MULTIPLE REPRESENTATIONS OF FUNCTIONS IN THE FRAME OF DISTANCE LEARNING

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Abstract: In this paper, the effectiveness of the use of multiple representations in teaching and learning functions is analyzed. The research was conducted in the frame of distance learning of calculus course at University of Novi Sad in 2021 (during Covid19 crisis), in the dynamic software environment. The students' work on the analysis the properties of functions, by using the multiple representations of functions, is presented, and analyzed. Results of the study implies that dynamic software environment during distance learning of mathematics, encourage solving tasks with using different approach by students and helps them to create their own sophisticated knowledge about the functions and their properties.

Keywords: calculus; distance learning; dynamic software; functions; multiple representations

1. Introduction

It is known that the students have lot difficulties with the calculus contents within their mathematics courses (Borba & Confrey 1996; Daher & Anabousy 2015; Tall 1992; Takačiet al. 2015). Examining the properties of the functions and their multiple representations, algebraic and graphical, turned out to be one of most difficult students' tasks (Borba & Confrey 1996; Ciltas & Tatar 2011; Daher & Anabousy 2015; Sierpinska et al. 2011; Wilhelmi et al. 2007). Since the functions are used in a lot of the students' further courses and their professional work, it is very important for students to overcome the difficulties and the conflicts in understanding the function concept, emphasizing the absolute value functions. In the last few years, different dynamic packages, as *Mathematica, GeoGebra, Maple, Cinderella*, etc. are applied in teaching and learning the calculus contents. Almost all dynamic software packages are appropriate for examining the families of functions, algebraically represented as the functions with parameters. They enable simultaneous dynamic multiple representations of the families of functions and their properties, providing an easy change of variables (Anabousi et al. 2014; Borba & Confrey 1996; Ciltas & Tatar 2011; Daher & Anabousy 2015).

The research was conducted with the students at University of Novi Sad. The research presented in this paper is the contribution to mathematics education during Covid19, in particular to the function concept, in order to improve distance learning of the calculus in the future.

2. Theoretical background

Constructivism

Constructivism is a very important learning theory, practicing by numerous authors in the late twentieth and early twenty-first centuries (Bodner 1986; Tobin & Tippins 1993; Von Glasersfeld 1995). It is based on students' actively constructing their own knowledge. The students use their previous knowledge and experiences as the foundation for understanding new facts within their learning materials. As stated in (Von Glasersfeld 1995), "The constructivist model implies that knowledge is quality if it is applicable, that is, if it is useful in achieving some goals." For that reason, students will construct only the knowledge they need to solve a particular problem. According to (Iran-Nejad 1995), (Sjoberg 2010) and (Taber 2011), constructivism stimulates students' curiosity and their discussions, which should be of crucial importance in the learning process.

Although the role of the teacher is extremely important, the emphasis is more on what the students do. In constructivist learning approach, it is very important that the teacher carefully guides the students through the learning process. It is necessary to introduce them to a new area and then help them adapt and master new concepts and features in the most understandable way possible. The teacher should also help students eliminate difficulties that arise during learning and problem-solving process. The teachers should create such a learning environment that will help students acquire new skills and knowledge (Tobin & Tippins 1993). The teacher must develop a dialogue with the students during learning. The dialogue should be of high quality and supported by accurate information because students form their new knowledge based on it. (Brooks & Brooks 1993; Taber 2011). The key role of the teacher is to pose a problem that will be of interest to students and to feel the need to solve it (Von Glasersfeld 1995).

Covid-19 education

The educational process during the Covid-19 crisis was rapidly turned to the achievements of modern technology (Adedoyin & Soykan 2020; Drijvers et al. 2021; Fidalgoet al. 2020; Mitrović et al. 2022; Stevanović et al. 2021). Different audio and video conferencing, data and application sharing, shared whiteboard, virtual "hand raising" and so on, have been applied in each educational institution over 2020, 2021, and at the beginning of 2022. Namely, educational process turned forward future suddenly, varying from school to school, university to univesity.

According to (Psotka 2022), these days, almost after (and during) the Covid-19 crisis, it is important to work on research regarding different experiences of the teaching process, about teachers' and students' experiences, in order to contribute to the improvement of future education. All prepared teaching materials, students' and teachers' experiences need to be analyzed to improve the future teaching process.

Multiple representations

Representations in mathematics, as representations at all, are usually considered as a process of modeling concrete objects in the real world into abstract concepts (Hwang & Hu 2013). The observer puts two concepts one against the other, revealing and comparing the similarities and differences between them. In this way one of the observed concepts is represented (Font et al. 2007). The classification of representations was the subjects of many research. Usually are considered two large groups of representations – internal representation, created in the mind of an individual, and external, created in his surroundings (Goldin & Shteingold 2001; Nakahara 2008). A quality of the multiple representations is significantly improved by using modern technology. The use of technology to work within multiple representations and to link them has a great attention of the contemporary researchers (Rau et al. 2015; Ozgun-Koca 2008; Sever & Yerushalmy 2007).

The advantage of using computers in the formation of multiple representations is especially evident when connecting different representations. Namely, there are software packages that allow simultaneous display of two to three representations of the same object. More recently, dynamic software packages which enable forming the multiple representations are especially important. These software packages enable dynamically linking the multiple representations and creating the system of multiple representations, where a change in one representation causes simultaneous change in other representations of the same object (Hwang & Hu 2013). Dynamic software also enables forming of so-called virtual manipulative representation. This representation is an interactive visual representation of a dynamic object, which allows students to manipulate objects, i.e., to change one property of the object and at the same time to observe how the change of that property affects the other properties of the object. There are several dynamic software packages which gives student opportunity to link different representations of mathematical objects. One of the more commonly used, because of its availability, simplicity, and performances, is *GeoGebra*. This software is used more often in teaching mathematics, from elementary school to university level. The application of *GeoGebra* was the topic of many researches in the last few years (Arzarello et al. 2012; Doruk et al. 2013; Takači et al. 2015).

Multiple representations of functions

Functions are mathematical concept that can be presented in many ways, i.e. by using different representations. One representation is usually not sufficient to represent all the properties of a function. Therefore, multiple representations are usually applyed to display the functions' properties (Božić & Takači 2021). Usually, the most importance is attached to the algebraic, graphic and tabelar representation, but in recent times, more and more verbal representation is gaining importance. The use of the computers brought innovations in a representation of functions. Multiple representations of functions in a computer-based environment have been topic of many papers (Goldin & Shteingold 2001; Borba & Confrey 1996; Doorman, et al. 2012).

Multiple representations of functions are, in the research above, usually considered in order to improve students' achievements and to help them to overcome difficulties they have with functions' examining and graphing (Božić & Takači 2021; Božić et al. 2022). Earlier researches have shown that students, when it comes to functions' examining, have the most difficulties in working with functions with variable parameters, namely families of functions, as well as with the transformations of functions (Borba & Confrey 1996; Daher & Anabousy 2015; Tall 1992).

The absolute value problems are usually related to the absolute value functions, equations and inequalities, and they are related to the absolute value functions. Most of the expressions, used in the absolute value equations and inequalities, can be observed as the transformations of the absolute value parent function, defined as $f(x) = | x |$ (Elis & Bryson 2011). Because of the mentioned, it is necessary for the students to deal with the absolute value functions and their properties, especially with the transformations of the functions. In (Elis & Bryson 2011) authors point out that the application of the conceptual approach, based on the combining different representations (including graphical and verbal), contributes to better achievements of the students, when it comes to the absolute value of the expression.

GeoGebra is one of the software packages which enables connecting of different representations and work within multiple representations of functions. It also enables forming the dynamic multiple representations of the functions, which are being formed by creating sliders, by which variable parameters are defined. The moving of the slider causes immediate changes of the parameter's value and, consequently, causes changes in algebraic and graphical representation, simultaneously.

3. Research question

The aim of this research is to examine the contribution of multiple representations, in the frame of distance learning during Covid-19 crisis, to the students' achievement in learning the properties of the absolute value functions. Due to the aim of the research, the research question is: How do students manage their learning by using multiple representations, in the dynamic software environment, in order to successfully examine the properties of absolute value functions?

4. Methodology

General background

In this research the experimental approach was applied. During Covid-19 crisis the distance learning was enforced all over the world. In this situation the dynamic software package *GeoGebra* was used for examining the properties of related families of functions, in particular with absolute value functions. The benefits of students' work in a *GeoGebra* dynamic environment are examined and analyzed within a Calculus course at the University of Novi Sad, in 2021 year.

Participants

The research was conducted with 60 undergraduate students, during their calculus course, at the University of Novi Sad in 2021, during Covid-19 crisis. During the distance learning, the teacher used Microsoft TEAMS and *GeoGebra* software for lecturing.

Instruments and procedures

At the beginning of the research, the teacher renewed the students' knowledge of the elementary properties of functions, and as usual, special attention was paid to the functions of absolute value. Also, the students were introduced in using *GeoGebra* software.

The teachers did exercises with students concerning the multiple representations (algebraic, graphical, and verbal) of the functions $f(x) = x^2$ and $g(x) = x^2 + a$, $h(x) = | x^2 + a |$, $x \in R$, $a \in R$, which are shown in fig 1. The functions f, g and h, and the parameter a, are inputted in *GeoGebra* dynamic software. Simultaneously, both their representations (algebraic and graphical) appeared, as is shown in fig 1. Graphics view 2 is used, by the teacher, as an interactive whiteboard, showing each change of inputs. The parameter a, as the slider was moving, causing the change of the function g. The teacher conducted the analysis of the dependence of the properties of the function g, especially its zeros, on the parameter a, as it is shown in fig 1.

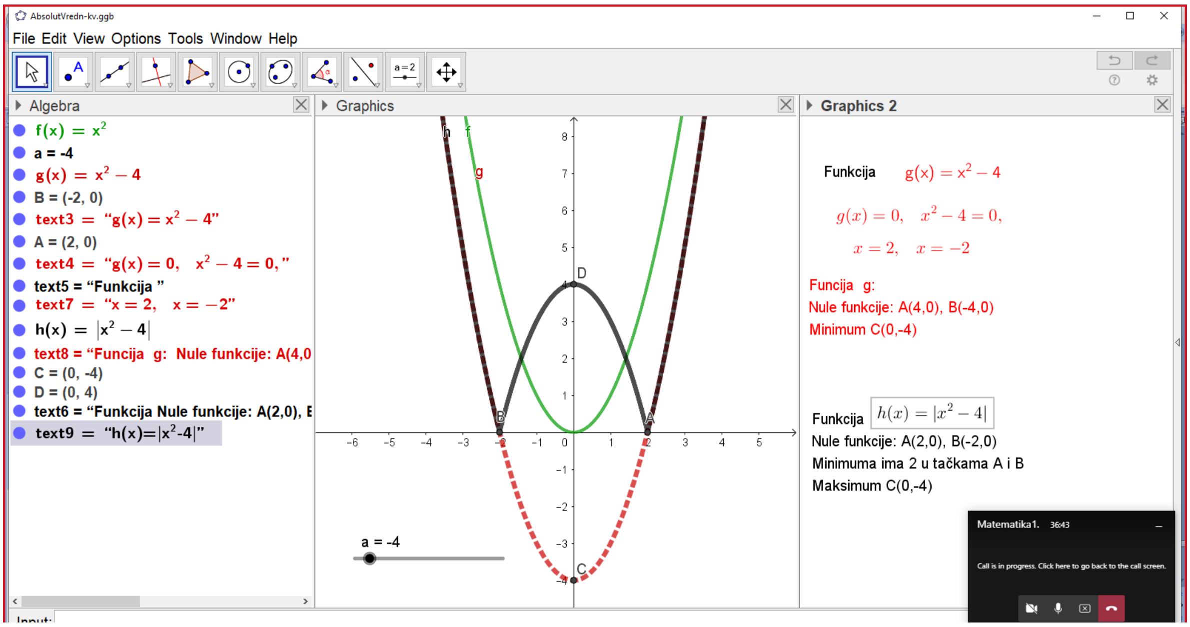


Figure 1. Teacher's explanation

During this lecture the students got the tasks (given in the Appendix A) concerning the properties of the absolute value functions. They had to finish and send them, individually, to the teacher, within 180 minutes.

The first task was given in order to recall the students' knowledge about the elementary properties of functions. In the second and the third task, the students are required to examine the influence of the variable parameters on the properties of functions, with the accent to absolute value. In all of these tasks the students had to analyze different functions, without, with one, and with two real parameters.

In the fourth task, one absolute value function with two real parameters is given and the students need to examine this function and its intersection with the line parallel with x-axis. In fact, the students had to deal with three parameters. Therefore, this task was the most difficult one. Firstly, the students had to examine the influence of the connected parameters a and b to the properties of functions, (at least four cases) and then to examine the influence of parameter p on the number of solutions of the given equation depending on function.

All these tasks are appropriate for the examining functions multiple represented (using graphical, algebraic, and verbal representations). In fact, the graphical representations of considering functions are helpful for required analysis.

In all given tasks, the students need to show their understanding of the dependence of the properties of absolute value functions of parameters by using their graphical representations, which can be obtained by using *GeoGebra.* Also, these graphical representations can be obtained without using the computer. Since, the functions are given with variable parameters, the students were suggested to use *GeoGebra* package, because its dynamic properties enable simultaneous change of representations, by changing parameters' values.

These tasks can be solved by using the algebraic representations of the given functions only, without the help of a computer, but the use of *GeoGebra* software is more convenient during the distance learning. After the exercises (and before the test), the teachers provided the explanations of the tasks for all students. Two weeks after their exercises, the students solved the test, at the University, without the computer. The test contained three tasks. Time for solving was 90 minutes. The tasks of the test are given in Appendix B.

In the first two tasks, the students had to use a graphical representation in order to analyze the absolute value functions $f(x) = | x^2 + a |$, $g(x) = | | x^2 + a | + b |$, $x, a, b \in R$. In fact, the purpose of these tasks was to test the students' knowledge of the influence of parameters on the properties of the absolute value functions, given by their graphs. Also, they had to examine the number of solutions of the absolute value equations. In the third task, the students had to deal with the absolute value equation given algebraically. The students' results of this test were analyzed.

5. Results

The tasks, envisaged for the students' exercises, are given in Appendix A. All students used *GeoGebra* software, during their exercises and they were very satisfied with its help, because it enables all tools necessary for examining functions. The absolute value functions were given without variable parameters, only in the first task. But in other tasks all students used one, two or three sliders to change the values of parameters. For that purpose, they fixed a value of one (or two) parameters and, by changing the second (third) one, they got the same situation as in the first part of the task. As it was expected, all students worked on this task very successfully.

The teachers remarked that the students showed a great deal of variety in their work within *GeoGebra* dynamic environment. There were no two identical worksheets, although the tasks were the same. They used different possibilities of *GeoGebra* package and got interesting algebraic and graphical representations of the absolute value functions, which they used for examining the properties of these functions. This indicates that the students applied constructivist approach during their work. Each student worked by his (her) own, analyzing different parameters and different functions, so they, consequently, got correct results, in different way.

In the following, the most interesting parts of works, from a mathematical point of view, are presented. For example, we present the picture of dynamic worksheet created by student S1 (fig. 2). He analyzed the similarities and the differences between the functions $f_1(x) = | x | + a$ and $f_4(x) = | | x | + a |$, by using "Graphics 1" and "Graphics 2" view, respectively. Both the graphical and algebraic representations of these functions are presented, for $a = 2$. Besides that, the student sent the verbal explanation where he correctly concluded that the observed functions are the same for the positive values of parameter *a*, but for negative values of this parameter they differ, and an example can be seen in fig. 2. It is expected that the examining and comparing the absolute value functions and their properties, such as the one conducted by student S1, increases the student's abilities in the application of graphical representation for solving the absolute value equations.

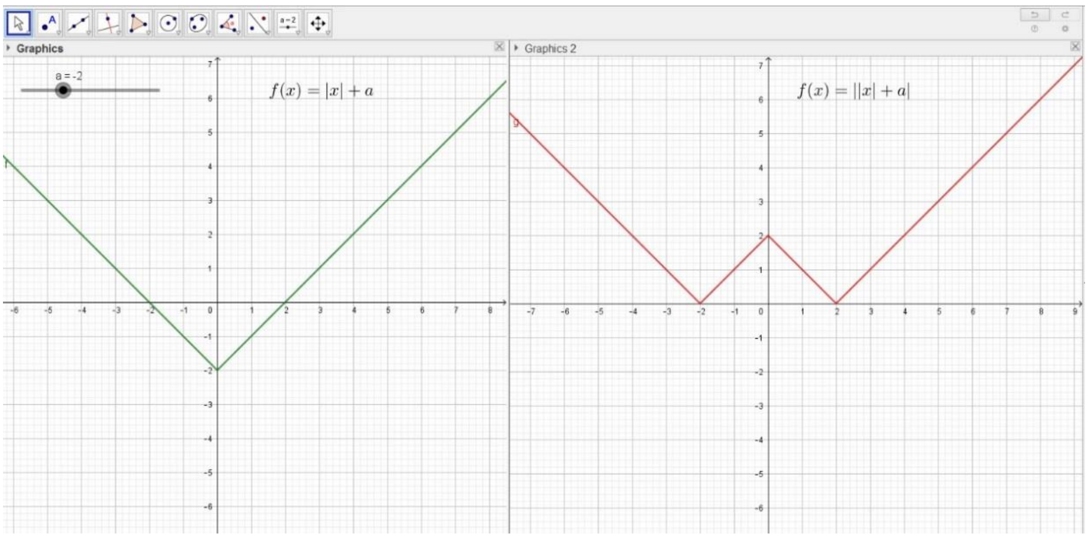


Figure 2. Student's S1 work – comparing two functions in *GeoGebra*

The next examples of the successful applications of the *GeoGebra* properties are given in Figures 3, 4, 5 and, where the student's S2 dynamic worksheets, used in solving the 4th task, are shown. This student analyzed four most important cases of the function $f(x) = | | (x + 2)^2 + a | + b |$ in dependence of the values of the real parameters *a*, *b*, and its connections. She started from the function $f_1(x) = (x + 2)^2$ and continues gradually with the graphs of functions

$$f_2(x) = (x + 2)^2 + a , \qquad f_3(x) = | (x + 2)^2 + a |, \qquad f_4(x) = | | (x + 2)^2 + a | + b |$$

and she distinguished the following cases:

1. Case 1: The function f has only one minimum for $a > 0$, $b > 0$ or $b < 0$, $| b | \le a$. The graphs of f_1, $f_2 = f_3$, f_4 for $a = 3$, $b = -1$ are shown in fig. 3.

2. Case 2: The function f has one maximum and two minimums for $a > 0$ and $|b| > a$ or $a \le 0$ and $b > a$. The graphs of f_1, f_2, f_3, f_4, for $a = -3$, $b = 1$ are shown in fig. 4.
3. Case 3: The function f has two maximums and three minimums for $a < 0$, $b < 0$, and $|b| \ge |a|$. The graphs of f_1, f_2, f_3, f_4, for $a = -1$, $b = -2$ are shown in fig 5.
4. Case 4: The function f has three maximums and four minimums for $a < 0$, $b < 0$, and $|b| < |a|$. The graphs of f_1, f_2, f_3, f_4, for $a = -5$, $b = -2$ are shown in fig. 6.

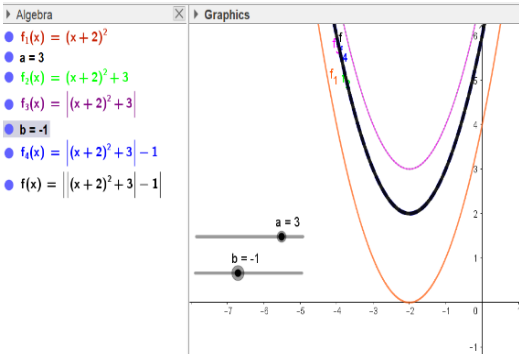


Figure 3. Student's S2 work – Case 1

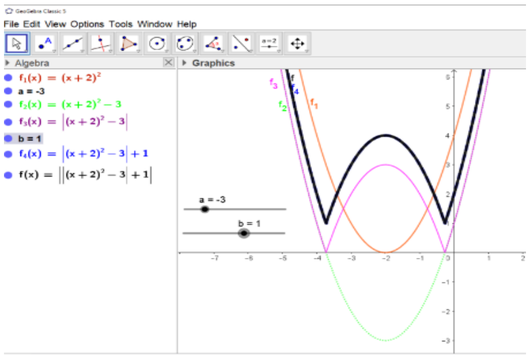


Figure 4. Student's S2 work – Case 2

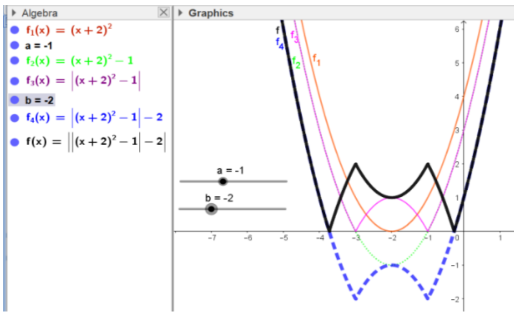


Figure 5. Student's S2 work – Case 3

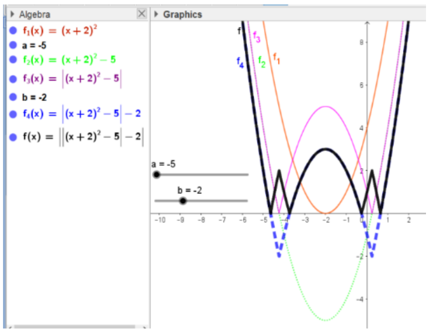


Figure 6. Student's S2 work – Case 4

In all these cases she paid special attention to the influence of the parameter p on the number of the solutions corresponding equation $f(x) = p$. Student S2 correctly applied options "intersection" in order to obtain the intersection points of the graphic of the function $y = f(x)$ and the line $y = p$ which x-coordinates represent the solutions of the equation $f(x) = p$. An example of students' S2 analysis of the equation $||(x + 2)^2 - 1| - 2| = p$, for $p = 1$ is shown in fig. 7. Student S2 use the ability of the *GeoGebra* package in order to show only the graph of the function f (not the graphs of the other functions f_1, f_2, f_3, f_4,) in order to make clear the intersections of f and the line $y = p$ the points A, B, C, D, and E.

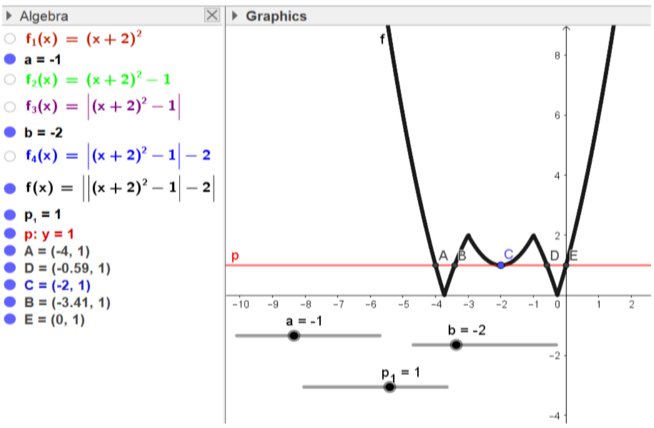


Figure 7. Student's S2 work

The teacher did not suggest the form of the finished students' tasks, so the students could prepare it by hand, by computer only, or by using their combinations. The students have prepared and submitted the final solutions of tasks in different ways. There were 37 final solutions prepared as the combination of hand and computer applications and 23 tasks were solved by computer only. The students who used their smart phones (24 students) submitted their final solution in the combination of both applications, while 23 students working with desktop or laptop computers had only computer applications of solutions. Nobody had only handwritten solutions.

An example of the successful examining the properties of the function $f(x) = | x^2 + a |$ (second task) depending on the parameter a, is given in fig. 8. Student S3 attached material as a combination of the electronic worksheet and handwritten document which is shown in fig. 8. We can remark that student S3 examined even more properties than it was required.

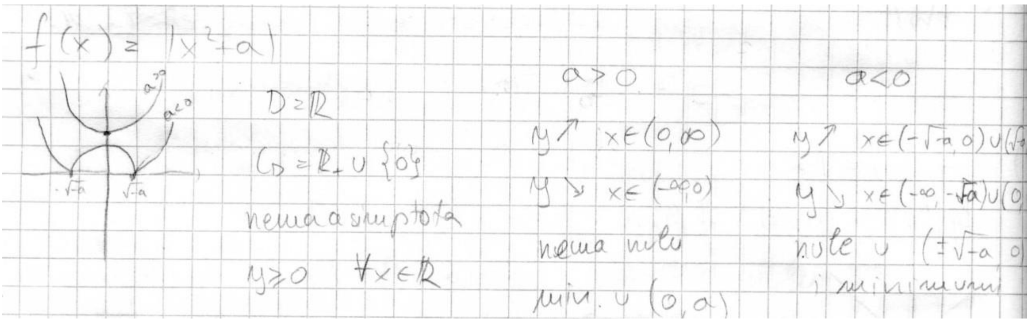


Figure 8. Student’s S3 work – examining the properties of the function

GeoGebra environment enables examining functions by using simultaneous graphical and algebraic representations and geometric notions, such as the corresponding intersection of objects. For example, the students S1 and S2 used the multiple representations, but with dominant graphical representation. Most of the students took care about all required properties of the functions and their dependence on the parameters, but some of them omitted some properties. An overview of the number of students who correctly explained the influence of parameters on certain properties (interesting for this research) of the functions is given in Table 1.

In determining the zeroes and the extreme values of the given functions the students needed to denote the points, but in determining the signs of functions, the students need to use the intervals. It is known that the students generally have problems with exact determining the corresponding intervals. Therefore, as it was expected, most of the students determined exactly the zeroes of the functions. The others just omitted some zeroes. There were no students who determined wrong coordinate for zero of function in all tasks. There were no students who did not determine exactly zeroes in first task but determined correctly zeroes in the other tasks.

Table 1. Number of the students who correctly explained the influence of parameters on certain property of the function

	Function’s property		
Task	Zeroes	Sign	Extremes
1.	52	36	48
2.	36	21	35
3.	18	15	26

Before the test, the teacher conducted a discussion with the students, using Microsoft TEAMS, analyzed their answers and suggested how to overcome the difficulties they encountered in their tasks.

Analysis of the test results

About two weeks after the exercises described in Section 4, the students' knowledge about the properties of absolute valued functions was tested. The test contained three tasks. Time for solving was 90 minutes. The students did not have the possibility of using the computers during the test. The tasks of the test are given in Appendix B. The percentages of students who gave the correct results on the test for first two tasks is given in Table2.

Table 2. The percentages of students who gave the correct results on the test (for first two tasks)

Task		1A	1B	2A	2B	2C	2D
Correct	a)	90%	78%	84%	68%	51%	36%
	b)	64%	58%	64%	56%	43%	18%

By analyzing the results given in Table, we can notice that the students had the best results in the first task. The first part of the task was done better than the second part, because some of the students did not care about the values $p < 0$. There were 34 students (57%) who remarked that all functions are even, and it will be useful for examining the number of solutions for given equations.

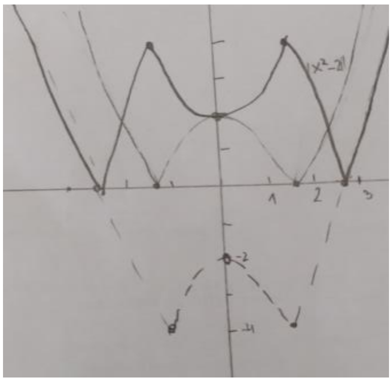


Figure 9. Student's H1 work – test, for $a = -2$, $b = -4$

In the second task, referring Case 2A, there were 25 students (42 % of them) who remarked that the solution is not unique and compared it with the Case 1A if one parameter is equal zero. In solving the cases 2C and 2D the students worked successively considering the graphs of functions:

$$y = x^2 + a,\ y = |\, x^2 + a \,|,\ y = |\, x^2 + a \,| + b,\ y = |\, |\, x^2 + a \,| + b \,|,$$

for different real values of a and b. For example, the student's H1 work is shown in fig 9. The student H1 drew the graphs of functions

$$y = |\, x^2 - 2 \,|,\ y = |\, x^2 - 2 \,| - 4,\ y = |\, |\, x^2 - 2 \,| - 4 \,|,$$

for $a = -2$, $b = -4$, with the same pencil but with different styles of lines. The student H2 drew all function for $a = -1$, $b = -4$, in different colors (fig. 10). The most difficult was the task Case 2D, as it was expected, and even 33% students just omitted it.

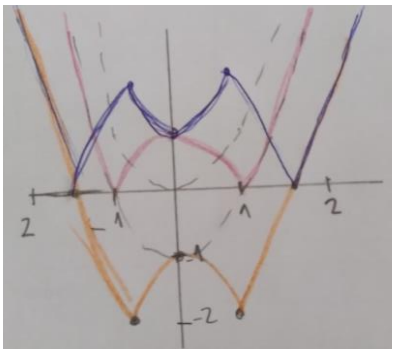


Figure 10. Student's H2 work – test, for $a = -1$, $b = -4$

It is interesting to note that all students had started to solve the third task. They explained that they there were no arbitrary parameters and therefore they had taught that it would not be difficult. The percentages of students who gave the correct results on the test for third task is presented in Table 3.

Table 3. The percentages of students who gave the correct results on the test for third task

Task	3a	3b	3c
correct	36%	28%	13%

The students obtained the solutions of the third task working either only algebraically or by using graphical representation (combined with the algebraic representation). There were 39 students (65%) who combined the graphical and the algebraic representation in the third task (by using the method from the previous tasks), but still only 8 students got the correct solutions of this task.

Even 21 students (35%) were solving this task in algebraic representation, but none of them did correctly the task 3c). The task 3a) was the easiest, but only 10 students got its correct solution. Only 5 of those students, who correctly solved 3a), got correct solution of 3b). There were no students who had correct task 3b), but not correct 3a).

There were 18 students (33%) who, working the third task, recognized that the graph of the function $y = | | x^2 - 4 | - 3 |$, appearing on the left hand side of equations in the third task, is already drawn in case 2D, and used it for solving this task, i.e., for discussing the number of the solutions of equation 3c ($| | x^2 - 4 | - 3 | = p$).

The students' H3 and H4 works are presented in fig. 11 and 12. They both used the graph in the case 2D. The task in fig. 11 is solved correctly. All interesting cases for the value p are analyzed and the numbers of solutions are given. In fig. 12 the most common students' answers are shown. These answers do not include all possible values of p. The student H4 did not even draw the graph of $y = p$. In fig. 13, the work of student H5 is shown. He solved the task 3c) correctly, by drawing graph by himself. There were two students who consider the line $x = 2$ instead of $y = 2$ in the task 3b. This indicates that they were not able to use the graphical representations of the given equations.

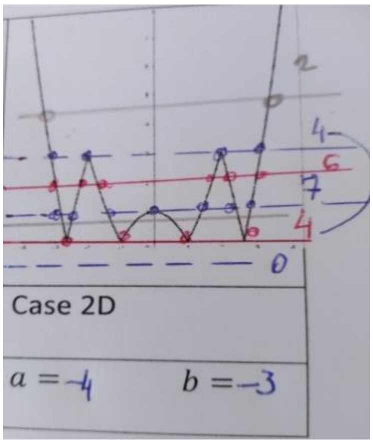


Figure 11. Student's H3 work – test

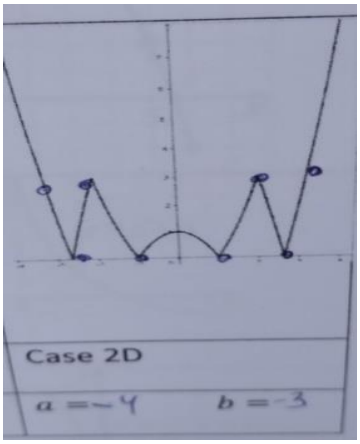


Figure 12. Student's H4 work – test

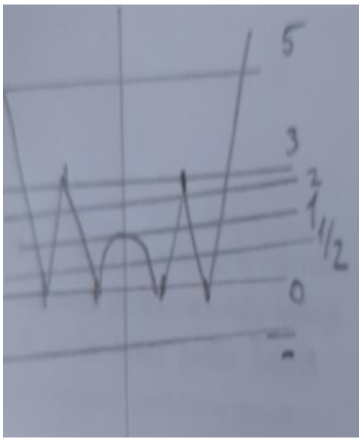


Figure 13. Student's H5 work – test

6. Discussion

In this research, the influence and effectiveness of the application of multiple representations in the dynamic environment on students' achievements in examining the properties of the absolute value functions are analyzed in the frame of distance learning during Covid19 crisis. The students had to use the multiple representations of the absolute value functions during their learning process and the test and to deal with the transformations of the absolute value functions. Such way of learning the absolute value functions is proved to be effective in learning the properties of these functions (Elis & Bryson 2011). In our research, we applied the multiple representations within the dynamic software environment. For providing the dynamic multiple representations of the absolute value functions, GeoGebra software package is applied, in accordance with (Stupel & Ben-Chaim 2014).

The students used GeoGebra dynamic software during the learning process in the frame of distance learning during Covid19-crisis. The students' solutions of the tasks, written works and electronic material, were reviewed by the teachers. The difficulties occurred (were related to the existence of more than one absolute value within the expression, similarly as in the research (Ciltas & Tatar 2011), and to the existence of more than one variable parameter, in accordance with the previous researches (Borba & Confrey 1996; Daher & Anabousy 2015).

In order to check the efficiency of the use of multiple representation in learning the absolute value function, the students got the test with the appropriate tasks. The results of the test were analyzed. Besides the difficulties occurred during the learning process, the students, solving the test, had difficulties with choosing the appropriate method for solving the equations (algebraic only or combined – graphical and algebraic), which is in accordance with the research (Stupel 2012) and (Stupel & Ben-Chaim 2014).

The analysis of the students' work on exercises and students' test showed great way of variety in their finished tasks. This indicates that they applied constructivist way of learning, according to (Bodner 1986), (Tobin & Tippins 1993), and (Von Glasersfeld 1995). The teacher created a learning environment (Microsoft Teams, with *GeoGebra* package) and encouraged discussion with the students about their difficulties during working on their exercises, in order to help students to acquire new skills and knowledge (Brooks & Brooks 1993; Iran-Nejad 1995; Sjoberg 2010; Taber 2011). The students also spoke highly of the use of *GeoGebra* during classes where they practiced tasks and deepened their mathematical knowledge. They especially liked being able to relate the algebraic notations of the functions to their graphs and see how changing a parameter in the algebraic notation of the function affected its graph.

The research presented in the paper contains students' experiences during Covid-19 in learning functions and, according to (Psotka 2022), this can represent the contribution to the improvement of future education.

Conclusions

In our research, we analyze the contribution of the dynamic multiple representations, to the improvement of the students' learning process, within function concept, in the frame of distance learning during Covid-19 crisis. The students organized their learning in different ways, depending on their previous knowledge and their learning skills and interests. This was expressed in a variety of students' ways of solving tasks, using *GeoGebra* software, which is discussed in the analysis of students' work.

From the above discussed, it can be concluded that the dynamic software environment during distance learning, encourage students to apply different ways and in solving tasks, i.e. to adjust the learning methods to their needs, and helps students to create their own sophisticated knowledge about the functions and their properties.

There exist some limitations of this research, of course. Namely, the tasks chosen for the analysis were appropriate for the graphical solving and it is shown that the dynamic software contributes to the students' achievements in this field. But, in the future research, other types of the absolute value problems should be considered in the environment of the modern technology.

Conflict of interests

The authors declare no conflict of interest.

REFERENCES

ADEDOYIN, O. B., & SOYKAN, E., 2020. COVID-19 pandemic and online learning: The challenges and opportunities. *Interactive Learning Environments*, 2(1), 1–13. https://doi.org/10.1080/10494820.2020.1813180

AL SALMAN, S., ALKATHIRI, M., KHALED BAWANEH, A., 2021. School off learning on: Identification of preference and challenges among school students towards distance learning during COVID19 outbreak. *International Journal of Lifelong Education*, 40(1), 53–71. https://doi.org/10.1080/02601370.2021.1874554

ANABOUSY, A., DAHER, W., BAYA'A, N., ABU-NAJA, M., 2014. Conceiving function transformations in different representations: Middle school student working with technology. *Mathematics Education*, 9(2), 99–114. https://doi.org/10. 29333/iejme/284

ARZARELLO, F., FERRARA, F., ROBUTTI, O., 2012. Mathematical modelling with technology: the role of dynamic representations. *Teaching Mathematics and Its Applications,* 31(1), 20-30.

BODNER, G.M., 1986. Constructivism: A theory of knowledge. *Journal of Chemical Education*, 63(10), 873–878. https://doi. org/10.1021/ed063p873

BORBA, M., CONFREY, J., 1996. A student's construction of transformations of functions in a multiple representational environment. *Educational Studies in Mathematics,* 31, 319-337.

BROOKS, J.G., BROOKS, M.G., 1993. *In search of understanding: The case for constructivist classrooms*. American Society for Curriculum Development.

BOŽIĆ, R., TAKAČI, Đ., 2021. The Benefits of Collaborative Learning in Examining Functions with Parameters in Dynamic Software Environment. *Mathematics and Informatics*. https://doi.org/10.53656/math2021-5-1-the

BOŽIĆ, R., TAKAČI, Đ., GRUJIN, M., 2022. The Influence of Collaborative Learning on Students' Achievement in Examining Functions with Parameters in Dynamic Software Environment. *Mathematics and Informatics*, 65(1). https://doi.org/10.53656/math2022-1-1-th

CILTAS, A., TATAR, E., 2011. Diagnosing Learning Difficulties Related to the Equation and Inequality that Contain Terms with Absolute Value. *International Online Journal of Educational Sciences*, 32(2), 461-473.

DAHER, W.M., ANABOUSY, A.A., 2015. Students' recognition of function transformations' themes associated with the algebraic representation. *REDIMAT,* 4(2)*,* 179-194.

DORUK, B.K., AKTUMEN, M., AYTEKIN, C., 2013. Pre-service elementary mathematics teachers' opinions about using GeoGebra in mathematics education with reference to 'teaching practices'. *Teaching Mathematics and Its Applications,* 32(3), 140-157.

DOORMAN, M., DRIJVERS, P., GRAVEMEIJER, K., BOON, P., REED, H., 2012. Tool use and the development of the function concept: from repeated calculations to functional thinking. *International Journal of Science and Mathematics Education* 10: 1243-1267.

DRIJVERS, P., THURM, D., VANDERVIEREN, E., KLINGER, M., MOONS, F., VAN DER REE, H., MOL, A., BARZEL, B., DOORMAN, M., 2021. Distance mathematics teaching in Flanders, Germany, and the Netherlands during COVID-19 lockdown. *Educational Studies in Mathematics*, 108(1-2), 35–64. https://doi.org/10.1007/s10649-021-10094-5

FIDALGO, P., THORMANN, J., KULYK, O., LENCASTRE, J. A., 2020, Students' perceptions on distance education: A multinational study. *International Journal of Educational Technology in Higher Education*, **17**, 18. https://doi.org/10.1186/s41239-020-00194-2

GOLDIN, G., SHTEINGOLD, N., 2001. System of representations and the development of mathematical concepts. In A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics,* pp. 1-23. Yearbook 2001. Reston, VA: NCTM.

ELLIS, M., BRYSON, J., 2011. A conceptual approach to absolute value equations and inequalities. *Mathematics Teacher*, 104(8), 592-598.

HWANG, W.Y., HU, S.S., 2013. Analysis of peer learning behaviors using multiple representations in virtual reality and their impacts on geometry problem solving. *Computers and Education,* Vol. 62, 308-319.

IRAN-NEJAD, A., 1995. Constructivism as substitute for memorization in learning: Meaning is created by learner. *Education*, 116, 16–32.

MITROVIĆ, S., BOŽIĆ, R., TAKAČI, Đ., 2022. Efficiency of blended learning of calculus content during the Covid19 crisis. *Interactive Learning Environments*. doi:10.1080/10494820.2022.2076129

NAKAHARA, T., 2008. Cultivating mathematical thinking through representation-utilizing the representational system*, APEC-TSUKUBA International Congress, Japan.*

OZGUN-KOCA, S.A., 2008. Ninth Grade Students Studying the Movement of Fish to Learn about Linear Relationships: The Use of Video-Based Analysis Software in Mathematics Classrooms. *The Mathematics Educator,* 18(1)*,* 15–25.

PSOTKA, J., 2022. Exemplary online education: For whom online learning can work better. *Interactive Learning Environments*, 30(2), 199–201. https://doi.org/10.1080/10494820.2022.2031065

RAU, M., MICHAELIS, J., FAY, N., 2015. Connection making between multiple graphical representations: A multi-methods approach for domain-specific grounding of an intelligent tutoring system for chemistry, *Computers and Education,* Vol. 82, 460-485.

SEVER, G., YERUSHALMY, M., 2007. To sense and to visualize functions: The case of graphs' stretching. In P. P. Demetra & P. George (Eds.), *The Fifth Conference of the European Society for Research in Mathematics Education (CERME5)*, pp. 1509-1518. Larnaca, Deprtment of Education, University of Cyprus.

SIERPINSKA, A., BOBOS, A., PRUNCUT, A., 2011. Teaching absolute value inequalities to mature students. *Educational Studies in Mathematics,* 78(3), 275–305.

SJOBERG, S., 2010. Constructivism and learning. In B. McGaw & P. Peterson (Eds.), *International encyclopaedia of education*, 3rd Edition, pp. 485-490. Elsevier.

STEVANOVIĆ, A., BOŽIĆ, R., RADOVIĆ, S., 2021. Higher education students' experiences and opinion about distance learning during the Covid-19 pandemic. *Journal of Computer Assisted Learning*, 37(6), 1682–1693 https://doi.org/10.1111/ jcal.12613

STUPEL, M., 2012. A special application of absolute value techniques in authentic problem solving. International Journal of Mathematical Education in Science and Technology, 44(4), 587-595.

STUPEL, M., BEN-CHAIM, D., 2014. Absolute value equations – what can we learn from their graphical representation? *International Journal of Mathematical Education in Science and Technology*, 45(6), 923-928.

TABER, K.S., 2011. Constructivism as educational theory: Contingency in learning, and optimally guided instruction. In J. Hassaskhan (Ed.), *Educational theory*, pp. 39–61. Nova Science Publishers, Hauppauge.

TAKAČI, D.J., STANKOV, G., MILANOVIC, I., 2015. Efficiency of learning environment using GeoGebra when calculus contents are learned in collaborative groups. *Computers and Education,*

Vol. 82, 421-431.
TALL, D., 1992. The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity, and Proof. In Grouws D.A. (Eds.) *Handbook of Research on Mathematics Teaching and Learning*, pp. 495-511. New York: Macmillan.
TOBIN, K., TIPPINS, D., 1993. Constructivism as a referent for teaching and learning. In K. Tobin (Ed.), *The practice of constructivism in science education*, pp. 3–22. Lawrence Erlbaum Associates.
TRENHOLM, S., PESCHKE, J., 2020. Teaching undergraduate mathematics fully online: a review from the perspective of communities of practice. *International Journal of Educational Technology in Higher Education*. https://doi.org/10.1186/s41239-020-00215-0
VON GLASERSFELD, E., 1995. *Radical constructivism: A way of knowing and learning*. Falmer Press.
WILHELMI, M.R., GODINO, J.D., LACASTA, E., 2007. Didactic effectiveness of mathematical definitions: The case of the absolute value. *International Electronic Journal of Mathematics Education,* 2(2), 72–90.

Appendix A. *The tasks solved by the students during their work on examining the absolute value functions*

1. Examine the properties (zeros, sign, extreme values) of the functions $| f(x) |$ and $f(| x |)$ where the functions f are given in the table below:

$f(x) = -5x + 2$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = x^{-1}$
$f(x) = 2^x$	$f(x) = 1/3^x$	$f(x) = \log x$	$f(x) = \sin x$

2. Examine the properties (zeros, sign, extreme values) of the given functions in dependence+ of real parameter a:

$f_1(x) = \| x \| + a$	$f_2(x) = \| x + a \|$	$f_3(x) = a\| x \|$
$f_4(x) = \| \| x \| + a \|$	$f_5(x) = \| x^2 + a \|$	

3. Examine the properties (zeros, sign, extreme values) of the given functions in dependence of real parameter a and b:

$g_1(x) = a\| x \| + b$	$g_2(x) = \| x + a \| + b$	$g_3(x) = \| \| x + a \| + b \|$
$g_4(x) = \| x^2 + a \| + b$	$g_5(x) = \| \| x^2 + a \| + b \|$	

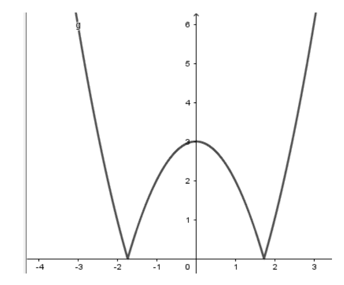
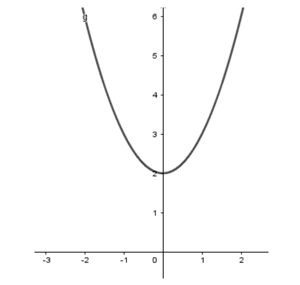
4. Examine the influence of the real parameters a, b, p on the properties (zeros, extreme values) of function $f(x) = | | (x - 2)^2 + a | + b |$. Determine the number of solutions of the equation $f(x) = p$.

Appendix B. *The tasks for the test.*

1. The function $f(x) = | x^2 + a | + b$ is given with real parameter a.

a) Determine the value of parameter a so, that the graphs below correspond to the function f and explain the answers for the cases 1A and 1B.

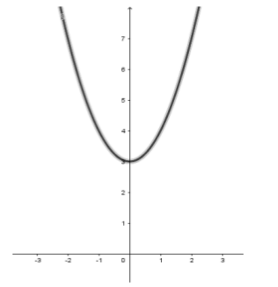
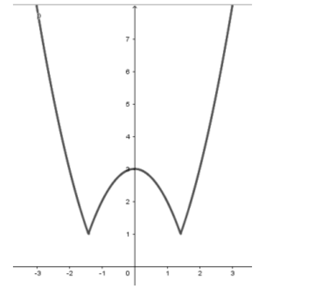
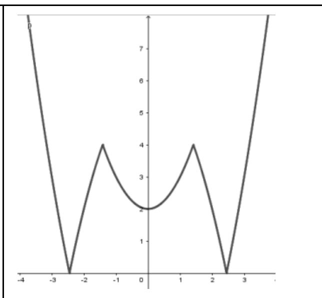
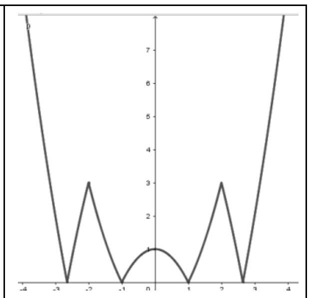
b) Discuss the number of solutions of equation $| x^2 + a | + p$, depending on p, for the cases below:

	
Case1A, a =	Case 1B, a =

2. The function $g(x) = | \, | x^2 + a \, | + b \, |$ is given with the real parameters a and b.

a) Determine the values of parameters a and b, so that the graphs below correspond to the function f and explain the answers.

b) Discuss the number solutions of equation $| \, | x^2 + a \, | + b \, | = p$ depending on real parameter p, for the cases below:

			
Case 2A	Case 2B	Case 2C	Case 2D
a = b =	a = b =	a = b =	a = b =

3. Determine the number of solutions of the equations

a) $| \, | x^2 - 4 \, | - 3 \, | = 0$, b) $| \, | x^2 - 4 \, | - 3 \, | = 2$, c) $| \, | x^2 - 4 \, | - 3 \, | = p, \; p \in R$.