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RESEARCH ARTICLE

On Null Cartan Rectifying Isophotic and Rectifying Silhouette Curves Lying on a Timelike Surface in Minkowski Space $\mathbb{E}^3_1$

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

In this paper, we introduce generalized Darboux frames of the first and the second kind along a null Cartan curve lying on a timelike surface in Minkowski space $\mathbb{E}^3_1$ and define null Cartan rectifying isophotic and rectifying silhouette curves in terms of the vector field that belongs to generalized Darboux frame of the first kind. We investigate null Cartan rectifying isophotic and rectifying silhouette curves with constant geodesic curvature k_g and geodesic torsion τ_g and obtain the parameter equations of their axes. We prove that such curves are the null Cartan helices and the null Cartan cubics. We show that the introduced curves with a non-zero constant curvatures k_g and τ_g are general helices, relatively normal-slant helices and isophotic curves with respect to the same axis. In particular, we find that null Cartan cubic lying on a timelike surface is rectifying isophotic and rectifying silhouette curve having a spacelike and a lightlike axis. Finally, we give some examples.

Keywords: *Generalized Darboux frame, null Cartan curve, rectifying isophotic curve, rectifying silhouette curve.*
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1. Introduction

In Euclidean space $\mathbb{E}^3$ *rectifying curves* are introduced by B.Y. Chen in [2] as regular curves whose position vector, with respect to some chosen origin, always lies in a rectifying plane $N^\perp$ spanned by the tangent and the binormal vector fields T and B. Consequently, the position vector of a rectifying curve α parameterized by arc-length s in $\mathbb{E}^3$, satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s),$$

where λ and μ are some differentiable functions. A rectifying curve has a property that the ratio of its torsion and the curvature is a non-constant linear function in s ([2]). Such curve is related with the Darboux vector (centrode) of its Frenet frame ([3],[5]). In particular, it is known that all conical rectifying curves in $\mathbb{E}^3$ are geodesics ([4]). Rectifying curves lying in the three dimensional sphere $\mathbb{S}^3(r)$ and in the three-dimensional hyperbolic space $\mathbb{H}^3(-r)$ are characterized in [19] and [20]. For some geometric properties of rectifying curves in Minkowski spaces, we refer to [12, 14, 15].

The notion of the *helix* in $\mathbb{E}^3$ is generalized in [18] in terms of the vector field that is constant with respect to the Frenet frame of the curve and forms a constant angle with some fixed direction. If the mentioned vector field lies in a normal, rectifying, or osculating plane, the curve is called a *normal, rectifying* and *osculating helix*, respectively.

Isophotic curves are defined by the property that the surface normal along such curves makes a non-zero constant angle with some fixed direction (*axis* of the curve). In particular, if the surface normal is orthogonal to

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the axis, the curve is called a *silhouette* (*contour generator*). It contains the points having the same light intensity from a given light source emitting parallel light rays ([7]).

Generalized Darboux frames of the first and the second kind along a pseudo null curve and a spacelike curve with a non-null principal normal lying on a lightlike surface in Minkowski space $\mathbb{E}_1^3$ are defined in [8] and [9] as the frames obtained by scaling the null normal vector field of the surface restricted to the curve. Such frames represent generalizations of the Darboux frame and have a property that the curve is geodesic, asymptotic and principal curvature line if and only if its generalized geodesic curvature, generalized normal curvature and generalized geodesic torsion is equal to zero, respectively at each point of the curve.

Isophotic curves, *relatively normal-slant helices* and *general helices* in Minkowski space $\mathbb{E}_1^3$ are characteristic curves defined by the property that the vector field of their Darboux frame $\{T, \zeta, \eta\}$ satisfies respectively the conditions $\langle \eta, U\rangle = constant \neq 0$, $\langle \zeta, V\rangle = constant$, $\langle T, W\rangle = constant$, where U, V, W are axes of the curves ([6, 11, 21, 22, 23, 24]. It is proved in [22] that null Cartan isophotic curves that are principal curvature lines, are relatively normal-slant helices having the same axis. In particular, null Cartan general helices with a non-zero constant geodesic curvature $k_g(s)$ and geodesic torsion $\tau_g(s)$, are null Cartan isophotic curves with respect to the same axis ([22]).

In this paper, we introduce *null Cartan rectifying isophotic* and *rectifying silhouette curves* in terms of the vector field $\tilde{\eta}$ that lies in the rectifying plane determined by Darboux frame's vector fields, belongs to the generalized Darboux frame of the first kind and satisfies the condition $\langle \tilde{\eta}, W\rangle = constant$, where W is a fixed axis of the curve. In relation to that, we first define the generalized Darboux frames of the first and the second kind along a null Cartan curve lying on a timelike surface obtained by scaling the null non-tangent vector field of its Darboux frame. We investigate the introduced curves with constant geodesic curvature $k_g(s)$, normal curvature $k_n(s)$ and geodesic torsion $\tau_g(s)$ and obtain parameter equations of their axes. In particular, we derive that such curves with constant curvatures $k_g(s)$, $k_n(s)$ and $\tau_g(s)$ are the null Cartan helices and the null Cartan cubics. We prove that an axis of a null Cartan rectifying isophotic curve with a non-zero constant curvatures $k_g(s)$ and $\tau_g(s)$, has the same direction as the Darboux vector of the Darboux frame. We obtain that null Cartan rectifying isophotic curves with non-zero constant curvatures $k_g(s)$ and $\tau_g(s)$, are general helices, relatively normal-slant helices and isophotic curves with respect to the same axis. We find that null Cartan cubic lying on a timelike surface, is rectifying isophotic and rectifying silhouette curve having a spacelike and a lightlike axis. Finally, we give some examples.

2. Preliminaries

Minkowski space $\mathbb{E}_1^3$ is a real vector space $\mathbb{R}^3$ equipped with an indefinite flat metric given by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3,$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}_1^3$. An arbitrary vector $x \neq 0$ in $\mathbb{E}_1^3$ can be *spacelike*, *timelike*, or *null* (*lightlike*) if $\langle x, x\rangle > 0$,$\langle x, x\rangle < 0$, or $\langle x, x\rangle = 0$, respectively ([25]). In particular, the vector $x = 0$ is said to be spacelike. The *norm* (*length*) of a vector x is given by $\|x\| = \sqrt{|\langle x, x\rangle|}$.

The *vector product* of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{E}_1^3$ is defined by

$$u \times v = (u_3v_2 - u_2v_3,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1).$$

An arbitrary curve $\beta : I \to \mathbb{E}_1^3$ can locally be *spacelike, timelike*, or *null (lightlike)*, if all of its velocity vectors β' are spacelike, timelike, or null, respectively ([25]). A null curve α in $\mathbb{E}_1^3$ parameterized by pseudo-arc function given by

$$s(t) = \int_0^t ||\alpha''(u)||^{\frac{1}{2}}\, du,$$

is called a *null Cartan curve* ([1]).

There exists a unique Cartan frame $\{T, N, B\}$ along a null Cartan curve α satisfying the conditions ([10])

$$\langle T, T\rangle = \langle B, B\rangle = \langle T, N\rangle = \langle N, B\rangle = 0,\ \langle N, N\rangle = 1,\ \langle T, B\rangle = \epsilon = \pm 1, \tag{2.1}$$

$$T \times N = \epsilon T, \quad N \times B = \epsilon B, \quad B \times T = N, \tag{2.2}$$

where $T = \alpha'(s)$ and $N(s) = \alpha''(s)$. The Cartan frame is *positively oriented* if it satisfies the relation

$$det(T, N, B) = [T, N, B] = 1.$$

The Cartan frame's equations read ([26])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \epsilon\tau & 0 & -\epsilon\kappa \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.3}$$

where $\kappa = \kappa(s) = 1$ is the *curvature* and $\tau = \tau(s)$ is the *torsion* of α. If $\kappa(s) = 1$ and $\tau(s) = 0$ for each s, α is called a *null Cartan cubic*. In particular, if $\kappa(s) = 1$ and $\tau(s) = constant \neq 0$ for each s, α is called a *null Cartan helix*.

A surface in Minkowski space $\mathbb{E}^3_1$ is called a *timelike*, if the induced metric on the surface is a non-degenerate of index one. A timelike ruled surface with parametrization $x(s,t) = \alpha(s) + tB(s)$ is called *B-scroll*, where α is a null Cartan curve and B is its binormal vector field.

Darboux frame along a null Cartan curve α lying on a timelike surface in $\mathbb{E}^3_1$ with parametrization $x(u,t)$ is a positively oriented frame $\{T, \zeta, \eta\}$, consisting of the null tangential vector field $T = \alpha'$, the unit spacelike normal vector field

$$\eta = \frac{x_u \times x_t}{||x_u \times x_t||}|_\alpha$$

and the null transversal vector field ζ, satisfying the conditions ([22])

$$\langle T, T\rangle = \langle \zeta, \zeta\rangle = \langle T, \eta\rangle = \langle \zeta, \eta\rangle = 0, \quad \langle \eta, \eta\rangle = 1, \quad \langle T, \zeta\rangle = \epsilon_1 = \pm 1, \tag{2.4}$$

$$T \times \zeta = \eta, \quad \zeta \times \eta = \epsilon_1 \zeta, \quad \eta \times T = \epsilon_1 T, \quad det(T, \zeta, \eta) = [T, \zeta, \eta] = 1. \tag{2.5}$$

Darboux frame's equations read ([22])

$$\begin{bmatrix} T' \\ \zeta' \\ \eta' \end{bmatrix} = \begin{bmatrix} \epsilon_1 k_g & 0 & k_n \\ 0 & -\epsilon_1 k_g & \tau_g \\ -\epsilon_1 \tau_g & -\epsilon_1 k_n & 0 \end{bmatrix} \begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix}, \tag{2.6}$$

where the functions

$$k_g(s) = \langle T'(s), \zeta(s)\rangle, \quad k_n(s) = \langle T'(s), \eta(s)\rangle, \quad \tau_g(s) = \langle \zeta'(s), \eta(s)\rangle. \tag{2.7}$$

are called *geodesic curvature*, *normal curvature* and *geodesic torsion* of α, respectively.

Definition 2.1. ([22]) A null Cartan curve lying on a timelike surface in $\mathbb{E}^3_1$ is called a *geodesic curve*, *asymptotic curve* and *principal curvature line* if it has $k_g(s) = 0$, $k_n(s) = 0$ and $\tau_g(s) = 0$ respectively for each s.

The Cartan and Darboux frame of α are related by

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \epsilon_1 k_g & 0 & k_n \\ -\frac{\epsilon}{2} k_g^2 & \epsilon\epsilon_1 & -\epsilon\epsilon_1 \frac{k_g}{k_n} \end{bmatrix} \begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix},$$

where the curvature functions of α satisfy the equation

$$\tau = \epsilon\epsilon_1 k_g' + \tau_g + \frac{\epsilon}{2} k_g^2. \tag{2.8}$$

Definition 2.2. ([22]) A null Cartan curve α with the Darboux frame $\{T, \zeta, \eta\}$ lying on a timelike surface in $\mathbb{E}^3_1$ is respectively called the *general helix*, *relatively normal-slant helix* and *isophotic curve*, if there exist constant vectors U, V and W in $\mathbb{E}^3_1$ such that respectively holds

$$\langle T, U\rangle = c_1, \quad \langle \zeta, V\rangle = c_2, \quad \langle \eta, W\rangle = c_3,$$

where c_1, c_2, $c_3 \in \mathbb{R}$. If $c_3 = 0$, α is called a *silhouette curve* with an axis W.

Darboux vector (*angular velocity vector*) of Darboux frame $\{T, \zeta, \eta\}$ has the form

$$D = \tau_g T - k_n \zeta + k_g \eta \tag{2.9}$$

and satisfies the *Darboux equations*

$$T' = D \times T, \quad \zeta' = D \times \zeta, \quad \eta' = D \times \eta. \tag{2.10}$$

The vector fields T, ζ and η satisfying relation (2.10) are called *constant vector fields with respect to the Darboux frame* ([18]). Throughout the next sections, let $\mathbb{R}_0$ denote $\mathbb{R} \setminus \{0\}$.

3. Generalized Darboux frames of the first and the second kind along a null Cartan curve

In this section, we introduce two frames along a null Cartan curve α lying on a timelike surface, derived by scaling the null non-tangent vector field ζ of the Darboux frame $\{T, \zeta, \eta\}$. The mentioned frames coincide with the Darboux frame of α in a special case. For this reason, we called them *generalized Darboux frames of the first and the second kind*. We give the relations between such frames and the Darboux frame and derive the generalized Darboux frame's equations. We first prove the next theorem.

Theorem 3.1. *Every null Cartan curve lying on a timelike surface in $\mathbb{E}^3_1$, with Cartan frame $\{T, N, B\}$ and Darboux frame $\{T, \zeta, \eta\}$, has the normal curvature $k_n(s) = -\epsilon\epsilon_1$, where $\langle T, B\rangle = \epsilon$ and $\langle T, \zeta\rangle = \epsilon_1$.*

Proof. Assume that α is a null Cartan curve lying on a timelike surface in $\mathbb{E}^3_1$. By using the relations (2.1), (2.3), (2.4) and (2.6), we find

$$\langle T', T'\rangle = \langle N, N\rangle = k_n^2 = 1.$$

Relations (2.1), (2.3) and (2.6) give

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \epsilon_1 k_g & 0 & k_n \\ -\frac{\epsilon}{2}k_g^2 & \epsilon\epsilon_1 & -\epsilon\epsilon_1\frac{k_g}{k_n} \end{bmatrix} \begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix}.$$

Since the Cartan frame and the Darboux frame along α are positively oriented, we have $[T, N, B] = [T, \zeta, \eta] = 1$. Hence we get

$$[T, N, B] = [T, \epsilon_1 k_g T + k_n\eta, -\frac{\epsilon}{2}k_g^2 T + \epsilon\epsilon_1\zeta - \epsilon\epsilon_1\frac{k_g}{k_n}\eta] = -[T, \epsilon\epsilon_1\zeta, k_n\eta] = 1.$$

The last relation gives $k_n(s) = -\epsilon\epsilon_1$. □

Corollary 3.1. *There are no null Cartan asymptotic curves lying on a timelike surface in $\mathbb{E}^3_1$.*

Throughout this section, let α denote a null Cartan curve parameterized by pseudo-arc s lying on a timelike surface in Minkowski space $\mathbb{E}^3_1$ with Darboux frame $\{T, \zeta, \eta\}$.

Definition 3.1. *Generalized Darboux frame of the first kind* of α is the frame $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ given by

$$\begin{aligned} \tilde{T}(s) &= \frac{1}{\mu(s)}T(s) - \epsilon_2\frac{\lambda^2(s)}{2\mu(s)}\zeta(s) - \epsilon_2\frac{\lambda(s)}{\mu(s)}\eta(s), \\ \tilde{\zeta}(s) &= \mu(s)\zeta(s), \\ \tilde{\eta}(s) &= \eta(s) + \lambda(s)\zeta(s), \end{aligned} \tag{3.1}$$

where $\lambda(s) \neq 0$ and $\mu(s) \neq 0$ for each s are some differentiable functions satisfying the Riccati differential equation

$$2\epsilon_1\lambda'(s) - 2\lambda(s)k_g(s) - \epsilon_1\lambda^2(s)\tau_g(s) = 0, \tag{3.2}$$

and the-first order linear differential equation

$$\mu'(s) - \mu(s)\lambda(s)\tau_g(s) = 0. \tag{3.3}$$

Definition 3.2. *Generalized Darboux frame of the second kind* of α is the frame $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ given by

$$\begin{aligned} \tilde{T}(s) &= \frac{1}{\mu_0}T(s), \\ \tilde{\zeta}(s) &= \mu_0\zeta(s), \\ \tilde{\eta}(s) &= \eta(s), \end{aligned} \tag{3.4}$$

where μ_0 is an arbitrary non-zero real number.

According to Definitions 3.1 and 3.2, the generalized Darboux frames (3.1) and (3.4) are not unique. In particular, if $\lambda(s) = 0$ and $\mu(s) = 1$ for each s, the vector fields of the generalized Darboux frame (3.1) coincide with the vector fields of the Darboux frame $\{T, \zeta, \eta\}$. Also, if $\lambda(s) = 0$ for each s, the generalized Darboux frame of the first kind reduces to the generalized Darboux frame of the second kind.

Remark 3.1. Note that a generalized Darboux frame of the second kind is obtained by rotating Darboux frame $\{T, \zeta, \eta\}$ of a null Cartan curve lying on a timelike surface about spacelike axis η for the constant hyperbolic angle $\theta = \ln \mu_0$, where $\mu_0 \in \mathbb{R}^+$. Up to isometries of $\mathbb{E}^3_1$, we may assume that at some point of the curve the vector fields T and ζ lie in the timelike plane with the equation $x_3 = 0$. Putting $\zeta(s) = (a_1(s), a_1(s), 0)$, $T(s) = (-\frac{1}{2}a_1(s), \frac{1}{2}a_1(s), 0)$ for some differentiable function $a_1(s)$ and by applying hyperbolic rotation R_θ for the hyperbolic angle θ about spacelike axis η spanned by $(0, 0, 1)$, we get

$$R_\theta(\zeta) = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1(s) \\ a_1(s) \\ 0 \end{bmatrix} = e^{\theta}\zeta(s) = \tilde{\zeta}(s) = \mu_0\zeta(s),$$

$$R_\theta(T) = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 \\ \sinh\theta & \cosh\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2}a_1(s) \\ \frac{1}{2}a_1(s) \\ 0 \end{bmatrix} = e^{-\theta}T(s) = \tilde{T}(s) = \frac{1}{\mu_0}T(s).$$

Hence $\mu_0 = e^{\theta}$. If $\mu_0 = 1$, the hyperbolic angle of rotation θ is equal to zero and in that case, the generalized Darboux frame of the second kind coincides with the Darboux frame of the null Cartan curve.

If $\mu_0 = -\omega_0 < 0$, $\omega_0 \in \mathbb{R}^+$, a generalized Darboux frame of the second kind is obtained by rotating the Darboux frame about spacelike axis η for the hyperbolic angle $\theta = \ln \omega_0$, and then by applying reflection in $\mathbb{E}^3_1$ with respect to a spacelike axis η.

It can be easily verified that generalized Darboux frames of the first and the second kind satisfy the conditions

$$\langle \tilde{T}, \tilde{T} \rangle = \langle \tilde{\zeta}, \tilde{\zeta} \rangle = \langle \tilde{T}, \tilde{\eta} \rangle = \langle \tilde{\zeta}, \tilde{\eta} \rangle = 0, \quad \langle \tilde{\eta}, \tilde{\eta} \rangle = 1, \quad \langle \tilde{T}, \tilde{\zeta} \rangle = \epsilon_1 = \pm 1, \tag{3.5}$$

$$\tilde{T} \times \tilde{\zeta} = \tilde{\eta}, \quad \tilde{\zeta} \times \tilde{\eta} = \epsilon_1 \tilde{\zeta}, \quad \tilde{\eta} \times \tilde{T} = \epsilon_1 \tilde{T}, \tag{3.6}$$

$$det(\tilde{T}, \tilde{\zeta}, \tilde{\eta}) = [\tilde{T}, \tilde{\zeta}, \tilde{\eta}] = 1, \tag{3.7}$$

We define the curvature functions with respect to the frames given by (3.1) and (3.4) as follows.

Definition 3.3. The curvature functions of α given by

$$\tilde{k}_n(s) = \langle \tilde{T}'(s), \tilde{\eta}(s) \rangle, \quad \tilde{k}_g(s) = \langle \tilde{T}'(s), \tilde{\zeta}(s) \rangle, \quad \tilde{\tau}_g(s) = \langle \tilde{\zeta}'(s), \tilde{\eta}(s) \rangle. \tag{3.8}$$

are respectively called *generalized normal curvature*, *generalized geodesic curvature* and *generalized geodesic torsion*.

By using (3.5) and (3.8), the next statement can be easily proved.

Theorem 3.2. *If α has a generalized Darboux frame given by (3.1) or (3.4), then the frame's equations read*

$$\begin{bmatrix} \tilde{T}' \\ \tilde{\zeta}' \\ \tilde{\eta}' \end{bmatrix} = \begin{bmatrix} \epsilon_1 \tilde{k}_g & 0 & \tilde{k}_n \\ 0 & -\epsilon_1 \tilde{k}_g & \tilde{\tau}_g \\ -\epsilon_1 \tilde{\tau}_g & -\epsilon_1 \tilde{k}_n & 0 \end{bmatrix} \begin{bmatrix} \tilde{T} \\ \tilde{\zeta} \\ \tilde{\eta} \end{bmatrix}. \tag{3.9}$$

In the next theorem, we obtain relations between the curvature functions k_g, k_n, τ_g and $\tilde{k}_g$, $\tilde{k}_n$ and $\tilde{\tau}_g$.

Theorem 3.3. *The curvature functions k_n, k_g, τ_g of α with respect to the Darboux frame and the curvature functions $\tilde{k}_n$, $\tilde{k}_g$, $\tilde{\tau}_g$ with respect to generalized Darboux frames of the first and the second kind, are related by*

$$\begin{aligned} \tilde{k}_n(s) &= \frac{1}{\mu(s)} k_n(s), \\ \tilde{k}_g(s) &= k_g(s), \\ \tilde{\tau}_g(s) &= \mu(s)\tau_g(s), \end{aligned} \tag{3.10}$$

where the function $\mu(s) \neq 0$ satisfies Riccati differential equation (3.3) if $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ is generalized Darboux frame of the first kind, or $\mu(s) \in \mathbb{R}_0$ if $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ is generalized Darboux frame of the second kind.

Proof. First assume that $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ is the generalized Darboux frame of the first kind given by (3.1). Differentiating the relation $\tilde{\zeta} = \mu\zeta$ with respect to s and using (2.6), we obtain

$$\tilde{\zeta}' = (-\epsilon_1 \mu k_g + \mu')\zeta + \mu\tau_g\eta. \tag{3.11}$$

According to relations (3.5) and (3.9), we have

$$\tilde{\zeta}' = -\epsilon_2\tilde{k}_g\tilde{\zeta} + \tilde{\tau}_g\tilde{\eta} = (-\epsilon_2\tilde{k}_g\mu + \tilde{\tau}_g\lambda)\zeta + \tilde{\tau}_g\eta. \tag{3.12}$$

Relations (3.11) and (3.12) give

$$\tilde{\tau}_g = \mu\tau_g, \quad \tilde{k}_g = \epsilon_2\lambda\tau_g - \epsilon_2\frac{\mu'}{\mu} + k_g. \tag{3.13}$$

In particular, differentiating the relation $\tilde{\eta} = \eta + \lambda\zeta$ with respect to s and using (2.6), we find

$$\tilde{\eta}' = -\epsilon_1\tau_g T + (\lambda' - \epsilon_1\lambda k_g - \epsilon_1 k_n)\zeta + \lambda\tau_g\eta.$$

Relations (3.1) and (3.9) imply

$$\tilde{\eta}' = -\epsilon_2\tau_g T + (\frac{\lambda^2}{2}\tau_g - \epsilon_2\mu\tilde{k}_n)\zeta + \lambda\tau_g\eta.$$

From the last two relations we get

$$\tilde{k}_n = -\epsilon_2\frac{\lambda'}{\mu} + \epsilon_2\frac{\lambda^2}{2\mu}\tau_g + \frac{1}{\mu}\lambda k_g + \frac{1}{\mu}k_n. \tag{3.14}$$

From (3.2) and (3.3) we have $\lambda' = \frac{\lambda^2}{2}\tau_g + \epsilon_1\lambda k_g$ and $\mu' = \mu\lambda\tau_g$. Substituting this in (3.13) and (3.14), we get (3.10).

Next, assume that $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ is the generalized Darboux frame of the second kind given by (3.4). Differentiating the relations $\tilde{\eta} = \eta$ and $\tilde{T} = \frac{1}{\mu_0}T$ with respect to s and using (2.6) and (3.4), we obtain

$$\eta' = -\epsilon_1\tau_g T - \epsilon_1 k_n\zeta = \tilde{\eta}' = -\epsilon_1\tilde{\tau}_g\frac{1}{\mu_0}T - \epsilon_1\tilde{k}_n\mu_0\zeta, \quad \tilde{T}' = \epsilon_1\tilde{k}_g\tilde{T} + \tilde{k}_n\tilde{\eta} = \frac{1}{\mu_0}(\epsilon_1 k_g T + k_n\eta). \tag{3.15}$$

Relation (3.15) yields

$$\tilde{k}_n = \mu_0 k_n, \quad \tilde{k}_g = k_g, \quad \tilde{\tau}_g = \frac{1}{\mu_0}\tau_g.$$

Therefore, relation (3.10) holds.

□

By using Theorem 3.3 we get a new characterization of null Cartan curves that are geodesic and principal curvature lines lying on a timelike surface.

Corollary 3.2. *A null Cartan curve α lying on a timelike surface in $\mathbb{E}^3_1$ is a geodesic line if and only if $\tilde{k}_g(s) = 0$ for each s, and a principal curvature line if and only if $\tilde{\tau}_g(s) = 0$ for each s.*

4. Null Cartan rectifying isophotic and rectifying silhouette curves lying on a timelike surface

Darboux frame $\{T, \zeta, \eta\}$ along a regular curve lying on a surface in Euclidean space $\mathbb{E}^3$ determines three mutually orthogonal planes $T^\perp$, $\zeta^\perp$, $\eta^\perp$ known as the *normal*, the *rectifying* and the *osculating plane*, respectively ([13]). In Minkowski space $\mathbb{E}^3_1$, Darboux frame $\{T, \zeta, \eta\}$ along a null Cartan curve α lying on a timelike surface determines lightlike *normal* plane $T^\perp = span\{T, \eta\}$, lightlike *rectifying* plane $\zeta^\perp = span\{\zeta, \eta\}$ and timelike *osculating* plane $\eta^\perp = span\{T, \zeta\}$ that are not mutually orthogonal as in the Euclidean case ([17]).

Let us consider the vector field $\tilde{\eta}$ along null Cartan curve α lying on a timelike surface, given by

$$\tilde{\eta}(s) = \eta(s) + \lambda(s)\zeta(s). \tag{4.1}$$

Such vector field lies in the rectifying plane $\zeta^\perp = span\{\zeta, \eta\}$ of α and belongs to its generalized Darboux frame of the first kind. This implies that $\lambda(s) \neq 0$ satisfies the Riccati differential equation (3.2). Note that $\tilde{\eta}$ satisfies Darboux equation of the form

$$\tilde{\eta}' = \tilde{D} \times \tilde{\eta},$$

where $\tilde{D}$ is the Darboux vector of the generalized Darboux frame of the first kind. Hence $\tilde{\eta}$ is a constant vector field with respect to such a frame. For constant vector fields with respect to the Frenet frame, see [18].

Definition 4.1. A null Cartan curve α lying on a timelike surface in $\mathbb{E}^3_1$ with generalized Darboux frame of the first kind $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ is called a *rectifying isophotic curve with an axis* W, if there exists a constant vector W in $\mathbb{E}^3_1$ such that holds

$$\langle \tilde{\eta}(s), W \rangle = c_0, \tag{4.2}$$

where $c_0 \in \mathbb{R}_0$. In particular, if $c_0 = 0$, α is called a *rectifying silhouette curve with an axis* W.

We first prove the next theorem.

Theorem 4.1. *A null Cartan curve lying on a timelike surface in* $\mathbb{E}^3_1$ *with constant curvatures* $k_g(s)$ *and* $\tau_g(s)$ *is a null Cartan helix, or a null Cartan cubic.*

Proof. Assume that the null Cartan curve lying on a timelike surface in $\mathbb{E}^3_1$ has constant curvatures $k_g(s)$ and $\tau_g(s)$. From the relations (2.3) and (2.6) we have $T'(s) = N(s) = \epsilon_1 k_g(s)T(s) + k_n(s)\eta(s)$. Differentiating the previous equation with respect to s, we get

$$N'(s) = (\epsilon_1 k_g'(s) + k_g^2(s) - \epsilon_1 k_n(s)\tau_g(s))T(s) - \epsilon_1 k_n^2(s)\zeta(s) + \epsilon_1 k_g(s)k_n(s)\eta(s).$$

Then we find $\langle N'(s), N'(s) \rangle = -2\epsilon\tau(s) = -2\epsilon_1 k_g'(s) - k_g^2(s) - 2\epsilon\tau_g(s)$. The last relation gives

$$\tau(s) = \tau_g(s) + \frac{\epsilon}{2}k_g^2(s) + \epsilon\epsilon_1 k_g'(s). \tag{4.3}$$

Hence α is a null Cartan helix if $\tau(s) = constant \neq 0$, or a null Cartan cubic if $\tau(s) = 0$. □

According to Theorem 4.1, null Cartan rectifying isophotic and rectifying silhouette curves lying on a timelike surface, with constant curvatures $k_g(s)$ and $\tau_g(s)$, are null Cartan helices and null Cartan cubics. We will not consider the general case when such curves have the curvature functions $k_g(s) \neq constant$ and/or $\tau_g(s) \neq constant$. In the sequel, we consider two cases (A) and (B).

(A) α is a rectifying isophotic curve with constant curvatures $k_g(s)$ and $\tau_g(s)$.

In this case, we distinguish four subcases.

(A.1) $k_g(s) = constant \neq 0$ and $\tau_g(s) = constant \neq 0$;

Theorem 4.2. *Let* α *be a null Cartan rectifying isophotic curve with a non-zero constant curvatures* $k_g(s)$ *and* $\tau_g(s)$ *lying on a timelike surface in* $\mathbb{E}^3_1$*. Then its axis has a parameter equation*

$$W(s) = -\frac{c_0\tau_g(s)}{k_g(s)}T(s) - \frac{\epsilon\epsilon_1 c_0}{k_g(s)}\zeta(s) - c_0\eta(s), \tag{4.4}$$

where $c_0 = \langle \tilde{\eta}(s), W \rangle \in \mathbb{R}_0$.

Proof. Assume that α is a null Cartan rectifying isophotic curve with an axis W and a non-zero constant curvatures $k_g(s)$ and $\tau_g(s)$. By solving the Riccati differential equation (3.2), we obtain

$$\lambda(s) = -\frac{2\epsilon_1 k_g(s)}{\tau_g(s)}.$$

Thus $\lambda(s) = \lambda_0 \in \mathbb{R}_0$ and

$$k_g(s) = -\frac{1}{2}\epsilon_1\lambda_0\tau_g(s). \tag{4.5}$$

Let us decompose an axis W of α by

$$W(s) = a(s)T(s) + b(s)\zeta(s) + c(s)\eta(s), \tag{4.6}$$

where $a(s)$, $b(s)$ and $c(s)$ are some differentiable functions in pseudo-arc s. Differentiating relation (4.6) with respect to s and using (2.6), we obtain the next system of the first-order differential equations

$$\begin{aligned} a' + \epsilon_1 a k_g - \epsilon_1 c\tau_g &= 0, \\ b' - \epsilon_1 b k_g - \epsilon_1 c k_n &= 0, \\ c' + a k_n + b\tau_g &= 0. \end{aligned} \tag{4.7}$$

Relations (2.4), (4.1), (4.2) and (4.6) give

$$c = c_0 - \epsilon_1 \lambda_0 a. \tag{4.8}$$

Substituting (4.8) in (4.7), we get

$$\begin{aligned} a' + \epsilon_1 a k_g - \epsilon_1 \tau_g (c_0 - \epsilon_1 \lambda_0 a) = 0, \\ b' - \epsilon_1 b k_g - \epsilon_1 k_n (c_0 - \epsilon_1 \lambda_0 a) = 0, \\ -\epsilon_1 \lambda_0 a' + a k_n + b \tau_g = 0. \end{aligned} \tag{4.9}$$

From the first and the third equation of (4.9) and using $\lambda_0 = -\frac{2\epsilon_1 k_g(s)}{\tau_g(s)}$, we find

$$a(s) = -\frac{c_0 \tau_g(s)}{k_g(s)}, \quad b(s) = \frac{c_0 k_n(s)}{k_g(s)}. \tag{4.10}$$

Substituting (4.10) in (4.8), we obtain

$$c = -c_0. \tag{4.11}$$

Finally, substituting (4.10) and (4.11) in (4.6), we get (4.4).

□

According to Theorem 4.2, we have

$$\langle T, W \rangle = \frac{\epsilon_1 c_0 k_n(s)}{k_g(s)} = constant \neq 0, \quad \langle \zeta, W \rangle = -\frac{\epsilon_1 c_0 \tau_g(s)}{k_g(s)} = constant \neq 0, \quad \langle \eta, W \rangle = -c_0 = constant \neq 0.$$

In this way, we obtain a remarkable property of a null Cartan rectifying isophotic curves.

Theorem 4.3. *Every null Cartan rectifying isophotic curve lying on a timelike surface in $\mathbb{E}^3_1$ with a non-zero constant curvatures $k_g(s)$ and $\tau_g(s)$ is a general helix, relatively normal-slant helix and isophotic curve with respect to the same axis.*

The following theorem states that an axis of null Cartan rectifying isophotic curve with a non-zero constant curvatures $k_g(s)$ and $\tau_g(s)$, has the same direction as the Darboux vector (2.9) of the Darboux frame.

Theorem 4.4. *An axis of a null Cartan rectifying isophotic curve α lying on a timelike surface in $\mathbb{E}^3_1$ with a non-zero constant curvatures $k_g(s)$ and $\tau_g(s)$ is collinear with the Darboux vector of the Darboux frame of α.*

Proof. By using (2.9) and (4.4), we easily find

$$W(s) = -\frac{c_0}{k_g(s)} D(s).$$

It can be verified that $W'(s) = 0$, so $W(s)$ is a constant vector. □

In particular, relations (4.1) and (4.2) imply the next property.

Corollary 4.1. *If a null Cartan curve lying on a timelike surface in $\mathbb{E}^3_1$ with a non-zero constant curvatures $k_g(s)$ and $\tau_g(s)$ is an isophotic curve and a relatively normal-slant helix with respect to the same axis, then it is a rectifying isophotic curve with respect to the same axis.*

(A.2) $k_g(s) = 0$ and $\tau_g(s) = constant \neq 0$;

In this subcase, substituting $k_g(s)$ and $\tau_g(s)$ in (4.7) and using (4.8), we obtain a contradiction.

Theorem 4.5. *There are no null Cartan rectifying isophotic curves lying on a timelike surface in $\mathbb{E}^3_1$ with the curvatures $k_g(s) = 0$ and $\tau_g(s) = constant \neq 0$.*

(A.3) $k_g(s) = constant \neq 0$ and $\tau_g(s) = 0$;

Substituting $k_g(s)$ and $\tau_g(s)$ in (4.7) and using (4.8), we also obtain a contradiction.

Theorem 4.6. *There are no null Cartan rectifying isophotic curves lying on a timelike surface in $\mathbb{E}^3_1$ with the curvatures $k_g(s) = constant \neq 0$ and $\tau_g(s) = 0$.*

(A.4) $k_g(s) = \tau_g(s) = 0$;

Theorem 4.7. *Let α be a null Cartan rectifying isophotic curve with the curvatures $k_g(s) = \tau_g(s) = 0$ lying on a timelike surface in $\mathbb{E}^3_1$. Then it has a spacelike axis with a parameter equation*

$$W(s) = c_0\epsilon_1 k_n(s)s\zeta(s) + c_0\eta(s), \tag{4.12}$$

where $c_0 = \langle \tilde{\eta}(s), W\rangle \in \mathbb{R}_0$.

Proof. Assume that α is null Cartan rectifying isophotic curve with an axis W and the curvatures $k_g(s) = \tau_g(s) = 0$. According to relation (4.3), α is a null Cartan cubic. Substituting $k_g(s) = \tau_g(s) = 0$ in (3.2), we get $\lambda(s) = \lambda_0 \in \mathbb{R}_0$. Next, substituting $k_g(s) = \tau_g(s) = 0$ and $\lambda(s) = \lambda_0 \in \mathbb{R}_0$ in (4.7) and (4.8), we obtain the system of three the first-order differential equations

$$\begin{aligned} a'(s) &= 0, \\ b'(s) - \epsilon_1 c(s)k_n(s) &= 0, \\ c'(s) + a(s)k_n(s) &= 0. \end{aligned} \tag{4.13}$$

The first equation of the previous system of equations implies $a(s) = a_0 = constant$. If $a_0 \neq 0$, we get a contradiction. If $a_0 = 0$, from (4.13) we find $b(s) = \epsilon_1 c_0 s k_n(s)$, $c(s) = c_0$. Substituting this in (4.6), we get (4.12). □

Corollary 4.2. *Every null Cartan cubic lying on a timelike surface in $\mathbb{E}^3_1$ that is rectifying isophotic curve, is an isophotic curve and relatively normal-slant helix with respect to the same axis.*

Example 4.1. Let us consider a timelike ruled surface in $\mathbb{E}^3_1$ with parametrization (see Figure 1)

$$x(s,t) = \alpha(s) + t(1, -\sin s, -\cos s),$$

where $\alpha(s) = (s, \sin s, \cos s)$ is a null Cartan helix with the curvature $\kappa(s) = 1$ and the torsion $\tau(s) = -\frac{1}{2}$. Since $T(s) = (1, \cos s, -\sin s)$ and $N(s) = (0, -\sin s, -\cos s)$, we obtain $T(s) \times N(s) = \epsilon T(s) = T(s)$. Thus $\epsilon = 1$.

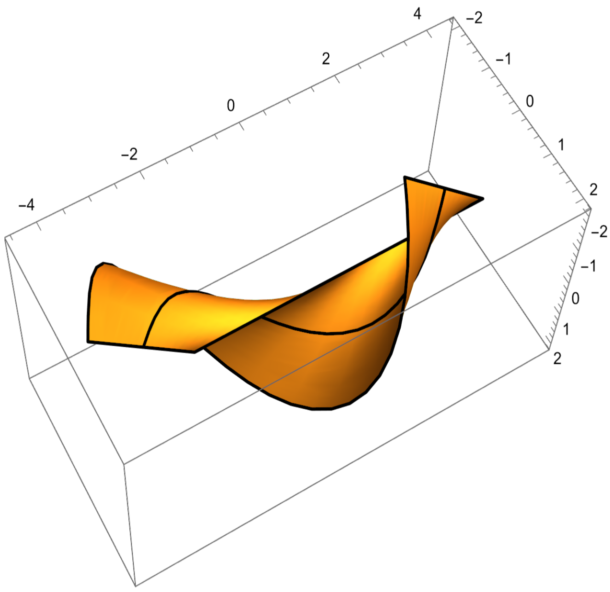

Figure 1. Null Cartan rectifying isophotic curve

Darboux frame along α reads

$$T(s) = (1, \cos s, -\sin s), \quad \zeta(s) = (1, -\sin s, -\cos s), \quad \eta(s) = (1, \cos s - \sin s, -\sin s - \cos s). \tag{4.14}$$

By using the last relation, we find $\eta \times T = \epsilon_1 T = -T$, so $\epsilon_1 = -1$. From relations (2.7) and (4.14), it follows that the curvature functions of α have the form

$$k_n(s) = 1, \quad k_g(s) = 1, \quad \tau_g(s) = -1. \tag{4.15}$$

Substituting $k_g(s) = 1$ and $\tau_g(s) = -1$ in (3.2), we obtain $\lambda(s) = -2$. Next, let us consider an axis $W(s)$ given by (4.4). By using (4.4), (4.14) and (4.15), we obtain $W(s) = c_0(T(s) + \zeta(s) - \eta(s)) = (c_0, 0, 0)$. Since

$$\tilde{\eta}(s) = \eta(s) + \lambda(s)\zeta(s) = (-1, \cos s + \sin s, \cos s - \sin s),$$

it follows $\langle \tilde{\eta}(s), W\rangle = c_0$. According to Definition (4.1), α is a rectifying isophotic curve. It can be easily verified that $\langle T(s), W\rangle = \langle \zeta(s), W\rangle = \langle \eta(s), W\rangle = -c_0$, so the statement of Theorem (4.3) also holds.

(B) α is a rectifying silhouette curve with constant curvatures $k_g(s)$ and $\tau_g(s)$.

Analogously as in case (A), we distinguish four subcases.

(B.1) $k_g(s) = constant \neq 0$ and $\tau_g(s) = constant \neq 0$;

Theorem 4.8. *There are no null Cartan rectifying silhouette curves lying on a timelike surface in $\mathbb{E}^3_1$ with non-zero constant curvatures $k_g(s)$ and $\tau_g(s)$.*

Proof. Assume that α is a null Cartan rectifying silhouette curve with an axis W given by (4.6) and a non-zero constant curvatures $k_g(s)$ and $\tau_g(s)$. Then (3.2) implies

$$\lambda(s) = -\frac{2\epsilon_1 k_g(s)}{\tau_g(s)} = \lambda_0 = constant \neq 0. \tag{4.16}$$

Substituting $c_0 = 0$ in (4.9), we obtain the system of differential equations

$$\begin{aligned} a' + \epsilon_1 a k_g + \lambda_0 \tau_g a = 0,\\ b' - \epsilon_1 b k_g + \lambda_0 k_n a = 0,\\ -\epsilon_1 \lambda_0 a' + a k_n + b\tau_g = 0. \end{aligned} \tag{4.17}$$

From the first equation of (4.17) and using (4.16), we find

$$a(s) = e^{\epsilon_1 s k_g(s)}. \tag{4.18}$$

From (4.18) and the third equation of (4.17) we obtain

$$b(s) = \frac{\epsilon_1}{\tau_g}(\epsilon - \frac{\lambda_0^2}{2}\tau_g)e^{\epsilon_1 s k_g(s)}. \tag{4.19}$$

On the other hand, by using the second equation of (4.17), (4.16) and (4.18), we get

$$b(s) = \epsilon\epsilon_1\lambda_0 s e^{\epsilon_1 s k_g(s)}. \tag{4.20}$$

Relations (4.19) and (4.20) give

$$\tau_g(s) = \frac{2\epsilon}{2\epsilon\lambda_0 s + \lambda_0^2}.$$

Thus $\tau_g(s)$ is a non-constant function, which is a contradiction. □

(B.2) $k_g(s) = 0$ and $\tau_g(s) = constant \neq 0$;

In this subcase, substituting $k_g(s)$ and $\tau_g(s)$ in (4.7) and using $c(s) = -\epsilon_1\lambda(s)a(s)$, we obtain a contradiction.

Theorem 4.9. *There are no null Cartan rectifying silhouette curves lying on a timelike surface in $\mathbb{E}^3_1$ with the curvatures $k_g(s) = 0$ and $\tau_g(s) = constant \neq 0$.*

(B.3) $k_g(s) = constant \neq 0$ and $\tau_g(s) = 0$;

In this subcase, substituting $k_g(s)$ and $\tau_g(s)$ in (4.7) and using $c(s) = -\epsilon_1\lambda(s)a(s)$, we also obtain a contradiction.

Theorem 4.10. *There are no null Cartan rectifying silhouette curves lying on a timelike surface in $\mathbb{E}^3_1$ with the curvatures $k_g(s) = constant \neq 0$ and $\tau_g(s) = 0$.*

(B.4) $k_g(s) = \tau_g(s) = 0$;

Theorem 4.11. *Let α be a null Cartan rectifying silhouette curve with the curvatures $k_g(s) = \tau_g(s) = 0$ lying on a timelike surface in $\mathbb{E}^3_1$. Then it has a lightlike axis with a parameter equation*

$$U(s) = b_0\zeta(s), \tag{4.21}$$

where $b_0 \in \mathbb{R}_0$.

Corollary 4.3. *Every null Cartan cubic that is rectifying silhouette curve lying on a timelike surface in $\mathbb{E}^3_1$, is a general helix, relatively normal-slant helix and isophotic curve with respect to the same axis.*

Example 4.2. Let us consider B-scroll in $\mathbb{E}^3_1$ with parametrization (see Figure 2)

$$x(s,t) = \alpha(s) + tB(s),$$

where

$$\alpha(s) = (\frac{s^3}{4} + \frac{s}{3}, \frac{s^3}{4} - \frac{s}{3}, \frac{s^2}{2})$$

is a null Cartan cubic.

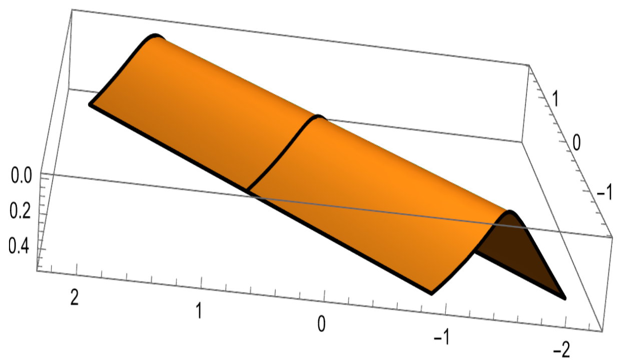

Figure 2. Null Cartan rectifying isophotic and rectifying silhouette curve

The Frenet frame along α has the form

$$T(s) = (\frac{3s^2}{4} + \frac{1}{3}, \frac{3s^2}{4} - \frac{1}{3}, s), \quad N(s) = (\frac{3s}{2}, \frac{3s}{2}, 1), \quad B(s) = (-\frac{3}{2}, -\frac{3}{2}, 0).$$

Then a straightforward calculation yields $T(s) \times N(s) = \epsilon T(s) = T(s)$, so $\epsilon = 1$.

Darboux frame of α reads

$$T(s) = (\frac{3s^2}{4} + \frac{1}{3}, \frac{3s^2}{4} - \frac{1}{3}, s), \quad \zeta(s) = -(\frac{3}{2}, \frac{3}{2}, 0), \quad \eta(s) = -(\frac{3s}{2}, \frac{3s}{2}, 1). \tag{4.22}$$

By using the previous relation, we get $\eta \times T = \epsilon_1 T = T$. Hence $\epsilon_1 = 1$. From relations (2.7) and (4.22), it follows that the curvature functions of α have the form

$$k_n(s) = -1, \quad k_g(s) = 0, \quad \tau_g(s) = 0.$$

Let us consider a spacelike axis W given by (4.12). By using (4.12) and (4.22), we obtain $W(s) = (0, 0, -c_0)$. Substituting $k_g(s) = \tau_g(s) = 0$ in (3.2), we get $\lambda(s) = \lambda_0 \in \mathbb{R}_0$. Thus

$$\tilde{\eta}(s) = \eta(s) + \lambda(s)\zeta(s) = -(\frac{3(s+\lambda_0)}{2}, \frac{3(s+\lambda_0)}{2}, 1).$$

Therefore, $\langle \tilde{\eta}, W \rangle = c_0$ which implies that α is a rectifying isophotic curve with an axis W. Next, let us consider a lightlike axis $U(s)$ given by (4.21). By using (4.21) and (4.22), we obtain $U(s) = -b_0(\frac{3}{2}, \frac{3}{2}, 0)$. Since $\langle \tilde{\eta}, U \rangle = 0$, α is rectifying silhouette curve with an axis U. Moreover, according to Corollary 4.3, α is a general helix, relatively normal-slant helix and isophotic curve with respect to the same axis.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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