),(T(x_0))_2^n)\Big)\\
&\subseteq\lim_{j\longrightarrow+\infty}\sup\Big(\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}\lambda((x_{j-1}),(x_0)_2^n)\Big),
\end{aligned}$$

and by (O4), we have $\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(T(x_0))_2^n)\subseteq\lim_{j\longrightarrow+\infty}\sup\ \lambda[\alpha_n(x_{j-1}),(T(x_0))_2^n\cdot\Omega_1+\beta_n(x_i)_{i=1}^n\cdot\Omega_2-\Gamma^{\Upsilon}_{\min}(T(x_0),x_0)_2^n]\subseteq\lambda\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}((T(x_0))_2^n,x_0)$. Therefore, $\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(T(x_0))_2^n)=\{0_{\mathcal{A}}\}$, and

$$0_{\mathcal{A}}\subseteq\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}((T(x_0))_1^n)\subseteq\lambda\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0)_1^n=\{0_{\mathcal{A}}\}.$$

Hence,

$$\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}((T(x_0))_1^n)=\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(T(x_0))_2^n)=\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0)_1^n.$$

From (O1), we conclude that $T(x_0)=x_0$. It follows that x_0 is a fixed point for the function T.

To prove the uniqueness property of a fixed point, let $y\in\mathcal{X}$ be another fixed point of T. Then, we have $T^ix_0=x_0$ and $T^iy=y$ for all $i\in\mathbb{N}$.

Choosing x_0 in the first part of the proof, we obtain

$$(x_0,(y)_2^n)\in\Upsilon\quad\text{or}\quad\cdots\quad\text{or}\quad((y)_1^{i-1},x_0,x_{i+1},(y)_{i+1}^n)\in\Upsilon\quad\text{or}\quad\cdots\quad\text{or}\quad((y)_1^{n-1},x_0)\in\Upsilon.$$

Since the function T is Υ-preserving. Then, $(T^j(x_0),(T^j(y))_2^n)\in\Upsilon$ or $\cdots$ or $((T^j(y))_1^{i-1},T^j(x_0),(T^j(y))_{i+1}^n)\in\Upsilon$ or $\cdots$ or $((T^j(y))_1^{n-1},T^j(x_0))\in\Upsilon$ for all $j\in\mathbb{N}$.

Thus, $\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(y)_2^n)=\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(T^j(x_0),(T^j(y))_2^n)\subseteq\lambda\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(y)_2^n)\subseteq\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(y)_2^n)$, which implies that $\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(y)_2^n)=0_{\mathcal{A}}$ and so $x_0=y$.

Finally, it is shown that if x_0 is a fixed point, then $\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0,(y)_2^n)=0$. Assume that x_0 is a fixed point of T. Then,

$$\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0)_1^n=\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(T^n(x_0),(T^n(y)_2^n)\subseteq\lambda\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0)_1^n\subset\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0)_1^n,$$

That is, $\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_0)_1^n = 0$. This completes the proof. □

Example 2. Suppose the space $\mathcal{X} = C^0(I)$ of continuous functions defined on $I = [0,1]$ with an n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric space over Banach algebra $\mathbb{R}$ given as follows:

$$\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_i(t))_{i=1}^n = \left[0_{\mathcal{A}},\ \sup_{t\in[a,b]}\left(\left|\frac{x_1(t)\ +\cdots+x_n(t)}{n}\right|\right)\right) \quad \text{for all } (x_i(t))_{i=1}^n \in (C^0(I))^n.$$

This space can also be equipped with an n-ary orthogonal given by $(x_i(t))_{i=1}^n \in (C^0(I))^n$,

$$(x_i(t))_{i=1}^n \in \Upsilon \quad \text{if and only if } x_1(t) \preccurlyeq x_2(t) \preccurlyeq \cdots \preccurlyeq x_n(t),$$

where $\alpha_n = \beta_n = 1$. Now, we consider the following Fredholm-type integral equation:

$$x(t) = \int_0^1 f(t,s,x(s))\mathrm{d}s \quad \text{for all } t,s \in [0,1], \quad \text{where } f \in C^0([0,1]^2 \times C^0(I), \mathbb{R}).$$

Then, $(C^0(I), \Gamma^{\Upsilon}_{(\alpha_n,\beta_n)})$ is an n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-complete hypermetric space.

Proposition 2.21. *Assume that, for all $(x_i)_{i=1}^n \in C^0([0,1], \mathbb{R})$,*

$$\left|\sum_{i=1}^n f(t,s,x_i(t))\right| \preccurlyeq \lambda\left|\sum_{i=1}^n x_i(t)\right| \quad \textit{for all } t,s \in [0,1],$$

where $z \in [0,1)$. Then the above integral equation has a unique solution.

Proof. Define $T : C^0(I) \longrightarrow C^0(I)$ by

$$T(x(t)) = \int_0^1 f(t,s,x(s))\mathrm{d}s \quad \text{for all } t,s \in [0,1], \quad \text{where } f \in C^0([0,1]^2 \times C^0(I), \mathbb{R}).$$

Note that the existence of a fixed point of the operator T is equivalent to the existence of a solution of the above integral equation. Furthermore, for all $(x_i)_{i=1}^n \in (C^0([0,1]))^n$, we have

$$\begin{aligned}
\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(T((x_i))_{i=1}^n &= \left[0_{\mathcal{A}}, \sup_{t\in[a,b]}\left(\left|\frac{\sum_{i=1}^n T(x_i(t))}{n}\right|\right)\right) \\
&= \left[0_{\mathcal{A}}, \sup_{t\in[a,b]}\left(\left|\int_0^1\left(\frac{\sum_{i=1}^n f(t,s,x_i(s))\mathrm{d}s}{n}\right)\right|\right)\right) \\
&\preccurlyeq \lambda\left[0_{\mathcal{A}},\ \sup_{t\in[a,b]}\left(\int_0^1\left|\left(\frac{\sum_{i=1}^n f(t,s,x_i(s))}{n}\right)\mathrm{d}s\right|\right)\right) \\
&\preccurlyeq \lambda\left[0_{\mathcal{A}}, \sup_{t\in[a,b]}\left(\int_0^1\left|\left(\frac{\sum_{i=1}^n x_i(t)}{n}\right)\mathrm{d}s\right|\right)\right) \\
&\preccurlyeq \lambda\left[0_{\mathcal{A}}, \sup_{t\in[a,b]}\left(\left|\frac{x_1(t)\ +\cdots+x_n(t)}{n}\right|\right)\right)\int_0^1 \mathrm{d}s \\
&\preccurlyeq \lambda\Gamma^{\Upsilon}_{(\alpha_n,\beta_n)}(x_i(t))_{i=1}^n.
\end{aligned}$$

Hence, the condition is satisfied. Therefore, all the conditions of the theorem are satisfied. Hence, the operator has a unique fixed point, which means that the above Fredholm integral equation has a unique solution. The proof is completed. □

3 Conclusion

The objective of this article is to study about $b_{(\alpha_n,\beta_n)}$-best approximations of $b_{(\alpha_n,\beta_n)}$-hypermetric spaces over Banach algebras and introduced certain fixed-point results of mappings in the setting of n-ary orthogonal $b_{(\alpha_n,\beta_n)}$-hypermetric spaces over Banach algebras. In the future, we can seek to generalize the Schauder and Brouwer fixed-point theorem on these spaces. This article is a candidate for a pioneering result, and many findings can be obtained in the near future. It can find application in engineering science [8].

Acknowledgment: The authors would like to thank the editor of the article and the referees.

Conflict of interest: The authors declare that there are no conflicts of interest.

References

[1] S. Aleksić, Z. Kadelburg, Z. D. Mitrović, and S. Radenović, *A new survey: Cone metric spaces*, J. Int. Math. Virtual Inst. **9** (2019), 93–121
[2] I. Bakhtin, *The contraction mapping principle in quasimetric spaces*, Func. An., Gos. Ped. Inst. Unianowsk **30** (1989), 26–37.
[3] V. Berinde, *Generalized Contractions in Quasimetric Spaces*, Seminar on Fixed Point Theory, Babes-Bolyai, University, vol. 3, issue 9, 1993, p. 3–9.
[4] N. Bourbaki, *Topologie Generale*; Herman, Paris, France, 197–4.
[5] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav. **1** (1993), 5–11.
[6] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.
[7] P. Debnath, N. Konwar, and S. Radenović, *Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Science*, Springer Verlag, Singapore, 2021.
[8] A. D. Nezhad, S. Beizavi, and S. H. Hekmatimoghaddam, *New Clustering Algorithm for Classification of Brain MRI Regions by using m-Universal Metric Technique*, Malaysian J. Math. Sci. **12** (2018), no. 1, 85–98.
[9] A. D. Nezhad, A. R. Forough, N. Mirkov, and S. Radenović, A new version of U_n-hypermetric space results, *Vojnotehni čki Glasnik/Military Technical courier*, vol. 69, Issue 3, 2021, p. 562–577.
[10] A. Dehghan Nezhad and H. Mazaheri, *New results in G-best approximation in G-metric spaces*, Ukrainian Math. J. **62** (2010), no, 4, 648–654.
[11] M. Eshagahi Gordji and H. Habibi, *Fixed point theory in generalized orthogonal metric space*, IJ. Linear Topol. Algebra, **6** (2017), 251–260.
[12] J. Fernandez, N. Malviya, S. Radenović, and K. Saxena, *F-cone metric spaces over Banach algebra*, Fixed Point Theory Appl. **7** (2017), 18.
[13] J. Fernandez, N. Malviya, A. Savic, M. Paunovic, and Z. D. Mitrovic, *The extended cone b-metric-like spaces over Banach algebra and some applications*, Mathematics **10** (2022), no. 149. DOI: https://doi.org/10.3390/math10010149.
[14] J. Fernandez, N. Malviya, Z. D. Mitrovic, A. Hussain, and V. Parvaneh. *Some fixed point results on Nb-cone metric spaces over Banach algebra*, Adv. Differ. Equ. **529** (2020), DOI: https://doi.org/10.1186/s13662-020-02991-5.
[15] L. Huang and X, Zhang, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332** (2007), no. 2, 1468–1476.
[16] T. Kamran, M. Samreen, and Q. Ul. Ain, *A generalization of b-metric space and some fixed point theorems*, Mathematics **5** (2017), 1–7, DOI: https://doi.org/10.3390/math5020019.
[17] K. A. Khan, *On the possibility of n topological spaces*, Int. J. Math. Arch. **3** (2012), no. 6, 2520–2523.
[18] K. A. Khan, *Generalized n-metric spaces and fixed point theorems*, J. Nonlinear Convex Anal. **15** (2014), no. 6, 1221–1229.
[19] Moosa Gabeleh and Naseer Shahzad, *Best proximity points, cyclic Kannan maps and geodesic metric spaces*, J. Fixed Point Theory Appl. **18** (2016), 167–188.

[20] H. Nawab, J. R. Roshan, V. Parvaneh, and A. Latif, *A unification of G-metric, partial metric, and b-metric spaces*, Abstract Appl. Anal. **2014** (2014), 1–14, DOI: https://doi.org/10.1155/2014/180698.

[21] S. Radenović and B. E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*, Comput. Math. Appl. **57** (2009), 1701–1707.

[22] W. Rudin, *Functional Analysis*, 2nd edn. McGraw-Hill, New York, 1991.

[23] V. Todorčević, *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*, Springer Nature, Switzerland AG, 2019.

[24] E. Zeidler, *Nonlinear functional analysis and its applications*, I - Fixed-Point Theorems, Springer-Verlag, New York, 1986.