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Energy of a fuzzy soft set and its application in decision-making

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Abstract

In our paper, we continue the study of the theory of fuzzy soft sets and their properties. Based on the singular values of the corresponding matrix, we define the energy of a fuzzy soft set, as well as the λ-energy of a fuzzy soft set, allowing us to introduce an effective method for decision-making. Then, we consider the limits of the defined energies, which are essentially non-negative numerical values. The paper demonstrates through examples how the introduced method can be successfully applied to many problems containing uncertainties. Additionally, the paper includes comparisons of the introduced method with other methods addressing similar problems.

Keywords: Soft set, energy, fuzzy set, singular values, fuzzy soft set.

1 Introduction

The information we receive daily is full of uncertainty, imprecision, and ambiguity. In fact, almost all the information we receive on a daily basis is more uncertain than precise. Every scientific discipline requires that concepts be precisely defined; otherwise, precise reasoning is not possible. For this reason, researchers are increasingly interested in modeling uncertainty because many problems in areas such as biology, economics, medical sciences, and others contain data that involve various types of uncertainty [33].

The usual mathematical tools are often not successful in solving these problems. In 1999, Molodtsov ([26] and [27]) introduced the theory of soft sets as a new mathematical tool for dealing with uncertainties and imprecisions. According to Molodtsov, the theory of soft sets has been very successful in various mathematical areas, including operations research, Riemann integration, game theory, measure theory, and, most importantly, machine learning. Since 1999, research in the theory of soft sets has become very common and has progressed rapidly. Shortly after the introduction of the concept of a soft set, many operations on soft sets were defined (see [6], [9], [23], [38], [39], [41], [44]), all to improve decision-making methods. With the definition of new operations, the need to consider soft structures, such as soft groups [2], arises, and then soft topological spaces where we could highlight works like [5], [12], [25], and [40], as the foundations of many studies involving soft topology and the theory of soft sets in general. Many generalizations and restrictions of soft topologies are essential concepts for various applications, for example, soft multi-set and soft multi-set topology [34], fundamental tools in artificial intelligence with numerous applications in decision-making under uncertainty. Soft sets can also be represented as intuitionistic fuzzy sets, and decision-making algorithms based on such sets are the subject of many research studies ([42] and [43]). Measurable soft sets ([28] and [35]), as well as measurable soft mappings [36], are basic concepts of the theory of soft measure, which has a wide application in the mentioned areas. It is noteworthy to mention the work [31], which considers the relationship between soft sets and information systems, showing that soft sets are a class of special information systems. However, despite the widespread application of the theory of soft sets, many researchers have advanced the theory by introducing the concept of fuzzy soft sets.

Justifiably, there is a large body of literature on fuzzy soft sets and their applications, including many generalizations. The comparison rating and value of choice are two different approaches applied in the theory of soft sets and fuzzy soft

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sets to decision-making problems. Maji et al. [22] introduced an approach based on the value of choice for decision-making problems based on soft sets, establishing the criterion that an object can be selected if it has the maximum value of choice. A different approach, based on the results of comparisons, is proposed by Roi et al. [37] for studying decision-making problems based on fuzzy soft sets. Feng et al. [14] presented a new approach to the problem of making correct decisions based on fuzzy soft sets by using level soft sets, so their new method can be successfully applied to some decision-making problems. Liu et al. [20] proposed a decision-making model based on fuzzy soft sets and an ideal solution approach. This new decision-making method uses the divide and conquer algorithm. The ideal solution is generated according to each attribute (for or against the attribute, with or without constraints) of fuzzy soft sets. Alcantud introduced two innovations [4] that produce a new approach to decision-making based on fuzzy soft sets. In the work [21], the concept of a soft expert set is generalized to a fuzzy soft expert set, which would be more efficient and useful to some extent. In the work [8], the concept of fuzzy parameterized fuzzy soft (fpfs) sets and their operations is defined, and then the fpfs-aggregation operator for forming the fpfs-decision-making method is defined, allowing the construction of more efficient decision-making processes. Furthermore, in the work [11], a fuzzy soft aggregation operator is introduced that allows the construction of a more efficient decision-making method. An example is given in that paper that shows that the method can be successfully applied to many problems containing uncertainties.

On the other hand, graph theory is one of the fundamental theories and mathematical tools in various disciplines, including various areas of computer science and chemistry. As one of its parameters, graph energy is an extremely important concept currently being studied by many researchers. The concept of graph energy was introduced by Ivan Gutman in 1978 in his work [17] as the sum of the absolute values of the eigenvalues of a graph. Since then, graph energy has been intensively studied (see [13], [18], [19], [29], [30], [46]). The fundamental concept in studying graph energy is the investigation of matrices and their properties, especially eigenvalues, singular values, and the trace of a square matrix. Exploring the extreme properties of these energies leads to solutions for numerous analytical problems deeply rooted in combinatorics.

Interpreting and analyzing the application of the trace of a square matrix (as a specific type of energy) is not straightforward, especially when the base matrix is not symmetric. One such interpretation has found applications in machine learning (see [1]), in addition to the already mentioned graphs. Fortunately, many square matrices encountered in machine learning appear in the form of $A \cdot A^T$, where A is the data matrix at our disposal. One of the motivations for this work is an interesting observation mentioned in [1], which is that the trace of the matrix $A \cdot A^T$ is equal to the energy of the matrix A. These energy-related concepts served as a motivation for defining the concept of the energy of fuzzy soft sets, which we will discuss later.

In Section 2, the fundamental concepts of soft set theory and fuzzy soft set theory are outlined. Section 3 introduces new concepts necessary for the new decision-making methods, specifically defining the concepts of the energy of a fuzzy soft set and the λ-energy of a fuzzy soft set. The possibility of applying the energy of a fuzzy soft set in decision-making, as well as in drawing correct conclusions, is demonstrated through concrete examples. The idea for this concept originated from observing the energy of graphs since a fuzzy soft set can also be represented in the matrix form, similar to an arbitrary graph. Section 4 provides proofs for certain statements that establish connections between defined energies and provides some of their limits. In Section 5, two examples illustrate the applicability of defined energies. The results obtained are analyzed, and a comparison is made with the results obtained using methods introduced in the papers [11], [14] and [32].

2 Preliminaries

In this section, we introduce the basic concepts of fuzzy set theory and fuzzy soft set theory. More detailed properties and characteristics can be found in the works [6], [10], [11], [23] and [24].

To begin, as done in [10], defining a soft set requires us to consider a universe, denoted as U, which is the set upon which we build the soft set. Next, we consider a set E, which represents the set of parameters. As is well-known, we denote the power set of U as $P(U)$. Let $A \subset E$.

Definition 2.1. [10] *A soft set, denoted as F_A, over the universe U, is a set defined by the mapping f_A such that $f_A : E \to P(U)$, where $f_A(x) = \emptyset$ if $x \notin A$.*

As is customary, the mapping f_A is called the approximate function of the soft set for each $x \in E$. We can also represent the soft set F_A over the universe U using ordered pairs. Such a representation is more intuitive, as the soft set F_A can be written as

$$F_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\}.$$

The set of all soft sets over the universe U is commonly denoted as $S(U)$.

Definition 2.2. [45] *A fuzzy set X over the universe U is defined by a function μ_X that represents the mapping $\mu_X : U \to [0,1]$, where μ_X is called the membership function of X, and the value $\mu_X(u)$ is referred to as the degree of membership of an element $u \in U$ in the fuzzy set X.*

Hence, a fuzzy set X over U can be represented as follows: $X = \{(\mu_X(u)/u) \mid u \in U, \mu_X(u) \in [0,1]\}$. The set of all fuzzy sets over the universe U is commonly denoted as $F(U)$.

Next, we define fuzzy soft sets, similar to how it was done in [11].

Definition 2.3. [11] *A fuzzy soft set, denoted as Γ_A, over the universe U is a set defined by the function γ_A, representing a mapping $\gamma_A : E \to F(U)$, such that $\gamma_A(x) = \emptyset$ if $x \notin A$.*

Similar to fuzzy sets, γ_A is called the fuzzy approximate function of the fuzzy soft set Γ_A, and the value $\gamma_A(x)$ is a set called the x-element of the fuzzy soft set for all $x \in E$. Therefore, a fuzzy soft set Γ_A over the universe U can be represented as a set of ordered pairs in the following way

$$\Gamma_A = \{(x, \gamma_A(x)) \mid x \in E, \gamma_A(x) \in F(U)\}.$$

The set of all fuzzy soft sets over the universe U is commonly denoted as $FS(U)$.

In the following, we will use notations: $\Gamma_A, \Gamma_B, \Gamma_C, \ldots$ for fuzzy soft sets, and $\gamma_A, \gamma_B, \gamma_C, \ldots$ for their fuzzy approximate functions, respectively.

There are also fuzzy soft sets with special characteristics, and for that reason, they have specific, distinct names.

Definition 2.4. [11] *Let Γ_A be a fuzzy soft set. If $\gamma_A(x) = \emptyset$ for all $x \in E$, then the fuzzy soft set Γ_A is called an empty fuzzy soft set, denoted as Γ_Φ.*

Definition 2.5. [11] *Let Γ_A be a fuzzy soft set. If $\gamma_A(x) = U$ for all $x \in E$, then the fuzzy soft set Γ_A is called an A-universal fuzzy soft set, denoted as $\Gamma_{\widetilde{A}}$.*

We can compare two fuzzy soft sets as follows.

Definition 2.6. [11] *Let Γ_A and Γ_B be fuzzy soft sets. The fuzzy soft set Γ_A is a fuzzy soft subset of Γ_B, denoted as $\Gamma_A \widetilde{\subseteq} \Gamma_B$, if $\gamma_A(x) \subseteq \gamma_B(x)$ for all $x \in E$.*

Definition 2.7. [11] *Let Γ_A and Γ_B be fuzzy soft sets. The fuzzy soft sets Γ_A and Γ_B are fuzzy soft equivalent, denoted as $\Gamma_A = \Gamma_B$, if and only if $\gamma_A(x) = \gamma_B(x)$ for all $x \in E$.*

Similarly to operations with classic sets, we can consider operations with fuzzy soft sets.

Definition 2.8. [11] *Fuzzy soft set $\Gamma_A^{\tilde{c}}$ is the complement of fuzzy soft set Γ_A, such that $\gamma_{A^{\tilde{c}}}(x) = \gamma_A^c(x)$ for all $x \in E$, where $\gamma_A^c(x)$ is the complement of the set $\gamma_A(x)$.*

Definition 2.9. [11] *The union of fuzzy soft sets Γ_A and Γ_B, denoted as $\Gamma_A \widetilde{\cup} \Gamma_B$, is defined by the fuzzy approximative function $\gamma_{A \widetilde{\cup} B}(x) = \gamma_A(x) \cup \gamma_B(x)$ for all $x \in E$.*

Definition 2.10. [11] *The intersection of fuzzy soft sets Γ_A and Γ_B, denoted as $\Gamma_A \widetilde{\cap} \Gamma_B$, is defined by the fuzzy approximative function $\gamma_{A \widetilde{\cap} B}(x) = \gamma_A(x) \cap \gamma_B(x)$ for all $x \in E$.*

The properties of the mentioned operations and relations defined with fuzzy soft sets can be found in [11] and you can learn more about soft sets and fuzzy sets, as well as their operations, in [6], [10], [23], [24] and [45].

3 Energy of a fuzzy soft set

In this section, we introduce the concept of the energy of a fuzzy soft set, which is a central idea in our work. Through an example, we will illustrate the potential application of the energy of a fuzzy soft set in decision-making and concluding. Let's consider the following example.

Example 3.1. *Let's assume that a company wants to fill a specific position. To fill this position, the company's management has four candidates, namely candidates from the set $U = \{u_1, u_2, u_3, u_4\}$. The hiring committee needs to decide whom to hire for the required position, and the committee will make a decision based on observed parameters or criteria that the candidates satisfy. The set of parameters to be considered is $E = \{x_1, x_2, x_3, x_4, x_5\}$. For $i = 1, 2, 3, 4, 5$, the parameters x_i represent, in order, "work experience," "knowledge of foreign languages," "communicativeness," "sociability," and "knowledge of specific computer software." After conducting interviews with each candidate and observing* *the parameters from the set $A = \{x_2, x_3, x_4, x_5\}$, the hiring committee has collected information, which is represented in the form of a fuzzy soft set*

$$\Gamma_A = \{(x_2, \{0.2/u_1, 0.5/u_2, 0.9/u_3\}), (x_3, \{0.7/u_1, 0.2/u_2, 0.4/u_3, 0.3/u_4\}),$$
$$(x_4, \{0.1/u_1, 0.6/u_2, 0.5/u_4\}), (x_5, \{0.2/u_1, 0.4/u_3, 0.8/u_4\})\}.$$

Based on the gathered information, the committee needs to decide which of the observed candidates would contribute the most to their company.

To make an informed decision, let's first recall the matrix representation of the soft set, which has been introduced in the paper [11].

Definition 3.2. [11] *Let Γ_A be a fuzzy soft set. Suppose $U = \{u_1, u_2, ..., u_m\}$, $E = \{x_1, x_2, ..., x_n\}$ and $A \subseteq E$. Then the fuzzy soft set Γ_A can be represented by the following table*

Γ_A	x_1	x_2	$\cdots$	x_n
u_1	$\mu_{\gamma_{A(x_1)}}(u_1)$	$\mu_{\gamma_{A(x_2)}}(u_1)$	$\cdots$	$\mu_{\gamma_{A(x_n)}}(u_1)$
u_2	$\mu_{\gamma_{A(x_1)}}(u_2)$	$\mu_{\gamma_{A(x_2)}}(u_2)$	$\cdots$	$\mu_{\gamma_{A(x_n)}}(u_2)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$\mu_{\gamma_{A(x_1)}}(u_m)$	$\mu_{\gamma_{A(x_2)}}(u_m)$	$\cdots$	$\mu_{\gamma_{A(x_n)}}(u_m)$

where $\mu_{\gamma_{A(x)}}$ is the membership function of γ_A.

If $a_{ij} = \mu_{\gamma_{A(x_j)}}(u_i)$ for every $i = 1, 2, ..., m$ and every $j = 1, 2..., n$, then the fuzzy soft set Γ_A is uniquely characterized by the matrix

$$[a_{ij}]_{mxn} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and is referred to as the fuzzy soft matrix of the fuzzy soft set Γ_A over the universe U, of size $m \times n$.

Now that we have established that each fuzzy soft set can be represented by its corresponding rectangular matrix A, we can determine the eigenvalues of the square matrix $A \cdot A^T$ and thus the singular values in a well-known manner (see [1]).

Definition 3.3. *An eigenvector of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution x of $Ax = \lambda x$, such an x is called an eigenvector corresponding to λ.*

Definition 3.4. *Let A be an $m \times n$ matrix. The singular values of A are the square roots of the nonzero eigenvalues of $A \cdot A^T$.*

Eigenvalues, as well as singular values of the matrix, can be used to characterize the representation matrix of the fuzzy soft set with numerical coefficients. These numerical coefficients will be referred to as the energies of the fuzzy soft set, analogous to graph theory.

Definition 3.5. *The energy of a fuzzy soft set Γ_A, denoted as $\mathbf{E}(\Gamma_A)$, is defined as $\mathbf{E}(\Gamma_A) = \sum_{i=1}^{m} \sigma_i$, where $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_m \geqslant 0$ are the singular values of the matrix A corresponding to the fuzzy soft set Γ_A.*

Let's go back to discussing the example from the beginning of this section.

Example 3.6. *The matrix representation of the fuzzy soft set from Example 3.1 is as follows*

$$A = \begin{bmatrix} 0 & 0.2 & 0.7 & 0.1 & 0.2 \\ 0 & 0.5 & 0.2 & 0.6 & 0 \\ 0 & 0.9 & 0.4 & 0 & 0.4 \\ 0 & 0 & 0.3 & 0.5 & 0.8 \end{bmatrix},$$

so

$$A \cdot A^T = \begin{bmatrix} 0.58 & 0.3 & 0.54 & 0.42 \\ 0.3 & 0.65 & 0.53 & 0.36 \\ 0.54 & 0.53 & 1.13 & 0.44 \\ 0.42 & 0.36 & 0.44 & 0.98 \end{bmatrix}.$$

Using the well-known methods of linear algebra, we can easily find that the singular values of the matrix are:

$$\sigma_1 = 1.4791991076, \sigma_2 = 0.7867909506, \sigma_3 = 0.5731297303, \sigma_4 = 0.4521603698,$$

so, the energy of the fuzzy soft set Γ_A *is:*

$$\mathbf{E}(\Gamma_A) = \sum_{i=1}^{4} \sigma_i = 1.4791991076 + 0.7867909506 + 0.5731297303 + 0.4521603698 = 3.2912795583.$$

Unlike the energy of a fuzzy soft set defined using the singular values of the matrix, another type of energy of a fuzzy soft set can be defined using the eigenvalues of a square matrix, similar to how graph energy was defined by Ivan Gutman in his work [17].

Definition 3.7. *The* λ*-energy of a fuzzy soft set* Γ_A*, denoted as* $\mathbf{LE}(\Gamma_A)$*, is defined as* $\mathbf{LE}(\Gamma_A) = \sum_{i=1}^{m} \sigma_i^2$*, where* $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_m \geqslant 0$ *are the singular values of the matrix* A *corresponding to the fuzzy soft set* Γ_A*.*

Since singular values that are equal to 0 do not affect the value of energy and λ-energy of the fuzzy soft set Γ_A, we will assume that singular values are positive.

Knowing the basic properties of matrices and eigenvalues, as well as singular values, the λ-energy of the fuzzy soft set can be determined in multiple ways, as follows:

$$\mathbf{LE}(\Gamma_A) = \sigma_1^2(A) + \sigma_2^2(A) + \cdots + \sigma_m^2(A) = \operatorname{tr}(A \cdot A^T) = \operatorname{tr}(A^T \cdot A) = \sum_{i,j} |a_{ij}|^2 = \lambda_1(A) + \lambda_2(A) + \cdots + \lambda_m(A),$$

where $\lambda_1(A), \lambda_2(A), \cdots, \lambda_m(A)$ are the eigenvalues of the square matrix $A \cdot A^T$.

Now, we can determine the λ-energy of the fuzzy soft set Γ_A from Example 3.1. Therefore, we have

$$\mathbf{LE}(\Gamma_A) = \sum_{i=1}^{4} \sigma_i^2 = \operatorname{tr}(A \cdot A^T) = 3.34.$$

The question arises about the relationship between the defined energies and what the upper bounds of these energies are. We will attempt to answer these questions in the next section. Additionally, by defining a fuzzy soft set with a reciprocal property, we can determine the lower bound of the energy of the fuzzy soft set.

Barik, Pati, and Sarma introduced the concept of a graph with a reciprocal property in their work [7]. Similarly, we will introduce a fuzzy soft set with a reciprocal property.

Definition 3.8. *The fuzzy soft set* Γ_A *has a reciprocal property if, for every singular value* σ *of the matrix* A *corresponding to the fuzzy soft set* Γ_A*, it holds that* $\frac{1}{\sigma}$ *is also a singular value of the matrix* A*.*

Then it's easy to see that the fuzzy soft set Γ_A with a reciprocal property holds

$$\mathbf{E}(\Gamma_A) = \sigma_1 + \sigma_2 + \cdots + \sigma_m = \frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \cdots + \frac{1}{\sigma_m}.$$

Questions related to the application of these energies and whether the same decisions are made using both energies will be discussed in Section 5, which deals with the applications of energies of fuzzy soft sets in decision-making. In Section 5, a comparison will be made with decision-making methods discussed in the works [11] and [32].

4 Properties of energy of a fuzzy soft set

The following theorem provides us with a relationship between the energy and the λ-energy of the fuzzy soft set Γ_A.

Theorem 4.1. *Let Γ_A be a fuzzy soft set, and let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be the singular values of the corresponding matrix of the fuzzy soft set Γ_A. Then the following inequality holds:*

$$\mathbf{E}(\Gamma_A) \leqslant \sqrt{m \cdot \mathbf{LE}(\Gamma_A)}.$$

Proof. Using the well-known inequality between the arithmetic mean and the quadratic mean applied to the singular values $\sigma_1, \sigma_2, \ldots, \sigma_m$ of the matrix A corresponding to the fuzzy soft set Γ_A, we obtain

$$\frac{\sigma_1 + \sigma_2 + \cdots + \sigma_m}{m} \leqslant \sqrt{\frac{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2}{m}}.$$

From this, we have

$$\sigma_1 + \sigma_2 + \cdots + \sigma_m \leqslant \sqrt{m \cdot (\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2)},$$

or equivalently

$$\mathbf{E}(\Gamma_A) \leqslant \sqrt{m \cdot \mathbf{LE}(\Gamma_A)}.$$

□

Using the membership function property $\mu_{\gamma_{A(x)}}$ of γ_A and the λ-energy of the fuzzy soft set Γ_A, along with Theorem 4.1, we can easily prove the following theorem, which provides upper bounds for the energy and λ-energy of the fuzzy soft set.

Theorem 4.2. *Let Γ_A be a fuzzy soft set where $U = \{u_1, u_2, ..., u_m\}$, $E = \{x_1, x_2, ..., x_n\}$, and $A \subseteq E$. Then the following inequalities hold:*

1. $\mathbf{LE}(\Gamma_A) \leqslant mn$;
2. $\mathbf{E}(\Gamma_A) \leqslant m\sqrt{n}$.

Proof. Since $a_{ij} = \mu_{\gamma_{A(x_j)}}(u_i) \leqslant 1$, for every $i = 1, 2, \ldots, m$ and every $j = 1, 2, \ldots, n$, we have:

$$\mathbf{LE}(\Gamma_A) = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2 \leqslant mn,$$

which proves the first part of the theorem.

Using Theorem 4.1 and the previously proven result, we obtain:

$$\mathbf{E}(\Gamma_A) \leqslant \sqrt{m \cdot \mathbf{LE}(\Gamma_A)} \leqslant \sqrt{m^2 n} = m\sqrt{n},$$

thus proving the second part of the theorem. □

The next theorem provides lower and upper bounds for the energy $\mathbf{E}(\Gamma_A)$.

Theorem 4.3. *Let Γ_A be a fuzzy soft set where $U = \{u_1, u_2, ..., u_m\}$, $E = \{x_1, x_2, ..., x_n\}$, and $A \subseteq E$. If $\sigma_1, \sigma_2, \ldots, \sigma_m$ are the singular values of the corresponding matrix of the fuzzy soft set Γ_A, such that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_m$, then the following holds*

$$\sigma_1 + \frac{\mathbf{LE}(\Gamma_A) - \sigma_1^2}{\sigma_2} \leqslant \mathbf{E}(\Gamma_A) \leqslant \sigma_1 + \sqrt{(m-1)\,(\mathbf{LE}(\Gamma_A) - \sigma_1^2)}.$$

Proof. Let's first prove the left inequality. Since $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_m > 0$, it follows that $0 < \frac{1}{\sigma_1} \leqslant \frac{1}{\sigma_2} \leqslant \ldots \leqslant \frac{1}{\sigma_m}$. Then,

$$\begin{aligned}\sigma_1 + \frac{\mathbf{LE}(\Gamma_A) - \sigma_1^2}{\sigma_2} &= \sigma_1 + \frac{\sigma_2^2 + \sigma_3^2 + \cdots + \sigma_m^2}{\sigma_2} = \sigma_1 + \sigma_2 + \frac{\sigma_3^2}{\sigma_2} + \cdots + \frac{\sigma_m^2}{\sigma_2} \\ &\leqslant \sigma_1 + \sigma_2 + \sigma_3^2 \cdot \frac{1}{\sigma_3} + \cdots + \sigma_m^2 \cdot \frac{1}{\sigma_m} = \mathbf{E}(\Gamma_A).\end{aligned}$$

Now let's prove the right inequality. Using the inequality between the arithmetic and quadratic means applied to the singular values $\sigma_2, \ldots, \sigma_m$, we get

$$\sigma_1 + \sqrt{(m-1)\left(\mathbf{LE}(\Gamma_A) - \sigma_1^2\right)} = \sigma_1 + \sqrt{m-1} \cdot \sqrt{\sigma_2^2 + \cdots + \sigma_m^2} \geqslant \sigma_1 + \sqrt{m-1} \cdot \frac{\sigma_2 + \cdots + \sigma_m}{\sqrt{m-1}} = \mathbf{E}(\Gamma_A).$$

□

If we assume that the singular values $\sigma_1, \sigma_2, \ldots, \sigma_m$ of the matrix A corresponding to the fuzzy soft set Γ_A are positive integers, we obtain the following theorem, which is equivalent to the theorem for graphs proven by Filipovski and Jajcay in the paper [15].

Theorem 4.4. *Assuming that the singular values $\sigma_1, \sigma_2, \ldots, \sigma_m$ of the matrix A corresponding to the fuzzy soft set Γ_A are positive integers such that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_m$, then*

$$\mathbf{E}(\Gamma_A) \leqslant \mathbf{LE}(\Gamma)^{\frac{\sigma_1}{\sigma_1+\sigma_m}} \cdot m^{\frac{\sigma_m}{\sigma_1+\sigma_m}}.$$

Proof. Based on Theorem 4.1, we have $\mathbf{LE}(\Gamma_A) \geqslant \dfrac{\mathbf{E}(\Gamma_A)^2}{m}$, so $\dfrac{\mathbf{LE}(\Gamma_A)}{\mathbf{E}(\Gamma_A)} \geqslant \dfrac{\mathbf{E}(\Gamma_A)}{m}$.

Since $\sigma_1, \sigma_2, \ldots, \sigma_m$ are positive integers, it follows that $\dfrac{\mathbf{E}(\Gamma_A)}{m} \geqslant 1$. Additionally, because $\sigma_1 \geqslant \sigma_m > 0$, we get $\dfrac{\sigma_1}{\sigma_m} \geqslant 1$. Now,

$$\left(\frac{\mathbf{LE}(\Gamma_A)}{\mathbf{E}(\Gamma_A)}\right)^{\frac{\sigma_1}{\sigma_m}} \geqslant \left(\frac{\mathbf{E}(\Gamma_A)}{m}\right)^{\frac{\sigma_1}{\sigma_m}} \geqslant \frac{\mathbf{E}(\Gamma_A)}{m}.$$

Hence, $\mathbf{LE}(\Gamma_A)^{\frac{\sigma_1}{\sigma_m}} \geqslant \dfrac{\mathbf{E}(\Gamma_A)^{\frac{\sigma_1+\sigma_m}{\sigma_m}}}{m}$, and this concludes the proof of the desired inequality. □

In the following theorem, we obtain a lower bound for a fuzzy soft set that has a reciprocal property, similar to what was done in the work by Filipovski and Jajcay for graphs in [16].

Theorem 4.5. *Let Γ_A be a fuzzy soft set with a reciprocal property, and let $\sigma_1, \sigma_2, \ldots, \sigma_m$ be the singular values of the corresponding matrix of the fuzzy soft set Γ_A such that $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_m > 0$. If $\sigma_1 \geqslant 4\sigma_m$, then*

$$\mathbf{E}(\Gamma_A) \geqslant m + \frac{1}{2}.$$

Proof. Using the reciprocal property of the fuzzy soft set Γ_A and the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}
\mathbf{E}(\Gamma_A)^2 &= (\sigma_1 + \sigma_2 + \cdots + \sigma_m)\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \cdots + \frac{1}{\sigma_m}\right) \\
&= (\sigma_m + \sigma_2 + \cdots + \sigma_1)\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \cdots + \frac{1}{\sigma_m}\right) \\
&\geqslant \left(\sqrt{\frac{\sigma_m}{\sigma_1}} + \sqrt{\frac{\sigma_2}{\sigma_2}} + \cdots + \sqrt{\frac{\sigma_{m-1}}{\sigma_{m-1}}} + \sqrt{\frac{\sigma_1}{\sigma_m}}\right)^2 \\
&= \left(\sqrt{\frac{\sigma_m}{\sigma_1}} + \sqrt{\frac{\sigma_1}{\sigma_m}} + m - 2\right)^2.
\end{aligned}$$

Since $\sigma_1 \geqslant 4\sigma_m$, it follows that $\sqrt{\frac{\sigma_m}{\sigma_1}} \geqslant \frac{1}{2}$ and $\sqrt{\frac{\sigma_1}{\sigma_m}} \geqslant 2$, so we have

$$\mathbf{E}(\Gamma_A)^2 \geqslant \left(m + \frac{1}{2}\right)^2,$$

which implies $\mathbf{E}(\Gamma_A) \geqslant m + \frac{1}{2}$. □

Remark 4.6. *The previous theorem can be generalized as follows. If* Γ_A *is a fuzzy soft set with a reciprocal property,* $\sigma_1, \sigma_2, \dots, \sigma_m$ *are the singular values of the corresponding matrix of the fuzzy soft set* Γ_A *such that* $\sigma_1 \geqslant \sigma_2 \geqslant \dots \geqslant \sigma_m > 0$ *and if* $\sigma_1 \geqslant p^2\sigma_m$*, for* $p^2 \geqslant 1$*, then*

$$\mathbf{E}(\Gamma_A) \geqslant p + \frac{1}{p} + m - 2.$$

Remark 4.7. *If we omit the assumption* $\sigma_1 \geqslant p^2\sigma_m$*, and simply use* $\frac{\sigma_1}{\sigma_m} \geqslant 1$ *(from* $\sigma_1 \geqslant \sigma_2 \geqslant \dots \geqslant \sigma_m > 0$*), we obtain* $\mathbf{E}(\Gamma_A) \geqslant \sqrt{\frac{\sigma_1}{\sigma_m}} + \sqrt{\frac{\sigma_m}{\sigma_1}} + m - 2$.

5 Application

In this section, we present the application of the energy and the λ-energy of a fuzzy soft set in decision-making. The interdependence of all factors in the system is an important aspect of decision-making because a system functions well when each factor contributes to the system in its own way. The individual qualifications of each factor in the system can sometimes oversaturate the system in certain segments while leaving other segments uncovered. This is because if no factor can cover a specific segment with its qualifications (even if a factor has only one qualification), then the system would face a problem. Specifically, this can be illustrated with examples. The examples we have chosen are specific cases presented in the papers [11], [32] and [14]. In our work, we make decisions in those examples based on the defined fuzzy soft set energy. Interestingly, using the methods that employ fuzzy soft set energies, we discovered a computational mistake that had slipped into the calculation conducted in the example in the paper [11]. Certainly, this mistake does not diminish the importance of the method described in the paper [11], nor does it question the correctness of the decision made.

A highly significant and useful method described in the paper [11] includes an algorithm for arriving at the most favorable decision. The algorithm is provided in 4 steps, which are as follows:

Step 1: Construct a fuzzy soft set Γ_A over the universe U;

Step 2: Determine the cardinal set $c\Gamma_A$ from the fuzzy soft set Γ_A;

Step 3: Determine Γ^* based on the formed fuzzy soft set Γ_A;

Step 4: Choose the most acceptable solution for us by determining the maximum value $\max \mu_{\Gamma_A^*}(u)$.

The algorithm mentioned above is illustrated by the following example.

Example 5.1. *Let's assume that a company wants to hire someone for a specific position. There are eight candidates who make up the set of possible solutions for the vacant position, denoted as* $U = \{u_1, ..., u_8\}$*. The hiring committee considers a set of parameters* $E = \{x_1, ..., x_5\}$*. For* $i = 1, ..., 5$ *parameter* x_i *represents "work experience," "computer skills," "age," "communication skills," and "sociability," respectively.*

After a thorough discussion, each candidate is evaluated based on their qualifications and limitations, all according to the selected subset of characteristics $A = \{x_2, x_3, x_4\}$, which are part of the set E. Finally, the committee applies the following algorithmic steps:

First, the fuzzy soft set Γ_A over U is constructed, and let's say this fuzzy soft set is of the form:

$$\Gamma_A = \{(x_2, \{0.3/u_2, 0.5/u_3, 0.1/u_4, 0.8/u_5, 0.7/u_7\}), (x_3, \{0.4/u_1, 0.4/u_2, 0.9/u_3, 0.3/u_4\}),$$

$$(x_4, \{0.2/u_1, 0.5/u_2, 0.1/u_5, 0.7/u_7, 0.1/u_8\})\}.$$

The committee analyzes the results obtained using the method presented in the paper [11] and obtains the following:

$$\Gamma_A^* = \{0.028/u_1, 0.058/u_2, 0.075/u_3, 0.021/u_4, 0.052/u_5, 0/u_6, 0.070/u_7, 0.004/u_8\}.$$

Finally, since

$$\max \mu_{\Gamma_A^*}(u) = 0.075,$$

the committee will choose candidate u_3.

Based on the results obtained using this method, we can linearly represent the order of candidates according to their assessed abilities as follows:

$$u_3 \succ u_7 \succ u_2 \succ u_5 \succ u_1 \succ u_4 \succ u_8 \succ u_6.$$

On the other hand, our goal is to present an algorithm for decision-making using the energy of fuzzy soft sets and the λ-energy of fuzzy soft sets. For this reason, let's introduce an algorithm for decision-making based on the defined concepts in our paper.

Step 1: Construct a fuzzy soft set Γ_A over U;

Step 2: Form fuzzy soft sets Γ_{A_i} over $U \setminus u_i$ for each $u_i \in U$;

Step 3: Determine the energies $\mathbf{E}(\Gamma_{A_i})$ (or λ-energies $\mathbf{LE}(\Gamma_{A_i})$) for each fuzzy soft set Γ_{A_i};

Step 4: Determine the minimum energy among all the energies of fuzzy soft sets obtained in step 3 and interpret the obtained result.

Let's return to the previous example, and using the method described in this paper, we can draw conclusions from the observed data. Namely, we'll form the corresponding fuzzy soft set from the data in the example (this step aligns with the step from the method in the paper [11]). Then, we'll form fuzzy soft sets Γ_{A_i} over $U \setminus u_i$ for each $u_i \in U$ and consider, respectively, their representation matrices $A_1, A_2, \ldots, A_8$, which are obtained by removing the first, second, and so on, row from the representation matrix A of the fuzzy soft set Γ_A, where the matrix A in this example is a matrix

$$A = \begin{bmatrix} 0 & 0 & 0.4 & 0.2 & 0 \\ 0 & 0.3 & 0.4 & 0.5 & 0 \\ 0 & 0.5 & 0.9 & 0 & 0 \\ 0 & 0.1 & 0.3 & 0 & 0 \\ 0 & 0.8 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \end{bmatrix}.$$

In the following lines, we describe how to calculate the energy of the fuzzy soft set Γ_{A_1}, and the other values can be determined analogously. So, the representation matrix of the fuzzy soft set Γ_{A_1} is the matrix

$$A_1 = \begin{bmatrix} 0 & 0.3 & 0.4 & 0.5 & 0 \\ 0 & 0.5 & 0.9 & 0 & 0 \\ 0 & 0.1 & 0.3 & 0 & 0 \\ 0 & 0.8 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \end{bmatrix},$$

so it is

$$A_1 \cdot A_1^T = \begin{bmatrix} 0.5 & 0.51 & 0.15 & 0.29 & 0 & 0.56 & 0.05 \\ 0.51 & 1.06 & 0.32 & 0.4 & 0 & 0.35 & 0 \\ 0.15 & 0.32 & 0.1 & 0.08 & 0 & 0.07 & 0 \\ 0.29 & 0.4 & 0.08 & 0.65 & 0 & 0.63 & 0.01 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.56 & 0.35 & 0.07 & 0.63 & 0 & 0.98 & 0.07 \\ 0.05 & 0 & 0 & 0.01 & 0 & 0.07 & 0.01 \end{bmatrix}.$$

The singular values of the matrix A_1 are $\sigma_1 = 1.4990363571, \sigma_2 = 0.8809273523, \sigma_3 = 0.5261672738$. Therefore, the energy of the fuzzy soft set Γ_{A_1} is equal to

$$\mathbf{E}(\Gamma_{A_1}) = \sum_{i=1}^{3} \sigma_i = 1.4990363571 + 0.8809273523 + 0.5261672738 = 2.9061309832.$$

Similarly, we can determine the λ-energy of the fuzzy set Γ_{A_1}. Therefore, we have

$$\mathbf{LE}(\Gamma_{A_1}) = \sum_{i=1}^{3} \sigma_i^2 = \operatorname{tr}(A_1 \cdot {A_1}^T) = 3.299995.$$

We can apply this procedure to the other fuzzy sets obtained from the base fuzzy set Γ_A. After a straightforward calculation, we obtain:

$\mathbf{E}(\Gamma_{A_2}) = 2.7858940753$, $\mathbf{LE}(\Gamma_{A_2}) = 2.999998$;
$\mathbf{E}(\Gamma_{A_3}) = 2.4292316804$, $\mathbf{LE}(\Gamma_{A_3}) = 2.44$;
$\mathbf{E}(\Gamma_{A_4}) = 2.9744732427$, $\mathbf{LE}(\Gamma_{A_4}) = 3.400005$;
$\mathbf{E}(\Gamma_{A_5}) = 2.6164029462$, $\mathbf{LE}(\Gamma_{A_5}) = 2.849999$;
$\mathbf{E}(\Gamma_{A_6}) = 3.0187533621$, $\mathbf{LE}(\Gamma_{A_6}) = 3.499998$;
$\mathbf{E}(\Gamma_{A_7}) = 2.517249968$, $\mathbf{LE}(\Gamma_{A_7}) = 2.520004$;
$\mathbf{E}(\Gamma_{A_8}) = 3.0115922883$, $\mathbf{LE}(\Gamma_{A_8}) = 3.489998$.

The obtained results can be linearly ranked, leading to the conclusion that:

$$\mathbf{E}(\Gamma_{A_6}) \geq \mathbf{E}(\Gamma_{A_8}) \geq \mathbf{E}(\Gamma_{A_4}) \geq \mathbf{E}(\Gamma_{A_1}) \geq \mathbf{E}(\Gamma_{A_2}) \geq \mathbf{E}(\Gamma_{A_5}) \geq \mathbf{E}(\Gamma_{A_7}) \geq \mathbf{E}(\Gamma_{A_3}),$$

or, in other words:

$$\mathbf{LE}(\Gamma_{A_6}) \geq \mathbf{LE}(\Gamma_{A_8}) \geq \mathbf{LE}(\Gamma_{A_4}) \geq \mathbf{LE}(\Gamma_{A_1}) \geq \mathbf{LE}(\Gamma_{A_2}) \geq \mathbf{LE}(\Gamma_{A_5}) \geq \mathbf{LE}(\Gamma_{A_7}) \geq \mathbf{LE}(\Gamma_{A_3}).$$

The conclusion we draw is that we have found that the energy of the fuzzy soft set $\mathbf{E}(\Gamma_{A_6})$ is the highest, which means that the characteristics of the candidate u_6 have the least impact on the system's value, or that the candidate u_6 contributes the least to the energy of the fuzzy soft set Γ_A, and therefore, such a candidate should be chosen. By analyzing the energies in this example, we can obtain a linear ranking of candidates based on their assessed abilities. Thus, by considering both the energy of the fuzzy soft set and the λ-energy of the fuzzy soft set, we get the same order of candidates:

$$u_3 \succ u_7 \succ u_5 \succ u_2 \succ u_1 \succ u_4 \succ u_8 \succ u_6.$$

If we compare the results obtained from all the mentioned methods, we can conclude that all methods provide the same or very similar results. We can verify this by comparing the defined methods with the methods introduced in the paper [32]. In the following example discussed in the paper [32], we will show that the method in which we use the energies of fuzzy soft sets and the λ-energies of fuzzy soft sets yields similar results. Based on the data on vaccine side effects that provide protection against COVID-19, a significant amount of data was collected after the treatment. The data is presented in Table 1 in the paper [32]. Of importance to illustrate the method are the steps 4 in the mentioned algorithms, as fuzzy soft sets are formed in them based on which we draw specific conclusions. Therefore, in step 4, a fuzzy soft set is formed,

$$\begin{aligned}\Gamma_A = \{&(x_1, \{0.21/u_1, 0.18/u_2, 0.07/u_3, 0.05/u_4, 0.05/u_5, 0.02/u_6, 0.04/u_7, 0.02/u_8, 0.32/u_{10}, 0.02/u_{11}, 0.02/u_{12}\}),\\ &(x_2, \{0.17/u_1, 0.16/u_2, 0.12/u_3, 0.04/u_4, 0.08/u_5, 0.03/u_6, 0.13/u_7, 0.05/u_9, 0.23/u_{14}\}),\\ &(x_3, \{0.05/u_1, 0.16/u_2, 0.08/u_3, 0.13/u_4, 0.02/u_5, 0.02/u_6, 0.21/u_7, 0.09/u_8, 0.07/u_9, 0.17/u_{14}\}),\\ &(x_4, \{0.11/u_1, 0.16/u_2, 0.14/u_3, 0.03/u_4, 0.02/u_5, 0.22/u_6, 0.06/u_7, 0.06/u_8, 0.04/u_9, 0.15/u_{14}\})\},\end{aligned}$$

and in the step 5, the matrix representation of that fuzzy soft set is given, i.e., the matrix A is obtained

$$A = \begin{bmatrix} 0.21 & 0.17 & 0.05 & 0.11 \\ 0.18 & 0.16 & 0.16 & 0.16 \\ 0.07 & 0.12 & 0.08 & 0 \\ 0.05 & 0.04 & 0.13 & 0.14 \\ 0.05 & 0.08 & 0.02 & 0.03 \\ 0.02 & 0.03 & 0.02 & 0.02 \\ 0.04 & 0.13 & 0.21 & 0 \\ 0.02 & 0 & 0.09 & 0 \\ 0 & 0.05 & 0.07 & 0 \\ 0.32 & 0 & 0 & 0.22 \\ 0.02 & 0 & 0 & 0.06 \\ 0.02 & 0 & 0 & 0.06 \\ 0 & 0 & 0 & 0.04 \\ 0 & 0.23 & 0.17 & 0.15 \end{bmatrix}.$$

We can apply the described decision-making algorithm using the energy of the fuzzy soft set and the λ-energy of the fuzzy soft set. As in the previous example, we calculate the singular values of the matrices and compute the individual energies of the fuzzy soft sets, resulting in the following results:

$\mathbf{E}(\Gamma_{A_1}) = 1.236146042$, $\mathbf{LE}(\Gamma_{A_1}) = 0.5309997$;
$\mathbf{E}(\Gamma_{A_2}) = 1.2440762548$, $\mathbf{LE}(\Gamma_{A_2}) = 0.5093999$;
$\mathbf{E}(\Gamma_{A_3}) = 1.2997654535$, $\mathbf{LE}(\Gamma_{A_3}) = 0.5928998$;
$\mathbf{E}(\Gamma_{A_4}) = 1.2739524942$, $\mathbf{LE}(\Gamma_{A_4}) = 0.577998$;
$\mathbf{E}(\Gamma_{A_5}) = 1.3206086462$, $\mathbf{LE}(\Gamma_{A_5}) = 0.6083998$;
$\mathbf{E}(\Gamma_{A_6}) = 1.3312753139$, $\mathbf{LE}(\Gamma_{A_6}) = 0.6164997$;
$\mathbf{E}(\Gamma_{A_7}) = 1.2410127198$, $\mathbf{LE}(\Gamma_{A_7}) = 0.5559998$;
$\mathbf{E}(\Gamma_{A_8}) = 1.3123925331$, $\mathbf{LE}(\Gamma_{A_8}) = 0.6101003$;
$\mathbf{E}(\Gamma_{A_9}) = 1.32398756$, $\mathbf{LE}(\Gamma_{A_9}) = 0.6111997$;
$\mathbf{E}(\Gamma_{A_{10}}) = 1.1274759697$, $\mathbf{LE}(\Gamma_{A_{10}}) = 0.4678$;
$\mathbf{E}(\Gamma_{A_{11}}) = \mathbf{E}(\Gamma_{A_{12}}) = 1.3259205994$, $\mathbf{LE}(\Gamma_{A_{11}}) = \mathbf{LE}(\Gamma_{A_{12}}) = 0.6146004$;
$\mathbf{E}(\Gamma_{A_{13}}) = 1.3291164305$, $\mathbf{LE}(\Gamma_{A_{13}}) = 0.617$;
$\mathbf{E}(\Gamma_{A_{14}}) = 1.1874119779$, $\mathbf{LE}(\Gamma_{A_{14}}) = 0.51430028$.

Comparing the obtained energy values and λ-energy values of fuzzy soft sets, we find that the following holds:

$$\begin{aligned}\mathbf{E}(\Gamma_{A_6}) &\geq \mathbf{E}(\Gamma_{A_{13}}) \geq \mathbf{E}(\Gamma_{A_{11}}) = \mathbf{E}(\Gamma_{A_{12}}) \geq \mathbf{E}(\Gamma_{A_9}) \geq \mathbf{E}(\Gamma_{A_5}) \geq \mathbf{E}(\Gamma_{A_8}) \\ &\geq \mathbf{E}(\Gamma_{A_3}) \geq \mathbf{E}(\Gamma_{A_4}) \geq \mathbf{E}(\Gamma_{A_2}) \geq \mathbf{E}(\Gamma_{A_7}) \geq \mathbf{E}(\Gamma_{A_1}) \geq \mathbf{E}(\Gamma_{A_{14}}) \geq \mathbf{E}(\Gamma_{A_{10}}),\end{aligned}$$

hence, we conclude that

$$\begin{aligned}\mathbf{LE}(\Gamma_{A_{13}}) &\geq \mathbf{LE}(\Gamma_{A_6}) \geq \mathbf{LE}(\Gamma_{A_{11}}) = \mathbf{LE}(\Gamma_{A_{12}}) \geq \mathbf{LE}(\Gamma_{A_9}) \geq \mathbf{LE}(\Gamma_{A_8}) \geq \mathbf{LE}(\Gamma_{A_5}) \\ &\geq \mathbf{LE}(\Gamma_{A_3}) \geq \mathbf{LE}(\Gamma_{A_4}) \geq \mathbf{LE}(\Gamma_{A_7}) \geq \mathbf{LE}(\Gamma_{A_1}) \geq \mathbf{LE}(\Gamma_{A_{14}}) \geq \mathbf{LE}(\Gamma_{A_2}) \geq \mathbf{LE}(\Gamma_{A_{10}}).\end{aligned}$$

Analyzing the obtained results, we can conclude that the fuzzy soft set Γ_{A_6} has the highest energy, meaning that side effect u_6 has the least impact on the overall energy of the system. Based on these results, we can deduce the following data ordering:

$$u_6 \prec u_{13} \prec u_{11} \approx u_{12} \prec u_9 \prec u_5 \prec u_8 \prec u_3 \prec u_4 \prec u_2 \prec u_7 \prec u_1 \prec u_{14} \prec u_{10}.$$

On the other hand, similar results are obtained when analyzing the λ-energies of soft sets, resulting in the following order:

$$u_{13} \prec u_6 \prec u_{11} \approx u_{12} \prec u_9 \prec u_8 \prec u_5 \prec u_3 \prec u_4 \prec u_7 \prec u_1 \prec u_{14} \prec u_2 \prec u_{10}.$$

It is of great importance to compare these results with the outcomes of the sMBR01 and CEC11 methods developed in the paper [32]. Using the sMBR01 method, the following order was obtained:

$$u_{13} \prec u_9 \prec u_{11} \approx u_{12} \prec u_8 \prec u_6 \prec u_5 \prec u_3 \prec u_7 \prec u_4 \approx u_{10} \prec u_{14} \prec u_1 \prec u_2,$$

while the CEC11 method yielded the following arrangement:

$$u_{13} \prec u_9 \prec u_{11} \approx u_{12} \prec u_8 \prec u_6 \prec u_5 \prec u_3 \prec u_4 \prec u_7 \prec u_1 \prec u_{14} \prec u_{10} \prec u_2.$$

Both the sMBR01 and CEC11 methods, as well as the methods using the energies of fuzzy soft sets, provide similar results. Specifically, when we examine and analyze the results, we conclude that the first 50% of side effects (namely, side effects $u_{13}, u_9, u_{11}, u_{12}, u_8, u_6$, and u_5) are the same for any of the considered methods.

In the paper [14], the authors also explore decision-making problems based on fuzzy soft sets. Through their analysis, they aimed not only to highlight the limitations of existing methods but also to propose improvements with the potential for broader application in real decision-making scenarios. By applying the energy-based method for fuzzy soft sets, as in Example 2.3 from their paper, we obtain results that are worth analyzing and comparing with the method from the paper [14]. Let's consider the fuzzy soft set of the mentioned example and form the corresponding matrix

$$A = \begin{bmatrix} 0.4 & 1 & 0.5 \\ 0.6 & 0.5 & 0.6 \\ 0.5 & 0.5 & 0.8 \\ 0.9 & 0.5 & 0.2 \\ 0.3 & 0.7 & 0.9 \end{bmatrix}.$$

In the familiar manner, we calculate the corresponding energies:
$\mathbf{E}(\Gamma_{A_1}) = 4.6$, $\mathbf{LE}(\Gamma_{A_1}) = 17.186404040778$
$\mathbf{E}(\Gamma_{A_2}) = 5.04$, $\mathbf{LE}(\Gamma_{A_2}) = 19.6548191378$
$\mathbf{E}(\Gamma_{A_3}) = 4.87$, $\mathbf{LE}(\Gamma_{A_3}) = 18.5293191286$;
$\mathbf{E}(\Gamma_{A_4}) = 4.91$, $\mathbf{LE}(\Gamma_{A_4}) = 21.35812429197$;
$\mathbf{E}(\Gamma_{A_5}) = 4.62$, $\mathbf{LE}(\Gamma_{A_5}) = 17.1221689022$.
We can then conclude that the following relations hold:

$$\mathbf{E}(\Gamma_{A_2}) \geq \mathbf{E}(\Gamma_{A_4}) \geq \mathbf{E}(\Gamma_{A_3}) \geq \mathbf{E}(\Gamma_{A_5}) \geq \mathbf{E}(\Gamma_{A_1}),$$

and

$$\mathbf{LE}(\Gamma_{A_4}) \geq \mathbf{LE}(\Gamma_{A_2}) \geq \mathbf{LE}(\Gamma_{A_3}) \geq \mathbf{LE}(\Gamma_{A_1}) \geq \mathbf{LE}(\Gamma_{A_5}).$$

Analyzing the obtained order of energies for fuzzy soft sets, we get:

$$u_1 \succ u_5 \succ u_3 \succ u_4 \succ u_2,$$

and by analyzing the order of λ-energies for fuzzy soft sets, we obtain:

$$u_5 \succ u_1 \succ u_3 \succ u_2 \succ u_4.$$

Therefore, the results obtained are noteworthy, and it would be beneficial to make comparisons with the results obtained in the paper [14], which will be discussed at the end of the section.

After obtaining the results, we can compare some previously known methods with our methods based on the energies of fuzzy soft sets. In Example 5.1 from the paper [11], we tested our methods based on the introduced energies of fuzzy soft sets. The conclusion obtained is the same using all three methods (the method with the energy of a fuzzy soft set, the method with the λ-energy of a fuzzy soft set, and the method using fs-aggregation introduced in the paper [11]). However, if we consider the overall linear order, we see that our two methods give the same solutions, while the method described in [11] provides a slightly different order. Specifically, our algorithm gives priority to the candidate u_5 over the candidate u_2, while the algorithm from [11] gives preference to the candidate u_2 over the candidate u_5, although this preference is almost negligible.

After this discussion, the question arises as to whether the algorithm based on the energy of fuzzy soft sets and the algorithm based on λ-energy of fuzzy soft sets sometimes give different solutions, i.e., whether they are fundamentally different algorithms, or do they always yield the same solutions, as was the case with the example 5.1 from [11]? To solve this question and justify the existence of both algorithms, let's analyze an example from [32]. By observing the results obtained by the methods sMBR01 and CEC11 from [32], as well as the results on the same example but applying the algorithms based on the energies of fuzzy soft sets, we find that the linear order of obtained solutions differs. At first glance, the differences are obvious, but if we look at half of the undesired effects in the given example, we conclude that it is the same set. Therefore, by analyzing the results obtained by applying the methods sMBR01 and CEC11, as well as the methods based on the energies of fuzzy soft sets, we come to four different linear orders.

Comparing decision-making methods based on the energies of fuzzy soft sets and the method based on the level soft set with weighted choice values described in [14], we notice a correctly made decision by the mentioned method and the method based on the λ-energy of a fuzzy soft set. The method based on the energy of a fuzzy soft set gives preference to the object u_1 over the object u_5, although this advantage is very small. Unlike the method described in [14], methods based on fuzzy energies provide a uniquely determined solution, and in exceptional cases, multiple solutions are possible, but in that case, the corresponding rows of observed matrices are equal (see the example from [32]), as expected.

Finally, in the next table we compare the focal criteria that we have discussed according to their main characteristics.

Procedure	Ranking methodology	Unique solution	Objections
[11]	fs-aggregation	Yes	Uses many new terms
[14]	Choice value of level soft set	No	Too rough criterion for decision-making
[32]	The decision set and the rank of the alternatives	No	Criterion is very controversial
Algorithms based on the energies	Scores based on comparison of energies	Yes	

This discussion shows that the solution obtained by our algorithms based on the energies of fuzzy soft sets does not completely match any proposed solution obtained by algorithms in the observed papers. The introduction and use of the corresponding energies in our work lead us to the same results when making decisions dealing with uncertainty, and these decisions relate to a large number of problems from the real environment.

As mentioned, using fuzzy soft sets enables decision-making in real-life scenarios with uncertainties and vagueness of data. Uncertainty primarily refers to situations involving imperfect or unknown information. Certainly, one of the main tasks is to determine the measure of uncertainty, which manifests in various forms. One of the most significant forms of uncertainty is fuzziness. The method based on energies of fuzzy soft sets is suitable for addressing such tasks because the uncertainty of parameters is determined in terms of the overall degree of is fuzziness.

In the context of uncertainty, the proposed method provides information for decision-making but does not provide information about the degree of confidence in the decision made. The level of confidence in the decision made is related to the minimum or maximum energy of the fuzzy soft set. Regarding this, the main current drawback of the method based on the energy of fuzzy soft sets is that it is not straightforward to determine the lower and upper bounds of the energy of the observed fuzzy soft set. Another limitation is that assigning energies is not bijective; that is, two fuzzy soft sets can have the same energies, all due to the fact that in this method, each parameter is considered an equal contributor.

6 Summary and conclusion

Almost without exception, every new property or characteristic of fuzzy soft sets has prompted additional research to establish comparisons, further generalizations, prove additional properties or produce applications of that property or characteristic. In addition to numerous applications of the theory of fuzzy soft sets, this paper defines additional parameters characterizing the nature of fuzzy soft sets, contributing to further advancements in the application of the theory of fuzzy soft sets, similar to the role of graph energy in graph theory. In this paper, we managed to integrate a characteristic from graph theory into the theory of fuzzy soft sets, which conceptually differs from graph theory. In line with that, the main result of the paper is the definition of the energy and λ-energy of a fuzzy soft set as a numerical characteristic that plays a crucial role in decision-making problems. There is ample room for future research in this area, as the limits of the introduced energies of fuzzy soft sets can be improved, and other properties of these defined energies can be explored. It would be interesting to discover the maximum value that the energy of a fuzzy soft set can achieve and construct a concrete example illustrating this concept. One of the main goals of future research is also the reconstruction (construction) of a fuzzy soft set with the desired energy. Certainly, in such research, attention should be paid to the uniqueness of such a fuzzy soft set, provided that existence is ensured. Future research must also be oriented towards determining the singular value decomposition of a fuzzy soft set, given that we introduced the singular values of the matrix representing a fuzzy soft set. Additionally, one direction for further research is the generalization of our method to (a, b)-fuzzy soft sets introduced in the paper [3], which offers a broader scope for dealing with uncertainties.

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