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Modelling and Analysis of an Aquifer System to Enhance Teaching in Control Systems

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Abstract: *This paper aims to demonstrate how analysing a practical example, such as an aquifer system, can enhance understanding of the Control Systems course teaching materials. The aquifer system was chosen because it is a multivariable system with multiple inputs and outputs, and thus has numerous variables that can be monitored over time. This makes it suitable for both time and frequency domain analysis using various MATLAB tools. Since the system is nonlinear, it was necessary to linearize it. A Taylor series expansion approach was used for this purpose. The linearized model was represented in both state-space form and as a block diagram. The state-space model was suitable for testing the controllability and observability of the system, while the block diagram model was used to assess stability and analyse the system's behaviour during transient and steady states.*

Keywords: *aquifer; nonlinear system; multivariable system; state-space model; block diagram model;*

NOMENCLATURE

q_{ik}[m^3/s] The flow rate of liquid into the *k*-th tank;

q_{ok}[m^3/s] The flow rate of liquid out of the *k*-th tank;

q_k[m^3/s] The flow rate between tanks;

q_{21}[m^3/s] The flow rate between Tank 1 and Tank 2;

d_k[m] Diameter of flow area between tanks;

d_{21} [m] Diameter of flow area between Tank 1 and Tank 2;

a_k[m^2] Cross-section of flow area between tanks; $a_k = d_k^2\pi/4$;

a_{21}[m^2] Cross-section of flow area between Tank 1 and Tank 2; $a_{21} = d_{21}^2\pi/4$;

D_k [m] Diameter of the *k*-th rank;

S_k[m^2] The cross-sectional area of *k*-th tank; $S_k = D_k^2\pi/4$;

h_k[m] Height of liquid in the *k*-th tank;

H_1[m] The reference value of the liquid in the first tank;

h_{ks} [m] Stationary value of the height of liquid in the *k*-th tank;

g[m/s^2] Ground acceleration;

1. INTRODUCTION

In recent years, there has been a lack of motivation among students to listen to classes in certain subjects. In student surveys that are carried out during the year, and are carried out for the purpose of self-evaluation, poorer grades are observed on the question of whether the knowledge acquired in this subject will be useful for their future professional work. Even in subjects where practical application is obvious, students are disinterested and do not see the purpose of learning certain teaching contents.

When talking about control systems, the first thing that comes to mind is physical systems in industrial processes. However, physical systems are also encountered in everyday life. For example, controlling the temperature and speed of a vehicle, as well as regulating temperature in apartments, refrigerators, and stoves. There are also numerous real systems in nature that are not physical but still represent control systems. One example is the human body. The regulation of body temperature and maintaining blood pH are examples of such systems. Also, one of the earliest models of physiological control systems is the pupil light reflex. This reflex involves the iris responding to changes in light intensity on the retina. As ambient light levels increase or decrease, the iris muscles adjust the pupil size to maintain a consistent amount of light reaching the retina. Furthermore, the natural process of groundwater storage and utilization is an example of a control system. These systems are highly complex, but like physical systems, they can be modelled and simplified for analysis and synthesis purposes [1, 2].

This paper aims to showcase how to apply some of the teaching materials from the Control Systems subject using the aquifer system as an example. We chose this example because the aquifer system is a multivariable system with multiple inputs and outputs, along with many variables that can be monitored over time. This makes it suitable for analysis in the state-space and is also interesting for applying MATLAB tools.

An aquifer represents a geological environment completely or partially saturated with free underground water, capable of accumulating and releasing free underground water that feeds springs, freely flows into rivers, lakes, and seas, and is captured by various water-receiving facilities (wells, water-receiving ditches, etc.) [3]. Aquifer systems are used to supply drinking water, for industrial purposes, or in agriculture for irrigation [1, 4, 5, 6, 7]. Such a system can be modelled with three interconnected reservoirs with corresponding liquid levels, called heads [1]. Groundwater naturally flows through aquifer material, which can be permeable or fractured rocks, or unconsolidated materials such as gravel, sand, or silt, altering water levels in reservoirs as it makes its way to the sea or river. The aquifer is made up of three main layers from top to bottom: the groundwater, the saturated zone, and the impermeable layer.

The deepest layer, the impermeable layer, prevents further downward movement of water, causing it to gather and move horizontally. This process plays a crucial role in the water cycle and the geological cycle.

In reality, single aquifers are rare in hydraulic systems. An aquifer is usually part of a larger system comprising multiple aquifers. This system consists of a series of aquifers separated by less permeable confining layers. The flow dynamics in such a system can be quite complex, depending on how well the individual aquifers are connected hydraulically [5].

In the first part, a description of the system, its modelling, and linearization will be provided. Subsequently, the model will be presented in both state-space form and as a block diagram. The results will highlight significant features of the system achieved through these modeling approaches. The state-space model will be used to test controllability and observability, while the block diagram model will be applied to assess stability and analyse and to improve the system's behaviour during transient and steady state.

2. THE SYSTEM MODEL

In Fig.1 a variant of the model of aquifer system consisting of three interconnected horizontally placed natural storage tanks is presented [1]. Natural water flow is towards the sea or river with the flows q_1, q_2, and q_3. The water level of the k-th tank is h_k, k=1, 2, 3. Engineered flow is from the Tank 2 to the Tank 1. If the water level in Tank 1, denoted as h_1, drops below the reference value H_1, it will be refilled with water from Tank 2. Conversely, if h_1 is higher than H_1, water will be pumped back to Tank 2, reducing the leakage towards the sea. The water flow between Tanks 1 and 2 is indicated as q_{21} and is dependent on the difference in water levels between H_1 and h_1.

The water in an aquifer system can be naturally replenished through processes such as rain, snow, irrigation return flow, and seawater intrusion, or artificially replenished through methods like filling tanks from wells [6, 7]. The artificial replenishment is illustrated using flows q_{ik}, k=1, 2, 3 in Fig. 1. Underground water can be utilized for industrial, domestic, and irrigation purposes. These are represented by flows q_{ok}, where k= 2, 3.

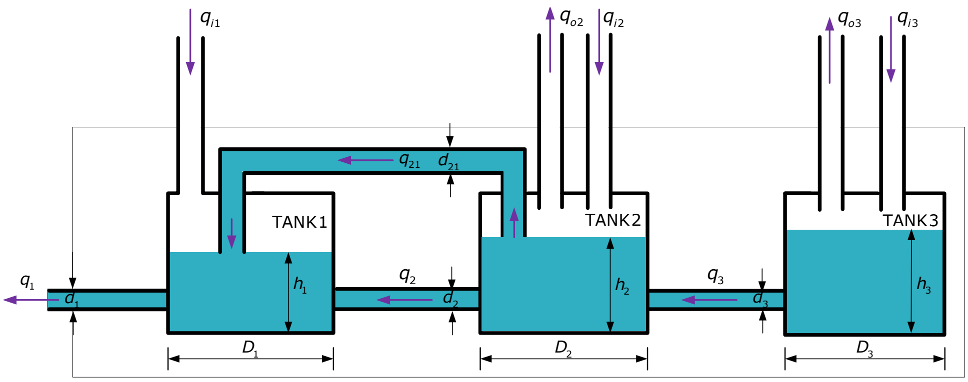

Figure 1. *Model of the aquifer system [1]*

2.1. Mathematical modelling of the system

The nonlinear mathematical model of the aquifer system can be derived using the mass balance equation. The equations that describe the dynamics of each aquifer are as follows:

$$S_1 \frac{dh_1}{dt} = q_{i1} + q_2 - q_1 + q_{21}, \tag{1}$$

$$S_2 \frac{dh_2}{dt} = q_{i2} - q_{o2} + q_3 - q_2 - q_{21}, \tag{2}$$

$$S_3 \frac{dh_3}{dt} = q_{i3} - q_{o3} - q_3. \tag{3}$$

Relations (1) -(3) can be written as:

$$S_1 \frac{dh_1}{dt} = q_{i1} + \alpha_2 \sqrt{h_2 - h_1} - \alpha_1 \sqrt{h_1} + \alpha_{21} \sqrt{H_1 - h_1}, \quad (4)$$

$$S_2 \frac{dh_2}{dt} = q_{i2} - q_{o2} + \alpha_3 \sqrt{h_3 - h_2} - \alpha_2 \sqrt{h_2 - h_1} - \alpha_{21} \sqrt{H_1 - h_1}, \quad (5)$$

$$S_3 \frac{dh_3}{dt} = q_{i3} - q_{o3} - \alpha_3 \sqrt{h_3 - h_2}, \quad (6)$$

where

$$\alpha_1 = a_1 \sqrt{2g},\ \alpha_2 = a_2 \sqrt{2g}, \quad \alpha_3 = a_3 \sqrt{2g},\ \alpha_{21} = a_{21} \sqrt{2g}. \quad (7)$$

2.2. The model linearization

After linearizing the model using a Taylor series expansion around the stationary state, the linearized model is obtained in the following form:

$$S_1 \frac{dh_1}{dt} = q_{i1} + B_2(h_2 - h_1) - B_1 h_1 + B_{21}(H_1 - h_1), \quad (8)$$

$$S_2 \frac{dh_2}{dt} = q_{i2} - q_{o2} + B_3(h_3 - h_2) - B_2(h_2 - h_1) - B_{21}(H_1 - h_1), \quad (9)$$

$$S_3 \frac{dh_3}{dt} = q_{i3} - q_{o3} - B_3(h_3 - h_2), \quad (10)$$

where

$$B_1 = \frac{a_1\sqrt{2g}}{2\sqrt{h_{1s}}},\ B_2 = \frac{a_2\sqrt{2g}}{2\sqrt{h_{2s} - h_{1s}}}, \quad B_3 = \frac{a_3\sqrt{2g}}{2\sqrt{h_{3s} - h_{2s}}},\ B_{21} = \frac{a_{21}\sqrt{2g}}{2\sqrt{H_1 - h_{1s}}}. \quad (11)$$

2.3. The state-space model of the system

According to (8)-(10), by selecting the liquid levels in the tanks as state variables

$$x_1 = h_1,\ x_2 = h_2,\ x_3 = h_3, \quad (12)$$

and by defining the input and output vectors as

$$\boldsymbol{u} = \begin{bmatrix} q_{i1} + B_{21}H_1 \\ q_{i2} - q_{o2} - B_{21}H_1 \\ q_{i3} - q_{o3} \end{bmatrix},\ \boldsymbol{y} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}, \quad (13)$$

the state-space model of the system (14) can be formed,

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}, \quad \boldsymbol{y} = \boldsymbol{Cx}, \quad (14)$$

where the state, input and output matrices respectively are:

$$\boldsymbol{A} = \begin{bmatrix} -\frac{B_1 + B_2 + B_{21}}{S_1} & \frac{B_2}{S_1} & 0 \\ \frac{B_2 + B_{21}}{S_2} & -\frac{B_3 + B_2}{S_2} & \frac{B_3}{S_3} \\ 0 & \frac{B_3}{S_3} & -\frac{B_3}{S_3} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \frac{1}{S_1} & 0 & 0 \\ 0 & \frac{1}{S_2} & 0 \\ 0 & 0 & \frac{1}{S_3} \end{bmatrix},\ \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (15)$$

Simulink state-space model of the aquifer system given with (13), (14), and (15), is shown in Fig.2.

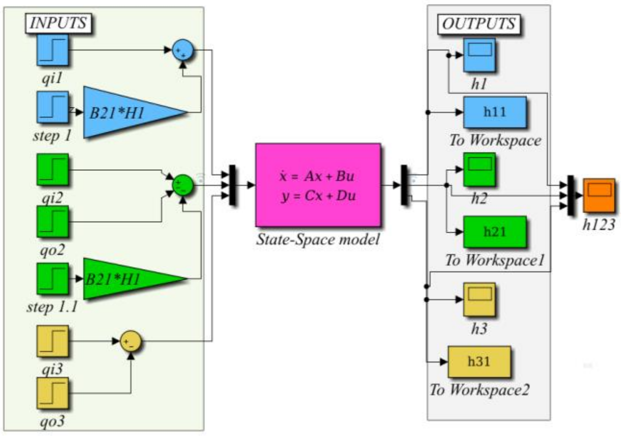

Figure 2. *The state-space model of the aquifer system.*

2.4. The transfer function model of the system

The transfer system model can be obtained by transforming equations (8), (9), and (10) into the Laplace domain:

$$h_1(T_1 s + 1) = (q_{i1} + B_{21}H_1)k_{11} + k_{12}h_2, \\ h_2(T_2 s + 1) = (q_{i2} - q_{o2} - B_{21}H_1)k_{21} + k_{22}h_1 + k_{23}h_3, \\ h_3(T_3 s + 1) = (q_{i3} - q_{o3})k_{31} + h_2, \quad (16)$$

where

$$T_1 = \frac{S_1}{B_1 + B_2 + B_{21}},\ T_2 = \frac{S_2}{B_2 + B_3},\ T_3 = \frac{S_3}{B_3}, \quad (17)$$

$$k_{11} = \frac{1}{B_1 + B_2 + B_{21}},\ k_{12} = \frac{B_2}{B_1 + B_2 + B_{21}}, \quad k_{21} = \frac{1}{B_2 + B_3},\ k_{22} = \frac{B_2 + B_{21}}{B_2 + B_3}, \quad k_{23} = \frac{B_3}{B_2 + B_3},\ k_{31} = \frac{1}{B_3}. \quad (18)$$

The block diagram, created using the transfer function model of the system (16), is illustrated in Fig.3.

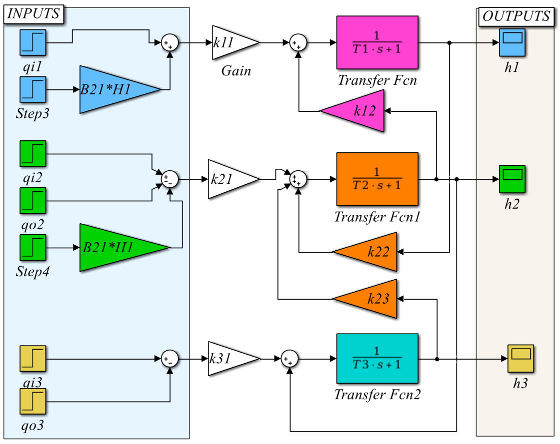

Figure 3. *The block diagram of the aquifer system in open loop.*

3. RESULTS

Table 1 shows the adopted parameters of the aquifer system. It was chosen $d_1=d_2=d_3=d_{21}=d$ and $D_1=D_2=D_3=D$ which implies $a_1=a_2=a_3=a_{21}=a$, and $S_1=S_2=S_3=S$.

Table 1. *Adopted parameters of the aquifer system*

Symbol	Value	Unit
h_{1s}	20	m
H_1	30	m
h_{2s}	30	m
h_{3s}	50	m
a	0.1963	m^2
S	314.1593	m^2

Table 2. *Adopted values of flows of the aquifer system*

Symbol	Value	Unit
q_{i1}	$15.768\cdot10^6$	m^3/year
q_{i2}	$189.216\cdot10^6$	m^3/year
q_{i3}	$94.608\cdot10^6$	m^3/year
q_{o2}	$47.304\cdot10^6$	m^3/year
q_{o3}	$69.379\cdot10^6$	m^3/year

Based on the state-space model (Fig. 2) and the corresponding block diagram (Fig. 3), the outputs of the aquifer system, which are the liquid heights in Tanks 1, 2, and 3, for the given inputs defined within Table 2, are shown in Fig. 4.

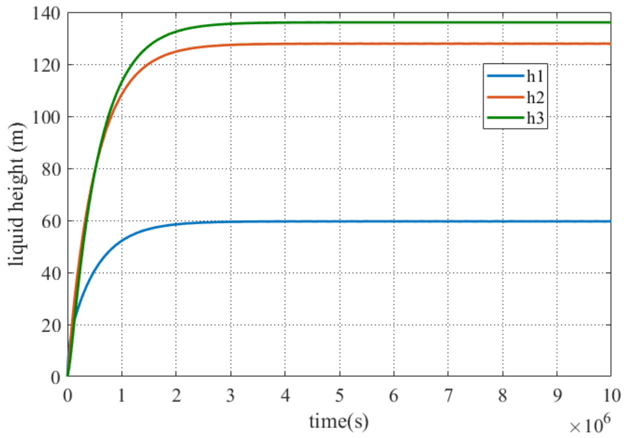

Figure 4. *The outputs of the aquifer system*.

It can be seen that the stationary state is reached after $5\cdot10^6$s which is equivalent to 57.87 days.

3.1. Results obtained on the basis of state-space models

The Kalman test is a method for determining the controllability of a system. According to this test, a linear multivariable time-invariant system is completely controllable if and only if the rank of the controllability matrix

$$\boldsymbol{Q}_c = \left[\boldsymbol{A} \quad \boldsymbol{AB} \quad \ldots \quad \boldsymbol{AB}^{n-1}\right] \tag{19}$$

is equal to n, where n is the number of state variables in the system.

$$\operatorname{rank} \boldsymbol{Q}_c = n \,. \tag{20}$$

After performing the Kalman test using the MATLAB package (n=3), it was found that the observed system is completely controllable. This means that it is possible to determine the values of the input vector $\boldsymbol{u}(t)$ that will transfer the system from an initial state $\boldsymbol{x}(0)$ to a desired state $\boldsymbol{x}(t)$, i.e. initial liquid levels to the desired liquid levels, within a finite time interval.

The Kalman test to determine observability is applied based on the following condition:

$$\boldsymbol{Q}_o = \left[\boldsymbol{C}^T \quad \boldsymbol{A}^T\boldsymbol{C}^T \quad \ldots \quad (\boldsymbol{A}^T)^{n-1}\boldsymbol{C}^T\right]^T, \tag{21}$$

$$\operatorname{rank} \boldsymbol{Q}_o = n \,. \tag{22}$$

Given that the matrix $\boldsymbol{C}$ in our case is $\boldsymbol{C}=\boldsymbol{I}$ (15), observability – which indicates that each state $\boldsymbol{x}(t)$ can be entirely determined by measuring the output vector $\boldsymbol{y}(t)$ within a finite time interval – does not need to be checked.

3.2. Results obtained based on the block diagram

Based on the block diagram in Fig. 3, transfer functions can be determined for the inputs and outputs of interest. The derived transfer functions are as follows:

From input 1 to output 1

$$G_{11}(s)=\frac{0.0001273\,s^2 + 5.382\cdot10^{-9}s + 2.76\cdot10^{-14}}{s^3 + 8.967\cdot10^{-5}\,s^2 + 1.607\cdot10^{-9}\,s + 2.684\cdot10^{-15}};$$

From input 1 to output 2

$$G_{12}(s)=\frac{4.459\cdot10^{-9}\,s + 5.52\cdot10^{-14}}{s^3 + 8.967\cdot10^{-5}\,s^2 + 1.607\cdot10^{-9}\,s + 2.684\cdot10^{-15}};$$

From input 1 to output 3

$$G_{13}(s)=\frac{5.52\cdot10^{-14}}{s^3 + 8.967\cdot10^{-5}\,s^2 + 1.607\cdot10^{-9}\,s + 2.684\cdot10^{-15}};$$

From input 2 to output 1

$$G_{21}(s)=\frac{2.229\cdot10^{-9}\,s + 2.76\cdot10^{-14}}{s^3 + 8.967\cdot10^{-5}\,s^2 + 1.607\cdot10^{-9}\,s + 2.684\cdot10^{-15}};$$

From input 2 to output 2

$$G_{22}(s)=\frac{0.0001273\,s^2 + 7.611\cdot10^{-9}\,s + 7.472\cdot10^{-14}}{s^3 + 8.967\cdot10^{-5}\,s^2 + 1.607\cdot10^{-9}\,s + 2.684\cdot10^{-15}};$$

From input 2 to output 3

$$G_{23}(s)=\frac{1.576\cdot10^{-9}\,s + 7.472\cdot10^{-14}}{s^3 + 8.967\cdot10^{-5}\,s^2 + 1.607\cdot10^{-9}\,s + 2.684\cdot10^{-15}};$$

From input 3 to output 1

$$G_{31}(s)=\frac{2.76\cdot10^{-14}}{s^3 + 8.967\cdot10^{-5}\,s^2 + 1.607\cdot10^{-9}\,s + 2.684\cdot10^{-15}};$$

From input 3 to output 2

$$G_{32}(s)=\frac{1.576\cdot10^{-9}\,s + 7.472\cdot10^{-14}}{s^3 + 8.967\cdot10^{-5}\,s^2 + 1.607\cdot10^{-9}\,s + 2.684\cdot10^{-15}};$$

From input 3 to output 3

$$G_{33}(s)=\frac{0.0001273\,s^2 + 9.841\cdot10^{-9}\,s + 1.023\cdot10^{-13}}{s^3 + 8.967\cdot10^{-5}\,s^2 + 1.607\cdot10^{-9}\,s + 2.684\cdot10^{-15}};$$

With MATLAB's sisotool, it is possible to obtain various diagrams to analyse the behaviour of the observed system in both transient and steady states. For example, Fig. 5 presents the Bode Plots, step response, root locus, and Nyquist Diagram for the transfer function $G_{13}(s)$ in closed loop system, shown in Fig.6.

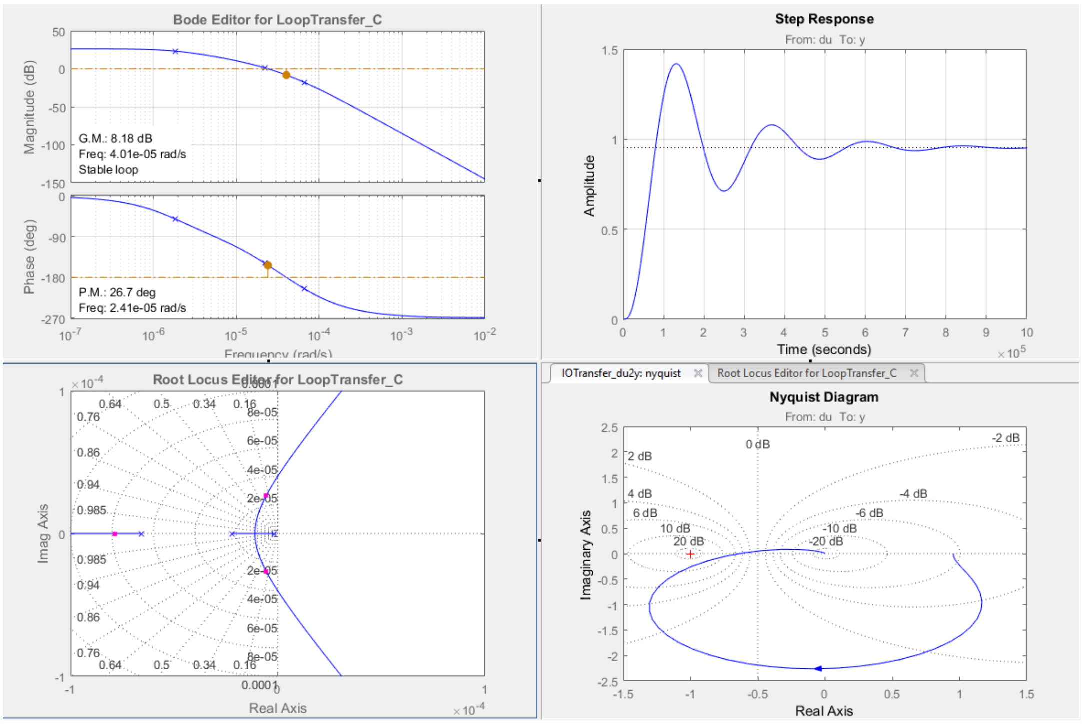

Figure 5. *Sisotool plots for the transfer function $G_{13}(s)$ in closed loop*

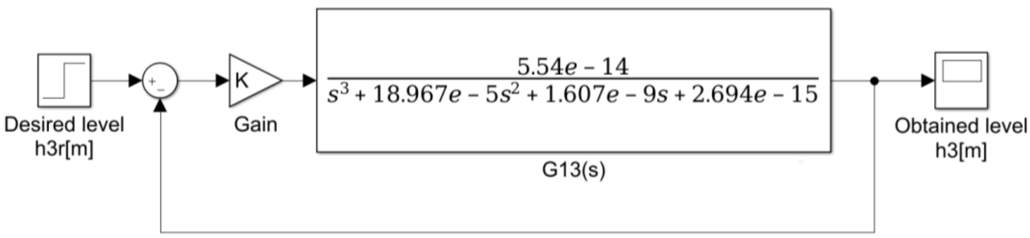

Figure 6. *Block diagram of the closed loop system $G_{13}(s)$*

Based on the obtained Bode Plots (for K=1), it can be concluded that the observed system with the transfer function $G_{13}(s)$ is stable in close loop and has a certain margin of stability. Using the Root Locus and Nyquist Diagram, the range of the gain K for which the closed-loop system remains stable can be determined. The analysis indicates that the system is stable for $0<K<2.7$, marginally stable at $K=2.7$ and unstable for $K>2.7$. Additionally, the step response diagrams (for $K=1$), reveals that the transient dynamics of the transfer function $G_{13}(s)$ are unfavourable, characterized by a steady state error of $e(\propto)=0.046$ (which is due to the lack of integrations in the transfer function $G_{13}(s)$), a rise time of $T_r=8\cdot10^4$s, a significant overshoot $P\%=48.8\%$, and large settling time $T_s=6.43\cdot10^5$s.

To enhance the quality of the transient response and steady-state behavior, a PID controller was designed. This was possible under the assumption that the flow q_{i1} (the output of the PID controller) could be controlled. By limiting it to a range of 0 to 2 m^3/s (Fig. 7), the response shown in Fig. 8 was obtained using an auto-tuning procedure. The auto-tuning procedure resulted in improved response dynamics and steady-state behavior, as depicted in Fig. 8. A zero steady-state error has been achieved $e(\propto)=0$, with a rise time of $T_r=1.377\cdot10^5$s, a overshoot of $P\%=23.6\%$, and a settling time of $T_s=5.297\cdot10^5$s.

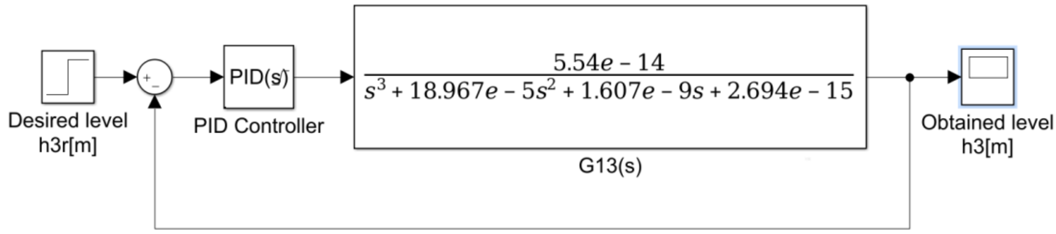

Figure 7. *Block diagram of the closed loop system $G_{13}(s)$ with PID controller*

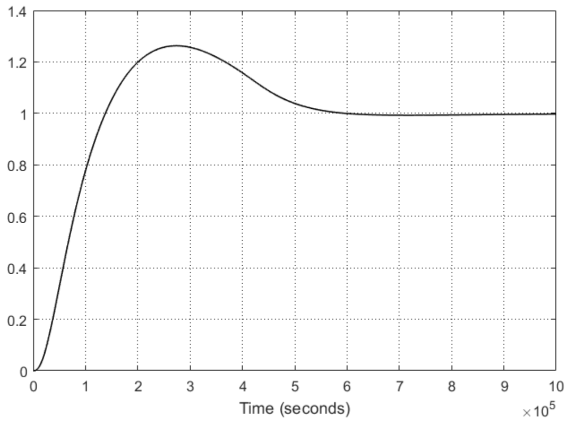

Figure 8. *Step function of the closed loop system $G_{13}(s)$ with PID controller*

4. CONCLUSION

This paper presents the application of teaching content from the control systems course using an example of an aquifer system. The system is multivariable, making it suitable for analysing the relationships between different variables within it. Both models presented, the state-space model and the block diagram model, are essential. The state-space model was used to analyse the controllability and observability of the system, while the block diagram model was utilized for analysing time and frequency responses. Based on the results shown, students can connect much of the teaching content covered in the control systems course with a real-world system. Given that this study models the system with three tanks, the analyses provided in this paper can be extended to similar systems with more tanks for various applications.

ACKNOWLEDGEMENTS

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