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University of Kragujevac

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10th International Congress
Motor Vehicles & Motors 2024
ECOLOGY - VEHICLE AND ROAD SAFETY - EFFICIENCY
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October 10th - 11th, 2024
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PREDGOVOR

U oktobru se na Fakultetu inženjerskih nauka Univerziteta u Kragujevcu tradicionalno održava skup istraživača i naučnika koji se bave proučavanjem motornih vozila, motora i drumskog saobraćaja. Od 1979. do 2004. godine održano je trinaest bienalnih MVM simpozijuma koji su 2006. prerasli u Međunarodni kongres MVM. Od tada je održano devet MVM kongresa, a oktobra 2024. godine Fakultet inženjerskih nauka je organizovao deseti međunarodni kongres MVM od 10. do 11. oktobra 2024. godine.

Na deseti kongres Motorna vozila i motori, MVM2024 dostavljen je veliki broj naučnih radova iz Srbije i inostranstva. Kongres tradicionalno podržavaju Ministarstvo za nauku, tehnološki razvoj i inovacije Republike Srbije, Univerzitet u Kragujevcu, Fakultet inženjerskih nauka i međunarodni časopis „Mobility and Vehicle Mechanics“.

Tema Kongresa MVM 2024 bila je „Ekologija – Bezbednost vozila i na putevima – Efikasnost“. Tokom ovog istraživačkog putovanja, učesnici su puno naučili kroz rad na različitim sekcijama, koje su pokrivale širok spektar tema u vezi sa inženjerstvom u automobilskoj industirji, od fundamentalnih istraživanja do industrijskih primena, naglašavaju interakciju između vozača, vozila i životne sredine i stimulišući naučnu interakcije i saradnju.

Međunarodni naučni odbor u saradnji sa organizacionim odborom izradio je podsticajan naučni program. Program je ponudio preko 54 prezentacije radova, uključujući predavanja po pozivu i radove u sekcijama. Prezentacije na ovom kongresu obuhvatile su aktuelna istraživanja u oblasti motornih vozila i motora sprovedena u 12 zemalja iz celog sveta.

Zadovoljstvo nam je bilo što su nam uvodničari bili profesor Emrulah Hakan Kaleli (sa Tehničkog univerziteta Yıldız, Turska), profesor Ralph Putz (sa Univerziteta Landshut UAS, Nemačka) i profesori Nenad Miljić i Slobodan Popović (sa Univerziteta u Beogradu, Srbija). Izazovi i rešenja u korišćenju vodonika kao goriva za motore sa unutrašnjim sagorevanjem, korišćenje aditiva nanoborne kiseline dodatog u motorno ulje, kao i evropska politika o budućoj mobilnosti na putevima su bile teme uvodnih predavanja.

Sigurni smo da je ovaj program pokrenuo živu diskusiju i podstakao istraživače na nova dostignuća.

10. Kongres MVM 2024. finansijski je podržalo Ministarstvo za nauku, tehnološki razvoj i inovacije Republike Srbije.

Zahvaljujemo se iskusnim i mladim istraživačima koji su prisustvovali i prezentovali svoju stručnost i inovativne ideje na našem kongresu.

Posebnu zahvalnost dugujemo članovima međunarodnog naučnog odbora i svim recenzentima za njihov značajan doprinos visokom nivou kongresa.

Naučni i organizacioni komitet Kongresa MVM2024

FOREWARD

In October, the Faculty of Engineering University of Kragujevac traditionally holds gatherings of researchers and academics who study motor vehicles, engines and road traffic. From 1979 to 2004, thirteen, biennal MVM Symposiums have been held and they grew into an International Congress MVM in 2006. Since then, ninth MVM Congresses have been held, and in October 2024, the Faculty of Engineering organized the tenth International Congress MVM from 10th to 11th October 2024.

A large number of scientific papers from the Serbia and abroad were submitted to the tenth Congress “MVM2024”. Congress is traditionally supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia, University of Kragujevac, Faculty of Engineering and the International Journal “Mobility and Vehicle Mechanics”.

The theme of the Congress MVM 2024 was "Ecology - Vehicle and Road Safety - Efficiency". Along this journey we learned from the various sessions, which broadly cover a wide range of topics related to automotive engineering from fundamental research to industrial applications, highlight the interaction between the driver, vehicle and environment and stimulate scientific interactions and collaborations.

The International Scientific Committee in collaboration with the Organising Committee built up a stimulating scientific program. The program offered over 54 presentations, including key-note speakers and paper sessions. The presentations to this conference covered current research in motor vehicle and motors conducted in 12 countries from all over the world.

We were pleased to have professor Emrullah Hakan Kaleli (from Yıldız Technical University, Türkiye), professor Ralph Pütz (from Landshut University UAS, Germany) and professors Nenad Miljić and Slobodan Popović (from University of Belgrade, Serbia) as the keynote speakers, addressing Challenges and solutions in using hydrogen as a fuel for internal combustion engines, using nanoboric acid (nBA) additive added in engine oil, as well as European policy on future road mobility.

We are sure this program will trigger lively discussion and will project researchers to new developments.

The 10th Congress MVM 2024 was financially supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia.

We would like to thank experienced and young researchers, for attending and bringing their expertise and innovative ideas to our conference.

Special thanks are due to the International Scientific Board Members and all reviewers for their significant contribution in the high level of the conference.

Scientific and Organizational committee of Congress MVM2024

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MVM2024-005

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FUNCTIONALLY GRADED MATERIALS IN AUTOMOTIVE INDUSTRY-MODELLING AND ANALYSIS OF FG PLATE ON ELASTIC FOUNDATION

ABSTRACT: Due to extensive application of composite materials in automotive industry projects, the focus of paper is on functionally graded materials as a modern composite. In the introductory section of the paper, the basic concept of the aforementioned materials is described, the main advantages and disadvantages are given, as well as examples of their application in various fields of automotive sector. A systematic and practical approach for modelling, designing and analysis of functionally graded plate on elastic foundations is discussed. Three simple elastic foundation models with constant parameters have been analysed: Winkler foundation, Pasternak foundation as well as Kerr foundation. The formulation of the Winkler/Pasternak/Kerr foundation models is studied analytically. On the basis of the described theoretical formulations, numerical examples of the bending of functionally graded plates on elastic foundation are done. The difference between three models of elastic foundation is also discussed by comparing their results. Finally, based on obtained results conclusions and the recommendations for further study are given.

KEYWORDS: functionally graded material, elastic foundation, bending analysis

INTRODUCTION

Functionally Graded Materials (FGM) are advanced composite materials characterized by gradual variations in composition, structure and properties through volume. It results in a change of the material's properties, such as mechanical, thermal, electrical, or optical characteristics. These variations are often engineered to optimize the material's performance in specific applications, combining the benefits of different materials in a single component. FGM are designed so that their properties change gradually, rather than abruptly, across the material. This can be in one direction (unidirectional gradient) or multiple directions, depending on the design requirements. The gradient can be in terms of composition, microstructure, or porosity. The composition of FGM can vary between two or more materials, such as metals, ceramics, polymers, or composites. For example, one side of an FGM could be made of

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a metal, providing ductility, while the other side could be ceramic, offering heat resistance. By varying the composition and structure, FGM can be tailored to meet specific functional requirements. This makes them ideal for applications where traditional homogeneous materials might fail or be less efficient.

Finally, it can be said that FGM has significant advantages over conventional materials as:

- **Stress reduction** - the gradual change in properties reduces stress concentrations, particularly in applications involving thermal or mechanical loads.
- **Improved durability** - FGM often have better wear, corrosion, and thermal resistance due to their tailored properties.
- **Design flexibility** - engineers can design components with specific property gradients to meet challenging operational conditions.

On the other hand, there are numerous challenges facing the process of designing and manufacturing FGM [1], [2]:

- **Complex manufacturing** - producing FGM can be more complex and expensive than traditional materials.
- **Design and simulation** - designing FGM requires advanced modelling and simulation tools to predict how the gradient will affect the material's overall behaviour.
- **Quality control** - ensuring uniformity and precision in the gradient can be challenging, requiring sophisticated inspection techniques.

The actuality of research into new materials with a special emphasis on functionally graded materials is indicated by a large number of review papers in recent years [3], [4], [5].

FUNCTIONALLY GRADED MATERIALS IN AUTOMOTIVE INDUSTRY

Functionally graded materials are increasingly being utilized in the automotive industry due to their ability to enhance the performance, safety, and efficiency of vehicles [6]. Some specific examples of FGM applications in the automotive sector are given below (Figure 1):

- *Brake discs and rotors* - FGM are used in brake discs and rotors to improve thermal management and reduce wear. The material gradient allows for high thermal conductivity on the outer surface to quickly dissipate heat, while the inner layers provide mechanical strength and wear resistance. This reduces the risk of thermal cracking and prolongs the life of the brake components.
- *Engine components* - FGM is employed in various engine components, such as pistons, cylinder heads, and valves. The gradient in material properties allows these components to withstand high temperatures and pressures while maintaining strength and durability. For example, a piston made with an FGM can have a high-temperature-resistant ceramic on the top surface, gradually transitioning to a lightweight metal alloy in the core to reduce overall weight and enhance fuel efficiency.
- *Exhaust systems* - FGM are used in exhaust systems, particularly in catalytic converters and exhaust manifolds. The gradient material design can help manage thermal stresses due to the high temperatures and corrosive environment. FGM can improve the durability and efficiency of the catalytic converter by having a ceramic layer on the interior for heat resistance and a metal exterior for structural support.
- *Turbine blades in turbochargers* - FGM are used in the turbine blades of turbochargers. The material gradient allows the turbine blades to withstand the extreme temperatures and stresses caused by high-speed rotation. The high-temperature-resistant ceramic on the surface gradually transitions to a tougher metal alloy, providing both heat resistance and mechanical strength.
- *Lightweight structural components* - FGM are utilized in various lightweight structural components, such as body panels and chassis parts. By using FGM, manufacturers can create components that are both strong and lightweight. For instance, a panel can have a tough, impact-resistant surface with a lightweight, stiff core, optimizing the strength-to-weight ratio. This contributes to improved fuel efficiency and vehicle performance.
- *Crashworthiness and impact protection* - FGM are used in areas of the vehicle that are designed to absorb energy during a crash, such as bumper systems and door panels. The material gradient can be tailored to gradually absorb and dissipate impact energy, improving the vehicle's crashworthiness. The outer layers can be made stiffer to absorb initial impact, while the inner layers are more compliant, helping to reduce the force transmitted to the occupants.
- *Heat shields* - FGM are used in heat shields that protect other components from the high temperatures of the engine or exhaust. The graded material can provide high thermal resistance on the side facing the heat source and a lower thermal conductivity on the opposite side, protecting the adjacent components from heat damage and reducing heat transfer to the vehicle cabin.
- *Advanced coatings for wear and corrosion resistance* - FGM are applied as coatings on various automotive parts, such as gears, shafts, and bearings. These coatings can provide a hard, wear-resistant outer layer with a tougher, more ductile inner layer, improving both the durability and fatigue resistance of the components. This is particularly important for parts subjected to cyclic loading and harsh environments.

- *Battery thermal management in electric vehicles* - FGM are being explored for use in the thermal management systems of batteries in electric vehicles (EVs). The graded material can help manage the temperature within the battery pack more effectively, ensuring uniform temperature distribution and preventing overheating, which is crucial for battery efficiency and safety.

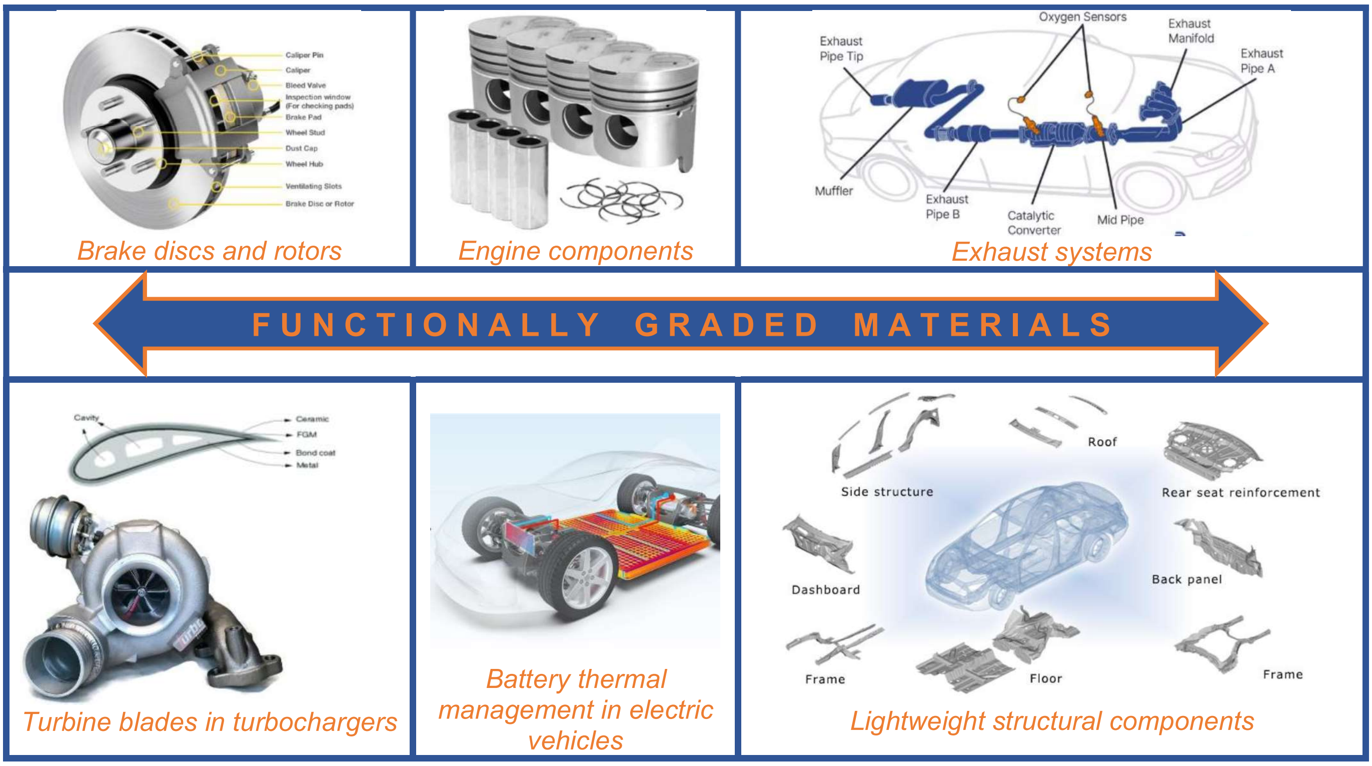


Figure 1 FGM applications in the automotive sector

MODELLING OF FG PLATE ON ELASTIC FOUNDATION

Modelling functionally graded (FG) plate on an elastic foundation involves understanding the gradation of material properties within the plate and accurately representing the foundation's support behaviour using appropriate models. The complexity of the model and the chosen foundation directly influence the accuracy and applicability of the analysis, making it essential to choose the right model based on the specific requirements of the problem.

Functionally graded materials are advanced composite materials where the material properties (e.g., Young's modulus, density, thermal conductivity) vary continuously along one or more dimensions, typically according to a specific gradient function. This gradual change in properties is often achieved by varying the volume fraction of two or more constituent materials. In an FGM plate, the material properties vary continuously across the thickness, making it possible to optimize the plate's performance under different loading conditions. For instance, one surface of the plate might be made of a metal for high strength, while the opposite surface might be ceramic for high-temperature resistance. The gradation is usually in the thickness direction (*z*-axis), with properties typically following a power-law, exponential, or sigmoid function [7]. A common power-law distribution of the volume fraction V(z) of the constituents can be expressed as:

$$V_{\mathrm{f}} = \left(\frac{1}{2} + \frac{z}{h}\right)^{p} \tag{1}$$

and mechanical properties of the FGM in the thickness direction of the plate

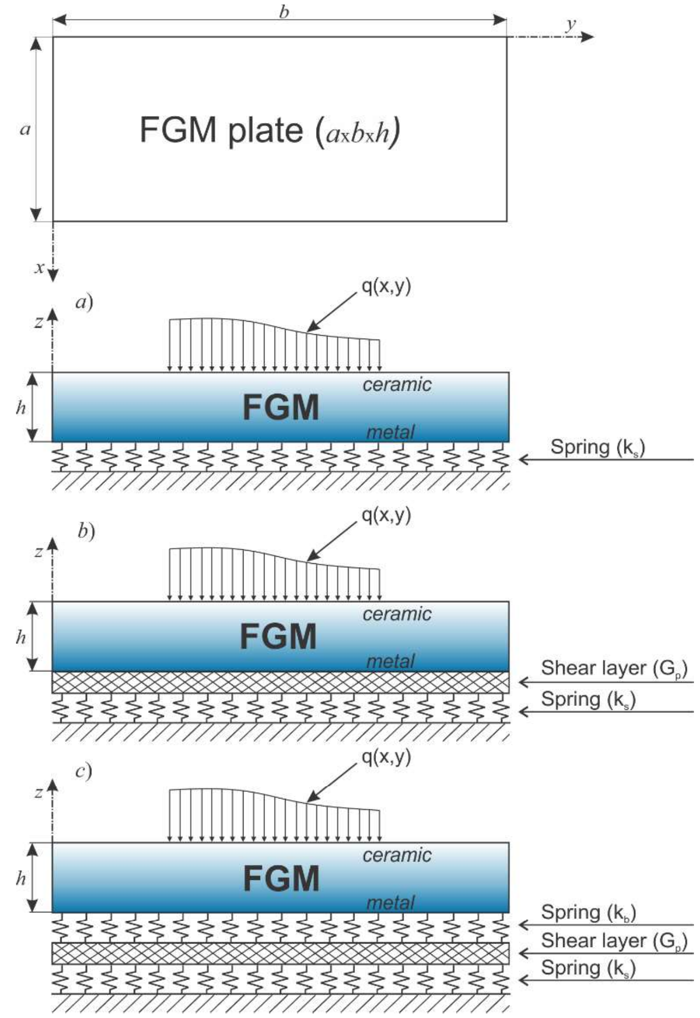
$$P(z) = P_{\mathrm{m}} + P_{\mathrm{cm}} V_{\mathrm{f}}, \qquad P_{\mathrm{cm}} = P_{\mathrm{c}} - P_{\mathrm{m}} \tag{2}$$

where is:

h - the total thickness of the plate
z - the coordinate through the thickness
p - the material gradation index that controls the material distribution
$P(z)$, P_{m}, P_{c} - mechanical properties of arbitrary cross section „z“, metal, ceramic, respectively

The elastic foundation provides the reactive support to the FG plate whereby different models can be used to describe the foundation behaviour (Figure 2), [8], [9], [10], [11]:

- **Winkler foundation** - represents the foundation as a series of independent, linear elastic springs. The foundation reaction is proportional to the local displacement of the plate, but it does not account for shear interactions between adjacent points.
- **Pasternak foundation** - adds a shear layer to the Winkler model, accounting for the shear interaction between adjacent points on the foundation. This model is more suitable when the foundation behaves like a continuous medium rather than discrete springs.
- **Kerr foundation** - incorporates the vertical displacement, shear interaction, and bending stiffness of the foundation, providing a more accurate representation, especially for thick or complex foundation systems.



$q_e(x,y)$ - foundation reaction

U_e - strain energy of the elastic foundation

$$q_e^{Winkler} = k_s w$$

$$U_e^{Winkler} = \int_A \frac{1}{2} k_s w^2 dA$$

$$q_e^{Pasternak} = k_s w - G_p \nabla^2 w \,, \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$U_e^{Pasternak} = \frac{1}{2} \int_A \left\{ k_s w^2 + G_p \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] \right\} dA$$

$$q_e^{Kerr} = \frac{k_s k_b}{k_s + k_b} w - \frac{G_p k_b}{k_s + k_b} \nabla^2 w$$

$$U_e^{Kerr} = \frac{1}{2} \int_A \left\{ \frac{k_s k_b}{k_s + k_b} w^2 + \frac{G_p k_b}{k_s + k_b} \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right] \right\} dA$$

Figure 2 Different models of FG plate on elastic foundation: a) Winkler foundation, b) Pasternak foundation, c) Kerr foundation

According to the higher-order shear deformation theory (HSDT), the initial step in defining the kinematic relations between displacement and strain involves the assumed forms of the displacement components:

$$u = u_0(x,y) - z w_{0,x} + f(z)\theta_x, \quad v = v_0(x,y) - z w_{0,y} + f(z)\theta_y, \quad w = w_0(x,y),$$
$$f(z) = z\left(\cosh\left(\frac{z}{h} \right) - 1{,}388 \right) \text{ - shape function} \tag{3}$$

To define the components of unit loads, it is essential to apply the relationships between displacements and strains according to the linear theory of elasticity. The elastic constitutive relations for FGM are given as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11}(z) & C_{12}(z) & 0 & 0 & 0 \\ C_{12}(z) & C_{22}(z) & 0 & 0 & 0 \\ 0 & 0 & C_{44}(z) & 0 & 0 \\ 0 & 0 & 0 & C_{55}(z) & 0 \\ 0 & 0 & 0 & 0 & C_{66}(z) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xz} \\ \gamma_{yz} \\ \gamma_{xy} \end{Bmatrix} \tag{4}$$

where the coefficients of the constitutive elasticity tensor could be defined through engineering constants:

$$C_{11}(z)=C_{22}(z)=\frac{E(z)}{1-\nu^2},\quad C_{44}(z)=C_{55}(z)=C_{66}(z)=\frac{E(z)}{2(1+\nu)},\quad C_{12}(z)=\frac{\nu E(z)}{1-\nu^2} \tag{5}$$

Due to the gradient variation of the plate structure along the z-coordinate, the modulus of elasticity can be defined based on equations (1) and (2) as follows:

$$E(z)=E_{\mathrm{m}}+E_{\mathrm{cm}}\left(\frac{1}{2}+\frac{z}{h}\right)^p,\qquad E_{\mathrm{cm}}=E_{\mathrm{c}}-E_{\mathrm{m}} \tag{6}$$

while ν=*const* - Poisson's ratio ν is considered constant because its variation in the thickness direction of the plate is minimal.

For the bending analysis to be carried out, it is assumed that the plate is subjected to sinusoidal transverse load *q(x,y)*. Work under external load is defined as:

$$V=-\frac{1}{2}\int_A qwdA,\quad \text{where is}\quad q(x,y)=q_0\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) \tag{7}$$

Plate strain energy deformation *U* are defined as:

$$\begin{aligned}U=\int_A &(N_{xx}\varepsilon_{xx}^{(0)}+N_{yy}\varepsilon_{yy}^{(0)}+N_{xy}\gamma_{xy}^{(0)}+M_{xx}k_{xx}^{(0)}+M_{yy}k_{yy}^{(0)}+M_{xy}k_{xy}^{(0)}\\ &+P_{xx}k_{xx}^{(1)}+P_{yy}k_{yy}^{(1)}+P_{xy}k_{xy}^{(1)}+R_x k_{xz}^{(2)}+R_y k_{yz}^{(2)})dA\end{aligned} \tag{8}$$

while strain energy of the elastic foundation U_e is defined according Figure 2 depending on the selected model of foundation.

Using the principle on minimum potential energy, the following equation is obtained:

$$\begin{aligned}\delta U+\delta V+\delta U_e=\delta\left(U+V+U_e\right)\equiv\delta\Pi=\int_A &(N_{xx}\delta\varepsilon_{xx}^{(0)}+N_{yy}\delta\varepsilon_{yy}^{(0)}+N_{xy}\delta\gamma_{xy}^{(0)}+M_{xx}\delta k_{xx}^{(0)}+M_{yy}\delta k_{yy}^{(0)}+M_{xy}\delta k_{xy}^{(0)}\\ &+P_{xx}\delta k_{xx}^{(1)}+P_{yy}\delta k_{yy}^{(1)}+P_{xy}\delta k_{xy}^{(1)}+R_x\delta k_{xz}^{(2)}+R_y\delta k_{yz}^{(2)})dA-\int_A q\delta wdA\\ &\underbrace{\underbrace{\underbrace{+\int_A\left\{\frac{k_s k_b}{k_s+k_b}w\delta w+\frac{G_p k_b}{k_s+k_b}\left(\frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial\delta w}{\partial y}\right)\right\}dA}_{Kerr}}_{Pasternak\,\rightarrow\,k_b\text{equal to infinity}}}_{Winkler\,\rightarrow\,k_b\text{equal to infinity}\quad and\quad G_p=0}=0\end{aligned} \tag{9}$$

By substituting the strain components and applying the calculus of variations, the following equilibrium equations are obtained:

$$\begin{aligned}\delta u_0:&\quad N_{xx,x}+N_{xy,y}=0\\ \delta v_0:&\quad N_{yy,y}+N_{xy,x}=0\\ \delta w_0:&\quad M_{xx,xx}+2M_{xy,xy}+M_{yy,yy}+N_{xx}w_{0,xx}+2N_{xy}w_{0,xy}+N_{yy}w_{0,yy}\\ &\qquad+q\underbrace{\underbrace{\underbrace{-\frac{k_s k_b}{k_s+k_b}w_0+\frac{G_p k_b}{k_s+k_b}\left(w_{0,xx}+w_{0,yy}\right)}_{Kerr}}_{Pasternak\,\rightarrow\,k_b\text{equal to infinity}}}_{Winkler\,\rightarrow\,k_b\text{equal to infinity}\quad and\quad G_p=0}=0\\ \delta\theta_x:&\quad P_{xx,x}+P_{xy,y}-R_x=0\\ \delta\theta_y:&\quad P_{xy,x}+P_{yy,y}-R_y=0\end{aligned} \tag{10}$$

where:

$$\mathbf{N}=\left\{N_{xx}\quad N_{yy}\quad N_{xy}\right\}^T,\ \mathbf{M}=\left\{M_{xx}\quad M_{yy}\quad M_{xy}\right\}^T,\ \mathbf{P}=\left\{P_{xx}\quad P_{yy}\quad P_{xy}\right\}^T,\ \mathbf{R}=\left\{R_x\quad R_y\right\}^T \tag{11}$$

represents the force, moments and higher order moment resultants.

To obtain analytical solutions for the system of equations (10), assumed solution forms and boundary conditions are adopted in accordance with Navier's solution as presented in [12].

NUMERICAL RESULTS AND DISCUSSION

To apply the theoretical results for simulating engineering problems in MATLAB, a code for analysing FGM plates has been developed, and various numerical examples have been conducted. This chapter will present the results obtained from the bending analysis of FG plates composed of metal (*Al*-Aluminium: E_m=0,7·10^5 [MPA], ν=0.3) and ceramic (Al_2O_3-Alumina: E_c=3,8·10^5 [MPA], ν=0.3) constituents.

Normalized values of a vertical displacement $\bar{w}$ (deflection) for different model of elastic foundation (Winkler/Pasternak/Kerr) and different volume fraction of constituents in FGM (index "p") are given in Table 1. Normalization of the aforementioned value has been conducted according to:

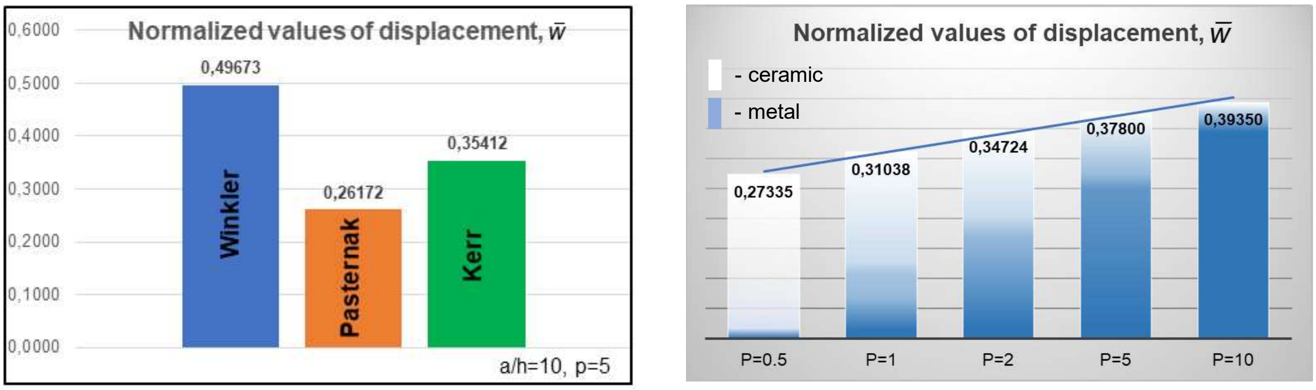
$$\bar{w} = \frac{10E_c h^3}{q_0 a^4} w\left(\frac{a}{2},\frac{b}{2}\right) \tag{12}$$

Table 1 Normalized values of displacement of FG square plate on elastic foundation

Theory	k_s	G_p	k_b	a/h	$\bar{w}$				
					FGM plate				
					p = 0,5	p = 1	p = 2	p = 5	p = 10
Winkler	100			5	0,35118	0,41476	0,48331	0,54504	0,57786
				10	0,32053	0,38260	0,44717	0,49673	0,52433
Pasternak	100	10		5	0,21481	0,23704	0,25795	0,27454	0,28263
				10	0,20294	0,22617	0,24728	0,26172	0,26919
Kerr	100	10	100	5	0,27335	0,31038	0,34724	0,37800	0,39350
				10	0,25441	0,29202	0,32819	0,35412	0,36792

Based on the results shown in Table 1, it can be concluded that compared to Winkler's model, the introduction of a shear layer (coefficient G_p) in Pasternak's model leads to a decrease in the normalized value of vertical displacement. On the other hand, the introduction of another row of springs (coefficient k_b) in Kerr's model leads to an increase in the normalized value of vertical displacement compared to Pasternak's model. In order to get a clear insight on the above mentioned effect of Winkler/Pasternak/Kerr coefficients, Figure 3a shows the comparative normalized displacement values for FG plate a/h=10 and p=5.

Figure 3b shows the influence of the volume fraction of constituents in FGM on vertical displacement. It should be kept in mind when p=0, the plate is homogenous, made of ceramics. On the other side, theoretically, when $p=\infty$ the plate becomes homogenous again, made of metal, although the plate can be considered homogenous even when p>10. From this it clearly follows that for 0<p<10 it is obtained by the FG of the plate. Analysing the diagram, one can clearly see the growth trend of the normalized displacement value with the increase in the volume fraction of metal in the FG plate.



a) b)

Figure 3 Comparative results of normalized values of displacement: a) different models of elastic foundation, b) different volume fraction of constituents in FGM

CONCLUSIONS

The paper highlights the significance and relevance of research on modern composite materials, with a particular focus on functionally graded materials. It underscores the broad range of engineering fields and automotive sectors where FGM are applied. The interaction of FGM plates with an elastic foundation was examined. The fundamental relations defining this interaction are presented by using Winkler's, Pasternak's, and Kerr's mathematical models. Assuming small strains, the kinematic relations between displacement and strain were established using the linear theory of elasticity involves the assumed forms of the displacement components based on higher-order shear deformation theory. Using the strain energy of the plate and the elastic foundation, the equilibrium equations were derived through the principle of minimum potential energy. A procedure for analytically solving these equilibrium equations was developed by applying Navier's assumed displacement forms for a simply supported rectangular FGM plate. These theoretical results were then utilized to obtain numerical results. The analysis of bending in FGM plates resting on an elastic foundation led to the following conclusions:

- the implementation of the shear layer in the Pasternak model results in a reduction of the normalized vertical displacement value when compared to the Winkler model. Conversely, the addition of an extra row of springs in the Kerr model leads to an increase in the normalized vertical displacement value compared to the Pasternak model.
- with the increase in volume fraction of metal in FGM, the value of normalized vertical displacement increases.

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