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THE INDEX FUNCTION OPERATOR FOR O-REGULARLY VARYING FUNCTIONS

DRAGAN DJURČIĆ[1], DANICA FATIĆ[1], AND NEBOJŠA ELEZ[2]

Abstract. The paper examines the functional transformation K of the class ORV_φ (see [3]) into the class of positive functions on interval $(0,+\infty)$ defined as follows:

$$K(f) = k_f, \tag{0.1}$$

where

$$k_f(\lambda) = \limsup_{x\to+\infty} \frac{f(\lambda x)}{f(x)}, \quad \lambda \in (0,+\infty),$$

and $f \in ORV_\varphi$.

Let $f \in IRV_\varphi$ or SO_φ (see [4]), K be the transformation (0.1) and for any $n \in \mathbb{N}$, $K_n(f) = \underbrace{K(K\cdots(K}_{\text{n}}(f))\cdots)$, then the function $p(s) = \lim_{n\to+\infty} K_n(f)(s)$, $s > 0$, is IRV_φ (and continuous) and SO_φ, respectively.

1. Introduction

The classic Karamata theory of regular variability has its beginnings in the 30s of the last century. Namely, studying the asymptotic properties of Riemann-Stieltjes (especially the Dirichlet and power series) Karamata observed the connection between the asymptotic properties of kernel of the Riemann-Stieltjes integral and the properties of that integral. Thus, asymptotic properties (serious and essential) for functions (and sequences) were perceived: regular variability and rapid variability; the study of the same in a qualitative sense and in applications began immediately (see [3]). These properties found a special place in the theory of summability, the theory of oscillations,

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the theory of Tauber properties, Fourier analysis, number theory, differential equations, etc (see [3]). In the 30s of the last century (and later) there were modifications of the classic Karamata theory of regular variability, depending on the needs of research. So, for example, the theory O-regular variability appears, which is an significant Tauber condition in very important Tauber-type theorems (see [1]). Recently, the classical Karamata theory of regular variability has a significant place in machine learning (especially in determining the direction of variation).

A function $f : [a, +\infty) \to (0, +\infty)$ is O-regularly varying function in the sense of Karamata (see [3] and [1]), if for some fixed $a > 0$ it is measurable and

$$\limsup_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = k_f(\lambda) < +\infty \tag{1.1}$$

holds, for every $\lambda > 0$. The function $k_f(\lambda)$, $\lambda > 0$, is called the index function of the function f and its characteristics give many of the asymptotic properties of the function f (see [2] and [4]). The function $k_f(\lambda)$, $\lambda > 0$, can be both measured and immeasurable. An example of the immeasurable index function is given by Rubel in [16] who has constructed the appropriate function f.

O-regularly varying functions in the sense of Karamata form a functional class ORV_φ, and elements of that class are very important objects in the qualitative analysis of divergent functional processes (see [3] and [1]).

1° Assume that $f \in ORV_\varphi$. Then $f \in IRV_\varphi$ (in some literature we can find that class IRV_φ is denoted by CRV_φ[4]) if $k_f(\lambda)$, $\lambda > 0$, is continuous. The importance of this class can be seen in the asymptotic analysis in points (e.g. [4, 5] and [7]).

2° Assume that $f \in ORV_\varphi$. Then $f \in ERV_\varphi$ (the class ERV_φ is so-called the Matuszewski class [14, 15] and [5], or the extended class of regularly varying functions in the sense of Karamata [3]), if $k_f(\lambda)$, $\lambda > 0$, for $\lambda = 1$ has finite one-sided derivatives. See [14] about the qualitative properties of the class ERV_φ.

3° Assume that $f \in ORV_\varphi$. Then $f \in RV_\varphi$ (RV_φ is a well-known class of regularly varying function in the sense of Karamata [9, 10]) if $k_f(\lambda)$, $\lambda > 0$, is differentiable function. An especially important subclass of RV_φ is the class SV_φ (slowly varying function in the sense of Karamata [1, 18]). For this function it holds that $k_f(\lambda) = 1$, for every $\lambda > 0$ (if $f \in SV_\varphi$).

It holds that (see [4])

$$SV_\varphi \subsetneq RV_\varphi \subsetneq ERV_\varphi \subsetneq IRV_\varphi \subsetneq ORV_\varphi. \tag{1.2}$$

Classes of functions in 1°, 2°, 3° are very important elements of Karamata's theory of regular variation (see [2] and [17]) and its applications (see [6, 8, 11–13] and [18]).

It is easy to prove the next lemma.

Lemma 1.1. *Assume that* $f \in ORV_\varphi$.

(*a*) *If* $k_f(\lambda)$, $\lambda > 0$, *is a measurable function, then* $k_f \in ORV_\varphi$.

(*b*) *If* $k_f(\lambda)$, $\lambda > 0$, *is a continuous function, then* $k_f \in IRV_\varphi$.

(c) *If* $k_f(\lambda)$, $\lambda > 0$, *has finite one-sided derivatives for* $\lambda = 1$, *then* $k_f \in ERV_\varphi$.
(d) *If* $k_f(\lambda)$, $\lambda > 0$, *is differentiable function, then* $k_f \in RV_\varphi$.
(e) *If* $k_f(\lambda)$, $\lambda > 0$, *is a constant function for* $\lambda > 0$, *then* $k_f \in SV_\varphi$.

2. The Main Result

Consider the functional transformation K on class ORV_φ into the class of positive functions defined on the interval $(0, +\infty)$, given as

$$K(f) = k_f. \tag{2.1}$$

The index function of function $f \in OR_\varphi$ (the operator $K(f) = k_f$) in the notation k_f carries with very important features for the function f. For example, upper and lower Karamata's index, also both Matuszewski's index for the observed function f. The characteristics of the index function k_f for the function $f \in ORV_\varphi$ describe the relation of the function f to the asymptotic equivalence relations and to the generalized inverse (see [3–8, 11] and [15]).

We can see that

$$K(K(f)) = K(f), \tag{2.2}$$

if $f \in RV_\varphi$.

The K transformation is called the index function operator. Its properties can be seen in Lemma 1.1. According to everything given above, it makes sense to consider the iterative process

$$K_n(f) = \underbrace{K(K\cdots(K}_{\text{n}}(f))\cdots), \tag{2.3}$$

for $n \in \mathbb{N}$ on the class ORV_φ, IRV_φ, ERV_φ, RV_φ.

Let us consider the properties of the operator (2.1) in the sense of iterative process (2.3) on the class IRV_φ. On the class RV_φ for the operator (2.1) the iterative process (2.3) is described by (2.2).

In probability and statistics there is a great need for important characterizations using Seneta's functions (O-regular variable functions with a bounded index function). They are essential generalizations of slow varying functions: each of them is the product of a slow varying and bounded function that is positive.

A function $f : [a, +\infty) \to (0, +\infty)$, $a > 0$, is called β-Seneta's function (see [17]), if there exist $\beta \geqslant 1$, $\beta \in \mathbb{R}$, such that

$$k_f(\lambda) \leqslant \beta, \tag{2.4}$$

for every $\lambda > 0$.

The class of β-Seneta's functions that satisfy (2.4) for given $\beta \geqslant 1$, $\beta \in \mathbb{R}$, we denote by SO^β_φ, and the class of all Seneta's functions by $SO_\varphi = \cup_{\beta\geqslant 1} SO^\beta_\varphi$.

This class is very important in approximation theory and probability theory (see [3] and [17]).

We have that $SV_\varphi \subseteq SO_\varphi \subseteq ORV_\varphi$. The class SO_φ can not be compared with classes IRV_φ and ERV_φ. We also have $SO_\varphi \cap (RV_\varphi \setminus SV_\varphi) = \emptyset$.

Lemma 2.1. *Let $f \in SO_\varphi$. If $k_f(\lambda)$, $\lambda > 0$, is a measurable function, then $K(f) \in SO_\varphi$. If $f \in SO_\varphi^\beta$ and $k_f(\lambda)$, $\lambda > 0$, is a measurable function, then $K(f) \in SO_\varphi^\beta$.*

Proof. If we give a proof for the second statement, then the first statement holds. Assume $f \in SO_\varphi^\beta$, for some real $\beta \geqslant 1$.

Then

$$0 < k_f(\lambda) = \limsup_{x\to+\infty} \frac{f(\lambda x)}{f(x)} \leqslant \beta < +\infty, \tag{2.5}$$

for every $\lambda > 0$. Also, the function $k_f(\lambda)$, $\lambda > 0$, is measurable and for every $s > 0$ and every $t > 0$ and

$$\begin{aligned} 0 < k_f(st) &= \limsup_{x\to+\infty} \frac{f(stx)}{f(x)} = \limsup_{x\to+\infty} \left(\frac{f(stx)}{f(tx)} \cdot \frac{f(tx)}{f(x)} \right) \\ &\leqslant \limsup_{x\to+\infty} \frac{f(stx)}{f(tx)} \cdot \limsup_{x\to+\infty} \frac{f(tx)}{f(x)} \\ &= k_f(s) \cdot k_f(t) \end{aligned}$$

is satisfied. Actually, for every $t > 0$, we have that

$$0 < \limsup_{s\to+\infty} \frac{k_f(ts)}{k_f(s)} \leqslant k_f(t) \leqslant \beta < +\infty.$$

Hence, $k_f(\lambda)$, $\lambda > 0$, belongs to the class SO_φ^β. □

From the above, we can conclude that for every $n \in \mathbb{N}$, $K_n(f) \in SO_\varphi$ is satisfied if the function $K_n(f)$ is measurable and $f \in SO_\varphi$.

Theorem 2.1. *Let $f \in ORV_\varphi$ and let operator K be given as in* (2.1). *Also, let functions $K_n(f)$, $n \in \mathbb{N}$, be given as in* (2.3) *are measurable. Then, for every $s > 0$, there is a function $p(s) = \lim_{n\to+\infty} K_n(f)(s)$ which belongs to class ORV_φ. Specially, if $f \in SO_\varphi^\beta \subsetneq ORV_\varphi$, then $p \in SO_\varphi^\beta$.*

Proof. Let $f \in ORV_\varphi$. Then according to Lemma 1.1 (a) function $K(f) \in ORV_\varphi$. Sequence of functions $K_n(f)(s)$, $s > 0$, is non-increasing sequence (supreme norm) of functions which are measurable and hold that $1 \leqslant K_n(f)(s) \cdot K(f)(\frac{1}{s}) < +\infty$ for every $n \in \mathbb{N}$ and every $s > 0$. That means, for every $s > 0$, sequence $(K_n(f)(s))$ converges to $0 < p(s) < +\infty$. The function $p(s)$, $s > 0$, is measurable as limit function of measurable functions.

As for every $s, t > 0$

$$p(s \cdot t) \leqslant p(s) \cdot p(t),$$

then for every $s > 0$

$$\limsup_{t\to+\infty} \frac{p(st)}{p(t)} = k_p(s) \leqslant p(s) < +\infty.$$

Hence, holds $p \in ORV_\varphi$. Specially, if $f \in SO_\varphi^\beta$, then according to Lemma 2.1 function $K(f) \in SO_\varphi^\beta$. Thus, for every $s > 0$, holds $k_p(s) \leqslant p(s) \leqslant K(f) \leqslant \beta$. Regarding, it is valid that $p \in SO_\varphi^\beta$. □

Corollary 2.1. *If we observe class of Seneta's functions SO_φ instead of S_φ^β, the Theorem* 2.1 *still holds.*

Theorem 2.2. *Let $f \in IRV_\varphi$ and the operator K be given as* (2.1)*. Then the function $p(s) = \lim_{n\to+\infty} K_n(f)(s)$, $s > 0$, exists for $s > 0$, is continuous, and belongs to the class IRV_φ.*

Proof. Let $f \in IRV_\varphi$, then according to Lemma 1.1 (b), the function $K(f) \in IRV_\varphi$ is continuous on $(0, +\infty)$. Also, for every $n \in \mathbb{N}$, the function $K_n(f) \in IRV_\varphi$ is continuous on $(0, +\infty)$. If $s = 1$, then $p(s) = 1$. If $s > 0$, $s \neq 1$, then for every $n \in \mathbb{N}$ it holds

$$0 < \frac{1}{K_n(f)(\frac{1}{s})} \leqslant \frac{1}{K_{n+1}(f)(\frac{1}{s})} \leqslant K_{n+1}(f)(s) \leqslant K_n(f)(s) < +\infty.$$

Hence, the function $p(s)$ is finite and positive for $s > 0$. As $p(s) \leqslant K_1(f)(s)$ for every $s > 0$ and $\lim_{s\to1} K_1(f)(s) = 1$, then $\limsup_{s\to1} p(s) \leqslant 1$. Therefore, the function p is measurable on $(0, +\infty)$ as the limit value of a continuous function, and for every $s, t > 0$ we have

$$\begin{aligned} p(st) &= \lim_{n\to+\infty} K_n(f)(st) \\ &\leqslant \lim_{n\to+\infty} K_n(f)(s) \cdot \lim_{n\to+\infty} K_n(f)(t) \\ &= p(s) \cdot p(t). \end{aligned}$$

It means that the function p is continuous on $(0, +\infty)$. Since $K(p)(s) \leqslant p(s)$ for every $s > 0$ and $\limsup_{s\to1} K(p)(s) \leqslant 1$, then $K(p)$ is continuous on $(0, +\infty)$. It holds that $p \in IRV_\varphi$. □

Remark 2.1. The continuinity of function p on $(0, +\infty)$ can be proved by using well-known Dini's theorem of uniform convergences.

Corollary 2.2. *Let $f \in ERV_\varphi$. Then $K(p) \in IRV_\varphi$, where the operator K is given by* (2.1) *and $p(s) = \lim_{n\to+\infty} K_n(f)(s)$, $s > 0$, ($K_n(f)$ is given by* (2.3)*, for every $n \in \mathbb{N}$).*

We finish with one open problem.

Remark 2.2. Does $p \in ERV_\varphi$ hold from Corollary 2.2?

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[1]Faculty of Technical Sciences,
University of Kragujevac,
Svetog Save 65, 32000 Čačak, Serbia
Email address: dragan.djurcic@ftn.kg.ac.rs
Email address: danica.fatic@ftn.kg.ac.rs

[2]Faculty of Philosophy,
University of East Sarajevo,
Alekse Šantića 1, 71420 Pale
Bosnia and Herzegovina
Email address: nebojsaelez@gmail.com