The structure of shift-invariant subspaces of Sobolev spaces

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This paper is dedicated to Academician Vasilii Sergeevich Valadimir on the occasion of the 100th anniversary of his birth.

Abstract. We analyze shift-invariant spaces V_s, subspaces of Sobolev spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, generated by the set of generators φ_i, $i \in I$, I is countable at most, by the use of range functions and characterize Bessel sequences, frames and Riesz basis of such spaces. Also V_s are described through Gramians and their direct sum decompositions. We show that an $f \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$ belongs to V_s if and only if its Fourier transform has the form $\widehat{f} = \sum_{i\in I} f_i g_i$, $f_i = \widehat{\varphi}_i \in L^2_s(\mathbb{R}^n)$, $\{\varphi_i(\cdot + k) : k \in \mathbb{Z}^n, i \in I\}$ is a frame and $g_i = \sum_{k\in\mathbb{Z}^n} a^i_k \mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle}$, with $(a^i_k)_{k\in\mathbb{Z}^n} \in \ell^2$. Moreover, connecting two different approaches to shift-invariant spaces V_s and $\mathcal{V}^2_s$, $s > 0$, under the assumption that the finite number of generators belongs to $H^s \cap L^2_s$, we give the characterization of elements in V_s through the expansions with coefficients in ℓ^2_s. The corresponding assertion holds for the intersections of such spaces and their duals in the case when the generators are elements of $\mathcal{S}(\mathbb{R}^n)$. Then, we show that $\cap_{s>0} V_s$ is the space consisting of functions which Fourier transforms equal products of functions in $\mathcal{S}(\mathbb{R}^n)$ and periodic smooth functions. The appropriate assertion is obtained for $\cup_{s>0} V_{-s}$.

Key Words and Phrases: Sobolev space; shift-invariant space; range function; frame; Bessel family.

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1 Introduction

Following the range function approach used in Bownik [8] - [10], based on [6], [7], [12], [19] (see also [20]), in this paper we investigate the structure of the shift-invariant subspaces of Sobolev spaces $H^s = H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, denoted as V_s, generated by at most countable family of generators, elements of $\mathcal{A}_s \subset H^s$ so that V_s is the closure of the span of integer translations of functions in $\mathcal{A}_s$, $s \in \mathbb{R}$. In the case $s = 0$, one arrives to L^2-theory. For the $L^2(\mathbb{R}^n)$, Bownik [8] gave a comprehensive analysis of the space V ($V = V_0$).

The analysis of shift-invariant spaces has a very rich foundation and historical background. It is extended in various directions as to shift-invariant (locally) compact groups (see [9], [13], [16]), shift-invariant subspaces of $L^{p,q}(\mathbb{R}^{n+1})$-spaces (see [14]) and to the powers of shift-invariant operators determining generators of V (see [2], [3]). Note that another approach, with the frames consisting of the finite set of generators and expansions with coefficients in ℓ^p-sequence spaces, $p \geq 1$, was developed in [4], [5] (see also references therein) and in [17] with the weighted ℓ^p sequences, ℓ^p_s sequences in [17]. This approach is connected with the one used in this paper in case $p = 2$.

In the first part of the paper we transfer the results of [8] from L^2-framework to H^s, $s \in \mathbb{R}$, and give the structure of elements in the shift-invariant spaces $V_s \subset H^s$, $s \in \mathbb{R}$, through the Fourier transform as it is stated in the Abstract. Our main results are given in the second part of the paper, in Section 5, where we compare results of the approach of Aldroubi and collaborators, cf. [4], [5], [17], with the ones related to the approach of [8] presented in this paper. Note that in case $s = 0$, the mapping $\mathcal{T} : L^2 \to L^2(\mathbb{T}^n, \ell^2)$, $\varphi \mapsto (t \mapsto (\widehat{\varphi}(t+k))_{k\in\mathbb{Z}^n}, t \in [0,1)^n)$, $\widehat{\varphi} = \mathcal{F}(\varphi)$, considered in [8], can be changed by $\widetilde{\mathcal{T}}\varphi \mapsto (t \mapsto (\varphi(t+k))_k, t \in [0,1)^n)$. This transform commute with the translation and implies another development of the theory. In the case $s \neq 0$ the definition of $\mathcal{T}_s, s \neq 0$ of this paper is the only possible transfer from the Sobolev spaces to the corresponding weighted sequence spaces. We also note that in this paper we give an analysis of shift-invariant spaces in the framework of distributions, in H^s, $s < 0$.

The paper is organized as follows. In Section 2 we follow the definitions of [8] now applied to subsets and shift-invariant subspaces of H^s, $s \in \mathbb{R}$. We define the mapping $\mathcal{T}_s$ which, for a.e. $t \in [0,1)^n$, maps an $f \in H^s$ to a sequence $(\widehat{g}(t+k)/(1+|k|^2)^{s/2})_{k\in\mathbb{Z}^n}$; $f \in H^s$ and $g \in L^2(\mathbb{R}^n)$ are connected by the relation $(1 - \frac{\Delta}{4\pi^2})^{s/2} f = g$ (Δ is Laplacian), $s \in \mathbb{R}$. For $f \in H^s$, $(1 - \frac{\Delta}{4\pi^2})^{s/2} f$ is defined as a Fourier multiplier $(1 - \frac{\Delta}{4\pi^2})^{s/2} f = \mathcal{F}^{-1}\big((1 + |\cdot|^2)^{s/2}\widehat{f}(\cdot)\big)$. With this, we are able to extend notions and theorems in [8] revisiting the proofs from that paper. Since the theory is complex enough in the L^2-case, we carefully analyse the range function J_s acting on spaces $V_s = \overline{\mathrm{span}}\big\{\text{shifts of elements in } \mathcal{A}_s \subset H^s\big\}$. Frames, Riesz basis and Bessel families are analysed in Section 3. Section 4 is devoted to the orthogonal sum decomposition of spaces V_s. We give in Section 5 the structure of spaces V_s, $s \in \mathbb{R}$. Especially for $s > 0$, we connect V_s-spaces with the

$\mathcal{V}_s^2$-spaces from [17] motivated by the results of Aldroubi and his collaborators, assuming that the finite number of generators belong to $H^s \cap L^2_s$ (with appropriate decrease at infinity) and characterize the elements of V_s through the expansions with coefficients in corresponding weighted sequence spaces ℓ^2_s under the assumption that $\mathcal{V}_s^2$, $s>0$ is closed in L^2_s. We have proved that the assumption $s>1/2$ implies $V_s=\mathcal{V}_s^2$ so that we have a new characterization of elements in V_s by the coefficients in ℓ^2_s. Even for $s=0$, our result seems new one. The corresponding corollaries related to the intersections of V_s-spaces $s>0$ and their duals are also given.

2 Notations and basic assertions

Throughout the paper we assume $s\in\mathbb{R}$ and $\mathbb{T}^n$ stands for $[0,1)^n$. Notation $T_y f(\cdot)=f(\cdot-y)$, means the shift by $y\in\mathbb{R}^n$. Define the Fourier transform $\widehat{f}$ of an integrable function f by $\mathcal{F}f(t)=\widehat{f}(t)=\int_{\mathbb{R}^n} f(x)\mathrm{e}^{-2\pi\sqrt{-1}\langle x,t\rangle}\,\mathrm{d}x$, $t\in\mathbb{R}^n$ ($\mathcal{F}^{-1}f(t)=\widehat{f}(-t)$), where $\langle x,t\rangle=\sum_{i=1}^n x_i t_i$, $x,t\in\mathbb{R}^n$. Note that the Fourier transform without 2π in the exponent is used in [5] and [17]. Let $\mu_s(\cdot)=(1+|\cdot|^2)^{s/2}$. Next,

$$\ell^2_s=\ell^2_s(\mathbb{Z}^n)=\Big\{(c_k)_{k\in\mathbb{Z}^n}\,:\,\sum_{k\in\mathbb{Z}^n}|c_k|^2\mu_s^2(k)<+\infty\Big\},\quad s\in\mathbb{R},$$

with the scalar product $\langle (c_k)_{k\in\mathbb{Z}^n},(d_k)_{k\in\mathbb{Z}^n}\rangle_{\ell^2_s}=\sum_{k\in\mathbb{Z}^n}c_k\overline{d}_k\mu_s^2(k)$. Recall, (cf. [1], [15]),

$$H^s=\Big\{f\in\mathcal{S}'(\mathbb{R}^n)\,:\,(1+|\cdot|^2)^{s/2}\widehat{f}(\cdot)\in L^2(\mathbb{R}^n)\\ \Leftrightarrow \|f\|_{H^s}=\Big(\int_{\mathbb{R}^n}|\widehat{f}(t)|^2\mu_s^2(t)\,\mathrm{d}t\Big)^{1/2}<+\infty\Big\},\quad s\in\mathbb{R},$$

and $\langle f,g\rangle_{H^s}=\int_{\mathbb{R}^n}\widehat{f}(t)\overline{\widehat{g}}(t)\mu_s^2(t)\,\mathrm{d}t$. Note, $L^2_s=L^2_s(\mathbb{R}^n)=\mathcal{F}(H^s)$, i.e. $f\in L^2_s$ if and only if $\widehat{f}\in H^s$, $s\in\mathbb{R}$. From the distribution theory we recall that the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consisting of rapidly decreasing functions, is dense in H^s, $s\in\mathbb{R}$. Also, that the pseudo-differential operator

$$(1-\tfrac{\Delta}{4\pi^2})^{s/2}f(x)=\mathcal{F}^{-1}\big(\widehat{f}(t)\mu_s(t)\big)(x),\quad x\in\mathbb{R}^n,$$

is an isometry of Sobolev spaces:

$$\Big(1-\frac{\Delta}{4\pi^2}\Big)^{s/2}:H^{m+s}\to H^m,\quad m,s\in\mathbb{R}.$$

The Hilbert space $H(\mathbb{T}^n, \ell^2_s)$ consists of all vector valued measurable square integrable functions $F : \mathbb{T}^n \to \ell^2_s$ with the norm

$$\|F\|_{H(\mathbb{T}^n,\ell^2_s)} = \left(\int_{\mathbb{T}^n} \|F(t)\|^2_{\ell^2_s} \, \mathrm{d}t \right)^{\frac{1}{2}} < +\infty.$$

In the case $s = 0$, it is denoted as $L^2(\mathbb{T}^n, \ell^2)$. Let $\mathcal{A} \subset L^2(\mathbb{R}^n)$. We denote

$$\mathcal{A}_s = \left\{ \varphi \in \mathcal{S}'(\mathbb{R}^n) : \widehat{\varphi} = \widehat{\psi}\mu_{-s} \text{ for some } \psi \in \mathcal{A} \right\}$$

and $E_s(\mathcal{A}_s) = \{T_k\varphi : \varphi \in \mathcal{A}_s, k \in \mathbb{Z}^n\}$. Clearly, $E_s(\mathcal{A}_s)$ is a subset of H^s. Denote by I a finite set or $\mathbb{N}$ and put $\mathcal{A}_I = \{\psi_i : i \in I\} \subset L^2$. We use notation $\mathcal{A}_{I,s} = \mathcal{A}_s$ if elements of $\mathcal{A}_s$ are φ_i, $i \in I$. If $I = \{1, 2, \dots, r\}$, we use notation $\mathcal{A}_{r,s}$ ($\mathcal{A}_r$ if $s = 0$) instead of notations $\mathcal{A}_{I,s}$ ($\mathcal{A}_I$ if $s = 0$).

It is said that a closed subspace $V_s \subset H^s$ is shift-invariant if $\varphi \in V_s$ implies $T_k\varphi \in V_s$, for any $k \in \mathbb{Z}^n$. For any subset $\mathcal{A}_s \subset H^s$, let

$$\begin{aligned} S_s(\mathcal{A}_s) =& \overline{\operatorname{span}}\left\{T_k\varphi : \varphi \in \mathcal{A}_s, k \in \mathbb{Z}^n\right\} \\ =& \overline{\operatorname{span}}\left\{(1 - \tfrac{\Delta}{4\pi^2})^{-s/2} T_k\psi : \psi \in \mathcal{A}, k \in \mathbb{Z}^n\right\}, \end{aligned}$$

where, for a given set M, $\overline{\operatorname{span}}(M)$ denotes the closure of all the linear combinations of elements in M. It is a shift-invariant space generated by $\mathcal{A}_s$. If $V_s = S_s(\{\varphi\})$, it is called a principal shift-invariant space (PSI) and $V_s = S_s(\{\varphi_1, \varphi_2, \dots, \varphi_r\})$ is called a finitely generated shift-invariant space (FSI). Note that if $s = 0$, then we have notation $S(\mathcal{A})$ and $E(\mathcal{A})$, as in [8].

Following the definition of the mapping $\mathcal{T} : L^2 \to L^2(\mathbb{T}^n, \ell^2)$ ([8]), we define $\mathcal{T}_s : H^s \to H(\mathbb{T}^n, \ell^2_s)$ ($\mathcal{T} = \mathcal{T}_s$, for $s = 0$) by

$$\mathcal{T}_s\varphi(t) = \left(\frac{\widehat{\psi}(t+k)}{\mu_s(k)} \right)_{k\in\mathbb{Z}^n}, \; t \in \mathbb{T}^n, \; \varphi \in H^s, \left(1 - \frac{\Delta}{4\pi^2}\right)^{s/2} \varphi = \psi (\in L^2(\mathbb{R}^n)).$$

Lemma 2.1. *Let $s \in \mathbb{R}$.*

a) $\mathcal{T}_s : H^s \to H(\mathbb{T}^n, \ell^2_s)$.

b) The following diagram of isometries commutes

$$\begin{array}{ccc} L^2 & \xrightarrow{\mathcal{T}} & L^2(\mathbb{T}^n, \ell^2) \\ \downarrow \alpha_s & & \downarrow \beta_s \\ H^s & \xrightarrow{\mathcal{T}_s} & H(\mathbb{T}^n, \ell^2_s), \end{array}$$

where $\alpha_s(g) = \mathcal{F}^{-1}\big(\widehat{g}(\cdot)/\mu_s(\cdot)\big)$ and $\beta_s((f_k(\cdot))_{k\in\mathbb{Z}^n}) = \big(\frac{f_k(\cdot)}{\mu_s(k)}\big)_{k\in\mathbb{Z}^n}$; in particular, $\beta_s\big(\big(\widehat{g}(\cdot + k)\big)_{k\in\mathbb{Z}^n}\big) = \big(\frac{\widehat{g}(\cdot+k)}{\mu_s(k)}\big)_{k\in\mathbb{Z}^n}$.

c) Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{T}_s T_j\varphi(\cdot) = \mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,j\rangle}\mathcal{T}_s\varphi(\cdot)$, $j \in \mathbb{Z}^n$.

Proof. *a*) We prove the assertion for an arbitrary function $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then, by the density arguments, the assertion holds for all the functions in H^s. So, let $\widehat{\varphi} = \widehat{\psi}\mu_{-s}$. Then

$$\|\mathcal{T}_s\varphi\|^2_{H(\mathbb{T},\ell^2_s)} = \int_{\mathbb{T}^n} \|\mathcal{T}_s\varphi(t)\|^2_{\ell^2_s}\,\mathrm{d}t = \int_{\mathbb{T}^n} \left\| \left(\frac{\widehat{\psi}(t+k)}{\mu_s(k)} \right)_{k\in\mathbb{Z}^n} \right\|^2_{\ell^2_s} \mathrm{d}t$$
$$= \|\widehat{\psi}\|^2_{L^2} = \int_{\mathbb{R}^n} |\widehat{\varphi}(t)|^2 \mu^2_s(t)\,\mathrm{d}t = \|\varphi\|^2_{H^s}.$$

b) This assertion is clear.

c) Since $\widehat{T_j\varphi}(\cdot) = \mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,j\rangle}\widehat{\varphi}(\cdot)$ it follows

$$\mathcal{T}_s T_j\varphi(\cdot) = \left(\frac{\widehat{\psi}(\cdot + k - j)}{\mu_s(k)} \right)_{k\in\mathbb{Z}^n} = \mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,j\rangle}\mathcal{T}_s\varphi(\cdot), \quad j \in \mathbb{Z}^n.$$

□

Remark 2.1. *Instead of* $\mathcal{T} : L^2 \to L^2(\mathbb{T}^n, \ell^2)$, $\varphi \mapsto ((t \mapsto (\widehat{\varphi}(t+k))_k, t \in \mathbb{T}^n)$, *in [8], one can use the mapping* $\widetilde{\mathcal{T}}\varphi \mapsto (t \mapsto (\varphi(t+k))_k, t \in \mathbb{T}^n)$, *which commutes with the translation* T_j, $j \in \mathbb{Z}^n$ *(with appropriate consequences on the theory). For the weighted* L^2_s *- spaces one can use a weighted version of* $\widetilde{\mathcal{T}}_s : \varphi \mapsto (t \mapsto (\frac{\varphi(t+k)}{\mu_s(k)})_k, t \in \mathbb{T}^n)$, *with the commutation of translation and* $\widetilde{\mathcal{T}}_s$. *Then, we have that* $\widetilde{\mathcal{T}}_s(\varphi) = \mathcal{T}_s(\mathcal{F}^{-1}\varphi)$, $\varphi \in L^2_s(\mathbb{R}^n)$.

Next, we reintroduce several notions following [8]. Concerning the measurability, since H^s is separable, strong and weak measurability are equivalent. So in both cases a sequence of measurable functions, if converges, it converges to a measurable function. With this, we recall the definitions and propositions related to the range function.

A mapping $J_s : \mathbb{T}^n \to \{\text{closed subspaces of } \ell^2_s\}$ is called the range function. It is measurable if the associated orthogonal projections $P_{J_s}(t) : \ell^2_s \to J_s(t)$, $t \in \mathbb{T}^n$, are weakly operator measurable; i.e., $t \mapsto \langle P_{J_s}(t)c, d\rangle_{\ell^2_s}$ is a measurable scalar function for each $c, d \in \ell^2_s$. For a given range function J_s (not necessarily measurable), the space

$$M_{J_s} = \{F \in H(\mathbb{T}^n, \ell^2_s) \,:\, F(t) \in J_s(t) \text{ for a.e. } t \in \mathbb{T}^n\}$$

is a closed subspace of $H(\mathbb{T}^n, \ell^2_s)$. If $M_{J_s} = M_{K_s}$ for some measurable range functions J_s and K_s with associated orthogonal projections P_{J_s} and Q_{K_s}, respectively, then $J_s(t) = K_s(t)$ for a.e. $t \in \mathbb{T}^n$. The proof is the same as in [8]. Suppose that J_s is a measurable range function. Let $\mathcal{P}_s$ be the projection

$$H(\mathbb{T}^n, \ell^2_s) \ni F \mapsto \mathcal{P}_s(F) \in M_{J_s}$$

so that for a.e. $t \in \mathbb{T}^n$, $\big(\mathcal{P}_s F\big)(t) \in J_s(t)$. Let

$$P_{J_s} : \mathbb{T}^n \to \big\{\text{space of projections of } \ell^2_s \text{ onto closed subspaces of } \ell^2_s\big\},$$

so that $P_{J_s}(t) : \ell^2_s \to J_s(t)$, for a.e. $t \in \mathbb{T}^n$.

The next assertions are generalizations of the corresponding ones in [8]. Their proofs for $s \neq 0$ are similar as in the case $s = 0$ and they are omitted.

Theorem 2.1. *Let J_s be a measurable range function.*

a) If $F \in H(\mathbb{T}^n, \ell^2_s)$, then $\big(\mathcal{P}_s F\big)(t) = P_{J_s}(t)\big(F(t)\big)$ for a.e. $t \in \mathbb{T}^n$.

b) A closed subspace $V_s \subset H^s$ is shift-invariant if and only if there exists some range function J_s such that

$$V_s = \big\{\varphi \in H^s \ : \ \mathcal{T}_s\varphi(t) \in J_s(t) \text{ for a.e. } t \in \mathbb{T}^n\big\}.$$

The correspondence between V_s and J_s is one-to-one under the convention that the range functions are identified if they are equal over $\mathbb{T}^n$ a.e. Furthermore, if $V_s = S_s(\mathcal{A}_{I,s})$ for some $\mathcal{A}_{I,s} \subset H^s$, then

$$J_s(t) = \overline{\operatorname{span}}\{\mathcal{T}_s\varphi(t) : \varphi \in \mathcal{A}_{I,s}\} \ \text{ for a.e. } t \in \mathbb{T}^n.$$

c) In the case that J_s is not measurable, there exists a unique measurable range function K_s such that $K_s(t) \subseteq J_s(t)$ for a.e. $t \in \mathbb{T}^n$, and $M_{J_s} = M_{K_s}$.

Recall that the spectrum of V_s is given by

$$\sigma(V_s) = \{t \in \mathbb{T}^n : J_s(t) \neq \{\mathbf{0}\}\}.$$

3 Bessel families and frames

We refer to [11] or any other book with the frame theory for the definitions of a Bessel family, a Riesz basis and a frame in a Hilbert space. Recall that, X is a fundamental frame in a Hilbert space $\mathcal{H}$ if span(X) is dense in $\mathcal{H}$.

We follow [8] and give analogue results related to frames, Bessel families and Riesz basis. We give assertions with the sketch of the proofs or without them since the arguments are already given in [8]. The next lemma is needed for the characterization of frames and quoted families in V_s. We give only a sketch of the proof in order to avoid repetition of all the arguments already given in [8].

Lemma 3.1. *a) Let $E_s(\mathcal{A}_{I,s})$ be a Bessel family. Then, for all $f \in \mathcal{A}_{I,s}$, one has*

$$\sum_{\varphi \in \mathcal{A}_{I,s}} \sum_{k \in \mathbb{Z}^n} \big|\langle T_k\varphi, f\rangle_{H^s}\big|^2 = \sum_{\varphi \in \mathcal{A}_{I,s}} \int_{\mathbb{T}^n} \Big|\big\langle \mathcal{T}_s\varphi(t), \mathcal{T}_s f(t)\big\rangle_{\ell^2_s}\Big|^2 \,\mathrm{d}t.$$

b) $\langle T_k\varphi, f\rangle_{H^s} = \langle T_k\psi, g\rangle_{L^2}$.

Proof. *a*) With $\psi, g \in L^2$ such that $\widehat{\varphi} = \widehat{\psi}\mu_{-s}$ and $\widehat{f} = \widehat{g}\mu_{-s}$, we have

$$
\begin{aligned}
&\sum_{\varphi\in\mathcal{A}_{I,s}} \sum_{k\in\mathbb{Z}^n} \left|\langle T_k\varphi, f\rangle_{H^s}\right|^2 \\
&\qquad = \sum_{\varphi\in\mathcal{A}_{I,s}} \sum_{k\in\mathbb{Z}^n} \left| \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi\sqrt{-1}\langle k,t\rangle} \widehat{\varphi}(t)\overline{\widehat{f}}(t)\mu_s^2(t)\,\mathrm{d}t \right|^2 \\
&\qquad = \sum_{\psi\in\mathcal{A}_I} \sum_{k\in\mathbb{Z}^n} \left| \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi\sqrt{-1}\langle k,t\rangle} \widehat{\psi}(t)\overline{\widehat{g}}(t)\,\mathrm{d}t \right|^2 \\
&\qquad = \sum_{\psi\in\mathcal{A}_I} \sum_{k\in\mathbb{Z}^n} \left| \sum_{j\in\mathbb{Z}^n} \int_{\mathbb{T}^n} \mathrm{e}^{-2\pi\sqrt{-1}\langle k,t\rangle} \widehat{\psi}(t+j)\overline{\widehat{g}}(t+j)\,\mathrm{d}t \right|^2 \\
&\qquad = \sum_{\psi\in\mathcal{A}_I} \sum_{k\in\mathbb{Z}^n} \left| \int_{\mathbb{T}^n} \mathrm{e}^{-2\pi\sqrt{-1}\langle k,t\rangle} \sum_{j\in\mathbb{Z}^n} \widehat{\psi}(t+j)\overline{\widehat{g}}(t+j)\,\mathrm{d}t \right|^2.
\end{aligned}
$$

Recall, if $A(t) = \sum_{j\in\mathbb{Z}^n} \widehat{\psi}(t+j)\overline{\widehat{g}}(t+j)$, $t \in \mathbb{T}^n$, then the coefficients of the periodic function $A(t) = A(t+\alpha)$, $t \in \mathbb{T}^n$, $\alpha \in \mathbb{Z}^n$, are determined by $c_k = \int_{\mathbb{T}^n} \mathrm{e}^{-2\pi\sqrt{-1}\langle k,t\rangle} A(t)\,\mathrm{d}t$, $k \in \mathbb{Z}^n$. Also, $\|A(t)\|^2_{L^2(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n} |c_k|^2$. We now apply these arguments to obtain

$$
\begin{aligned}
\sum_{\varphi\in\mathcal{A}_{I,s}} \sum_{k\in\mathbb{Z}^n} \left|\langle T_k\varphi, f\rangle_{H^s}\right|^2 &= \sum_{\psi\in\mathcal{A}_I} \int_{\mathbb{T}^n} \left| \sum_{j\in\mathbb{Z}^n} \widehat{\psi}(t+j)\overline{\widehat{g}}(t+j) \right|^2 \mathrm{d}t \\
&= \sum_{\psi\in\mathcal{A}_I} \int_{\mathbb{T}^n} \left| \sum_{j\in\mathbb{Z}^n} \frac{\widehat{\psi}(t+j)}{\mu_s(j)} \cdot \frac{\overline{\widehat{g}}(t+j)}{\mu_s(j)} \mu_s^2(j) \right|^2 \mathrm{d}t \\
&= \sum_{\varphi\in\mathcal{A}_{I,s}} \int_{\mathbb{T}^n} \left| \left\langle \mathcal{T}_s\varphi(t), \mathcal{T}_s f(t) \right\rangle_{\ell^2_s} \right|^2 \mathrm{d}t.
\end{aligned}
$$

This completes the proof of *a*).

b) The assertion follows from

$$
\begin{aligned}
\langle T_k\varphi, f\rangle_{H^s} &= \int_{\mathbb{R}^n} \widehat{T_k\varphi}(t)\overline{\widehat{f}}(t)\mu_s^2(t)\,\mathrm{d}t = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi\sqrt{-1}\langle t,k\rangle} \widehat{\varphi}(t)\overline{\widehat{f}}(t)\mu_s^2(t)\,\mathrm{d}t \\
&= \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi\sqrt{-1}\langle t,k\rangle} \widehat{\psi}(t)\overline{\widehat{g}}(t)\,\mathrm{d}t = \int_{\mathbb{R}^n} \widehat{T_k\psi}(t)\overline{\widehat{g}}(t)\,\mathrm{d}t = \langle \widehat{T_k\psi}, \widehat{g}\rangle_{L^2} \\
&= \langle T_k\psi, g\rangle_{L^2}.
\end{aligned}
$$

□

With this two assertions we have:

Lemma 3.2. *Let $s, s_0 \in \mathbb{R}$. Then $\{\mathcal{T}_s\varphi(t) : \varphi \in \mathcal{A}_{I,s}\} \subset \ell^2_s$ is a frame for $J_s(t)$ or a Riesz basis with bounds A, B or a Bessel family with bound B for a.e. $t \in \mathbb{T}^n$, if and only if $\{\mathcal{T}_{s_0}\varphi(t) : \varphi \in \mathcal{A}_{I,s_0}\}$ is a frame or a Riesz basis for $J_{s_0}(t)$ with bounds A, B or a Bessel family with bound B for a.e. $t \in \mathbb{T}^n$, respectively. Moreover, $\{\mathcal{T}_s\varphi(t) : \varphi \in \mathcal{A}_{I,s}\}$ is a fundamental frame for a.e. $t \in \mathbb{T}^n$, if and only if $\{\mathcal{T}_{s_0}\varphi(t) : \varphi \in \mathcal{A}_{I,s_0}\}$ is a fundamental frame for a.e. $t \in \mathbb{T}^n$.*

Since s and s_0 in the previous lemma are two arbitrary real numbers, we reformulate the previous lemma into the next theorem.

Theorem 3.1. *$E_s(\mathcal{A}_{I,s})$ is a frame or a Riesz basis for $V_s = S_s(\mathcal{A}_{I,s})$ with bounds A, B or a Bessel family with bound B for every $s \in \mathbb{R}$ (equivalently, by the previous two lemmas, for some $s \in \mathbb{R}$), if and only if $\{\mathcal{T}_s\varphi(t) : \varphi \in \mathcal{A}_{I,s}\} \subset \ell^2_s$ is a frame for $J_s(t)$ with bounds A, B or a Bessel family with bound B for a.e. $t \in \mathbb{T}^n$, for every $s \in \mathbb{R}$ (equivalently for some $s \in \mathbb{R}$), respectively. Moreover, $E_s(\mathcal{A}_{I,s})$ is a fundamental frame for every $s \in \mathbb{R}$, if and only if $\{\mathcal{T}_s\varphi(t) : \varphi \in \mathcal{A}_{I,s}\} \subset \ell^2_s$ is a fundamental frame for a.e. $t \in \mathbb{T}^n$, for every $s \in \mathbb{R}$.*

Let $\mathcal{A}_{I,s} = \{\varphi_i : i \in I\} \subset H^s$. Set

$$z^i_s = (z^i_s(k))_{k\in\mathbb{Z}^n} \subset \ell^2_s, \quad i \in I, \tag{3.1}$$

where $z^i_s(k)$ is defined by $z^i_s(k) = \frac{\widehat{\psi}_i(t+k)}{\mu_s(k)}$, for fixed $t \in \mathbb{T}^n$, and $\psi_i \in L^2$ such that $\widehat{\varphi}_i = \widehat{\psi}_i\mu_{-s}$, $i \in I$.

Let $(z^i_s)_{i\in I}$ be given. One defines operator N_s by

$$N_s(c) = \left(\sum_{i\in I} c_i z^i_s(k)\right)_{k\in\mathbb{Z}^n}, \tag{3.2}$$

for sequence $c = (c_i)_{i\in I}$ with compact support (this means that only finitely many members are different from zero), and then extend it as a continuous mapping $N_s : \ell^2(I) \to \ell^2_s(\mathbb{Z}^n)$. It has the adjoint operator $N^*_s : \ell^2_s(\mathbb{Z}^n) \to \ell^2(I)$ given by

$$N^*_s(a) = \left(\langle a, z^i_s\rangle_{\ell^2_s}\right)_{i\in I}, \quad a = (a_k)_{k\in\mathbb{Z}^n} \in \ell^2_s(\mathbb{Z}^n). \tag{3.3}$$

It is evident that N_s is bounded if and only if N^*_s is bounded if and only if $(z^i_s)_{i\in I}$ is a Bessel family. Thus, $\|N^*_s\|^2 \leq B$ implies that $\{z^i_s : i \in I\}$ is a Bessel family with the same constant B.

The Gramian G_s of the system $\{z^i_s : i \in I\}$ (see (3.1)), defined by $G_s = N^*_s N_s$, defines a mapping $G_s : \ell^2(I) \to \ell^2(I)$, and its dual Gramian $\widetilde{G}_s : \ell^2_s(\mathbb{Z}^n) \to \ell^2_s(\mathbb{Z}^n)$ is defined by $\widetilde{G}_s = N_s N^*_s$, where N_s and N^*_s are given by (3.2) and (3.3), respectively.

Remark 3.1. *It is evident that* G_s *and* $\widetilde{G}_s$ *are self-adjoint and that* $\|N_s\|^2 = \|N_s^*\|^2 = \|G_s\| = \|\widetilde{G}_s\|$.

Remark 3.2. *Let* $\{e_i : i \in I\}$ *be the usual basis of* $\ell^2(I)$. *Since* $\langle G_s e_i, e_j\rangle_{\ell^2} = \langle N_s e_i, N_s e_j\rangle_{\ell^2_s} = \langle z_s^i, z_s^j\rangle_{\ell^2_s}$, $i, j \in I$, *and* $\langle \widetilde{G}_s e_k, e_\ell\rangle_{\ell^2_s} = \langle N_s^* e_k, N_s^* e_\ell\rangle_{\ell^2} = \sum\limits_{i\in I} z_s^i(k)\overline{z_s^i}(\ell)$, *we have*

$$G_s(t) = \Big(\big\langle \mathcal{T}_s\varphi_i(t), \mathcal{T}_s\varphi_j(t)\big\rangle_{\ell^2_s}\Big)_{i,j\in I} = \bigg(\sum_{k\in\mathbb{Z}^n} \widehat{\psi}_i(t+k)\overline{\widehat{\psi}_j}(t+k)\bigg)_{i,j\in I}$$

and for the dual Gramian

$$\widetilde{G}_s(t) = \bigg(\sum_{i\in I} \frac{\widehat{\psi}_i(t+k)}{\mu_s(k)} \cdot \frac{\overline{\widehat{\psi}_i}(t+\ell)}{\mu_s(\ell)}\bigg)_{k,\ell\in\mathbb{Z}^n}.$$

Theorem 3.2. *Let* $\mathcal{A}_{I,s} = \{\varphi_i : i \in I\} \subset H^s$.

a) $E_s(\mathcal{A}_{I,s})$ *is a Bessel family with the bound* B *if and only if*

$$\operatorname*{esssup}_{t\in\mathbb{T}^n} \|G_s(t)\|_{\ell^2} \le B$$

if and only if $\operatorname*{esssup}\limits_{t\in\mathbb{T}^n} \|\widetilde{G}_s(t)\|_{\ell^2_s} \le B$.

b) $E_s(\mathcal{A}_{I,s})$ *is a frame with positive constants* A, B *if and only if*

$$A\|a\|^2_{\ell^2_s} \le \langle \widetilde{G}_s(t)a, a\rangle_{\ell^2_s} \le B\|a\|^2_{\ell^2_s}, \tag{3.4}$$

where $a \in \operatorname{span}\{\mathcal{T}_s\varphi_i(t) : i \in I\}$ *for a.e.* $t \in \mathbb{T}^n$, *if and only if*

$$\sigma(\widetilde{G}_s(t)) \subseteq \{0\} \cup [A, B] \quad \text{for a.e. } t \in \mathbb{T}^n. \tag{3.5}$$

Furthermore, $E_s(\mathcal{A}_{I,s})$ *is a fundamental frame with constants* A, B *if and only if* $\sigma(\widetilde{G}_s(t)) \subseteq [A, B]$ *for a.e.* $t \in \mathbb{T}^n$.

c) $E_s(\mathcal{A}_{I,s})$ *is a Riesz family with constants* A, B *if and only if*

$$A\|c\|^2_{\ell^2} \le \langle G_s(t)c, c\rangle_{\ell^2} \le B\|c\|^2_{\ell^2}, \;\; c \in \ell^2(I) \text{ for a.e. } t \in \mathbb{T}^n, \tag{3.6}$$

if and only if

$$\sigma(G_s(t)) \subseteq [A, B] \text{ for a.e. } t \in \mathbb{T}^n. \tag{3.7}$$

Furthermore, $E_s(\mathcal{A}_{I,s})$ *is a Riesz basis if and only if* (3.7) *holds and* $0 \notin \sigma(\widetilde{G}_s(t))$ *for a.e.* $t \in \mathbb{T}^n$.

Proof. We will use the analysis similar to that in [8], for $s = 0$.

a) The assertion follows from Theorem 3.1 and Remarks 3.1 and 3.2.

b) Since

$$\langle \widetilde{G}_s(t)a, a\rangle_{\ell^2_s} = \langle N_s^* a, N_s^* a\rangle_{\ell^2} = \sum_{i\in I} \left|\langle a, z_s^i\rangle_{\ell^2_s}\right|^2, \quad a \in \ell^2_s(\mathbb{Z}^n),$$

by Theorem 3.1, the first equivalence is obtained. Since $\widetilde{G}_s(t)$ is self-adjoint operator, it follows that

$$\ker \widetilde{G}_s(t) \oplus \overline{\operatorname{rank} \widetilde{G}_s(t)} = \ell^2_s(\mathbb{Z}^n).$$

Furthermore, $\ker \widetilde{G}_s(t) = \ker N_s^* = J_s(t)^\perp$, where J_s is the range function of $S_s(\mathcal{A}_{I,s})$, implies $\operatorname{rank} \widetilde{G}_s(t) = J_s(t)$ for a.e. $t \in \mathbb{T}^n$. The equivalence (3.4)⇔(3.5) is obtained considering the restriction of operator $\widetilde{G}_s(t)$ on $J_s(t)$. Additionally, if $\ker \widetilde{G}_s(t) = J_s(t)^\perp = \{\mathbf{0}\}$ for a.e. $t \in \mathbb{T}^n$, then $E_s(\mathcal{A}_{I,s})$ is a fundamental frame.

c) The first equivalence follows from

$$\langle G_s c, c\rangle_{\ell^2} = \langle N_s c, N_s c\rangle_{\ell^2_s} = \left\| \sum_{i\in I} c_i z_s^i \right\|^2_{\ell^2_s}, \quad c = (c_i)_{i\in I} \in \ell^2(I),$$

and Theorem 3.1. The equivalence (3.6)⇔(3.7) is due to the fact that G_s is a non-negative definite operator. Additionally, if $\ker \widetilde{G}_s(t) = J_s(t)^\perp = \{\mathbf{0}\}$; i.e., $0 \notin \sigma(\widetilde{G}_s(t))$ for a.e. $t \in \mathbb{T}^n$, then $E_s(\mathcal{A}_{I,s})$ is a Riesz basis. □

4 The decomposition

This section, for $s \neq 0$ gives the same kind of decomposition as in the case $s = 0$. So, the proof are omitted. We follow [8] and define the dimension function of V_s, denoted by $\dim_{V_s}$. Let J_s be a range function and $V_s = \mathcal{T}_s^{-1} M_{J_s}$. A mapping $\dim_{V_s} : \mathbb{T}^n \to \mathbb{N} \cup \{0, +\infty\}$ defined by $\dim_{V_s}(t) = \dim J_s(t)$ is called the dimension function of V_s .

Let $V_s = S_s(\varphi)$, $\varphi \in H^s$ and $\varphi_0 \in V_s$. It is said that φ_0 is a tight frame generator or quasi-orthogonal generator of V_s if

$$\|f\|^2_{H^s} = \sum_{k\in\mathbb{Z}^n} \left|\langle T_k \varphi_0, f\rangle_{H^s}\right|^2, \quad \text{for all } f \in V_s.$$

By Theorem 2.1 and Lemma 3.1 the following conditions are equivalent:

(1) φ_0 is a quasi-orthogonal generator of $V_s = S_s(\varphi)$,

(2) $\|\mathcal{T}_s\varphi_0(t)\|_{\ell^2_s} = \mathbf{1}_{\sigma_{V_s}}(t)$ for a.e. $t \in \mathbb{T}^n$.

Now, we can prove the decomposition theorem. The same construction given in the proof of Theorem 3.3 in [8] with the change

$$\eta_k(t) = \begin{cases} \frac{P_{J_s}(t)\mathrm{e}_{\pi(k)}}{\|P_{J_s}(t)\mathrm{e}_{\pi(k)}\|_{\ell^2_s}}, & t \in A_k, \\ 0, & \text{otherwise}, \end{cases}$$

leads to the proof of the next theorem.

Theorem 4.1. *Suppose that V_s is a shift-invariant subspace of H^s. Then, V_s can be decomposed as an orthogonal sum*

$$V_s = \bigoplus_{i \in \mathbb{N}} V_s^i,$$

where V_s^i, $i \in \mathbb{N}$, are principal shift-invariant spaces with quasi-orthogonal generators φ_i, $i \in \mathbb{N}$, and $\sigma_{V_s^{i+1}} \subset \sigma_{V_s^i}$, for all $i \in \mathbb{N}$. Moreover, $\dim_{V_s^i}(t) = \|\mathcal{T}_s\varphi_i(t)\|_{\ell^2_s}$, $i \in \mathbb{N}$, and

$$\dim_{V_s}(t) = \sum_{i \in \mathbb{N}} \|\mathcal{T}_s\varphi_i(t)\|_{\ell^2_s}, \quad \textit{for a.e. } t \in \mathbb{T}^n.$$

Remark 4.1. *The decomposition of the shift-invariant space V_s is not always unique, but this decomposition always gives us an optimal number of non-trivial components V_s^i, $i \in \mathbb{N}$.*

5 Structural theorems

Recall [18], [21], that $\mathcal{D}_{L^2}(\mathbb{R}^n) = \bigcap_{s \geq 0} H^s$ and $\mathcal{D}'_{L^2}(\mathbb{R}^n) = \bigcup_{s \geq 0} H^{-s}$.

We construct a dual frame E_s^d for $E_s(\mathcal{A}_{I,s})$. Note V_s is a closed subspace of H^s, $s \in \mathbb{R}$, so it is also a separable Hilbert space. The dual frame $\{\theta_k^i : k \in \mathbb{Z}^n, i \in I\}$ is determined by

$$\theta_k^i = L^{-1}(T_k\varphi_i), \quad k \in \mathbb{Z}^n,\, i \in I,$$

where L is the frame operator,

$$L(f) = \sum_{i \in I} \sum_{k \in \mathbb{Z}^n} \langle f, T_k\varphi_i \rangle_{H^s} T_k\varphi_i, \quad f \in V_s.$$

Theorem 5.1. *Assume that $E_s(\mathcal{A}_{I,s})$ is a frame for V_s and that $\{T_k\theta^i : k \in \mathbb{Z}^n, i \in I\}$ is its dual frame. Then $\mathcal{F}(V_s)$ is the set of Fourier transforms of elements $f \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$ so that*

$$\widehat{f}(\cdot) = \sum_{i \in I} \widehat{\varphi}_i(\cdot) \sum_{k \in \mathbb{Z}^n} a_k^i \mathrm{e}^{-2\pi\sqrt{-1}\langle \cdot, k \rangle},$$

where $(a_k^i)_{k\in\mathbb{Z}^n} \in \ell^2$ *is given by*

$$a_k^i = \int_{\mathbb{R}^n} \widehat{f}(x) \mathrm{e}^{2\pi\sqrt{-1}\langle k,x\rangle} \overline{\widehat{\theta^i}}(x) \mu_s^2(x)\,\mathrm{d}x, \quad k \in \mathbb{Z}^n, i \in I. \tag{5.1}$$

Equivalently, it is equal to the space of elements $f \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$ *which Fourier transforms have the form*

$$\widehat{f} = \sum_{i\in I} f_i g_i, \quad g_i \in L^2_{per}(\mathbb{R}^n),$$

where $f_i = \widehat{\varphi}_i \in L^2_s(\mathbb{R}^n)$*,* $i \in I$*, and* g_i*,* $i \in I$*, have the expansions*

$$g_i(\cdot) = \sum_{k\in\mathbb{Z}^n} a_k^i \mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle},$$

with a_k^i *determined by* (5.1).

Proof. Recall that the frame operator is a bijection. Moreover, it is a self-adjoint shift-preserving operator (commutes with the shift). Since

$$L^{-1}\big(\varphi_i(x-k-j)\big) = \theta^i_{k+j}(x) \text{ and } L^{-1}\big(\varphi_i(x-k-j)\big) = \theta^i_j(x-k),$$

we have $\theta^i_{k+j}(x) = \theta^i_j(x-k)$, $x \in \mathbb{R}^n$, $k, j \in \mathbb{Z}^n$, $i \in I$. So, for $j = 0$ we obtain

$$\theta^i_k(x) = T_k\theta^i(x), \quad x \in \mathbb{R}^n,\, k \in \mathbb{Z}^n,\, i \in I.$$

As in [8], the corresponding range operator is given by $\widetilde{G}_s(t)_{|J_s(t)}$, for a.e. $t \in \mathbb{T}^n$, where $\widetilde{G}_s = N_s N_s^*$ (see (3.2) and (3.3)) is the dual Gramian for $\{\mathcal{T}_s\varphi(t) : i \in I\}$ for a.e. $t \in \mathbb{T}^n$. We know that for every $f \in V_s$ there holds

$$\begin{aligned} f &= \sum_{i\in I}\sum_{k\in\mathbb{Z}^n} \big\langle f(x), T_k\varphi_i(x)\big\rangle_{H^s} T_k\theta^i \\ &= \sum_{i\in I}\sum_{k\in\mathbb{Z}^n} \big\langle f(x), T_k\theta^i(x)\big\rangle_{H^s} T_k\varphi_i = \sum_{i\in I}\sum_{k\in\mathbb{Z}^n} a_k^i T_k\varphi_i \end{aligned}$$

in the sense of convergence in V_s, where

$$a_k^i = \int_{\mathbb{R}^n} \widehat{f}(x) \mathrm{e}^{2\pi\sqrt{-1}\langle k,x\rangle} \overline{\widehat{\theta^i}}(x) \mu_s^2(x)\,\mathrm{d}x, \quad k \in \mathbb{Z}^n, i \in I.$$

Since $\big\{T_k\theta^i : k \in \mathbb{Z}^n, i \in I\big\}$ is a frame, we have

$$A\|f\|^2_{H^s} \le \sum_{i\in I}\sum_{k\in\mathbb{Z}^n} |a_k^i|^2 \le B\|f\|^2_{H^s}, \quad A > 0,\, B > 0.$$

Thus, $f \in V_s$ if and only if

$$f = \sum_{i \in I} \sum_{k \in \mathbb{Z}^n} a_k^i T_k \varphi_i,$$

where $(a_k^i)_{k \in \mathbb{Z}^n} \in \ell^2$ are given by (5.1).

Since the space of periodic L^2-functions, $L^2_{per}(\mathbb{R}^n)$, is defined by

$$L^2_{per}(\mathbb{R}^n) = \Big\{ g : g(\cdot) = \sum_{k \in \mathbb{Z}^n} a_k \mathrm{e}^{-2\pi\sqrt{-1}\langle \cdot, k \rangle},\ (a_k)_{k \in \mathbb{Z}^n} \in \ell^2 \Big\},$$

we have proved the assertion. □

5.1 Relations with $\mathcal{V}_s^2$

Instead of notation V_s^p in [17] (and V_p in [5]) we use $\mathcal{V}_s^2$, for $p = 2$. We recall some results of [17], where we have considered a weighted version of spaces V^p, $p \in [1, +\infty)$, analysed in [5].

So, assume that $p = 2$. In the case $s = 0$, we assume that $\psi^i \in \mathcal{L}^\infty$, $i = 1, \dots, r$, where

$$\mathcal{L}^\infty = \Big\{ \psi : \|\psi\|_{\mathcal{L}^\infty} = \sup_{t \in \mathbb{T}^n} \sum_{j \in \mathbb{Z}^n} |\psi(t + j)| < +\infty \Big\}.$$

By [5],

$$\mathcal{V}^2 = \Big\{ f : f = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^n} c_k^i T_k \psi^i,\ (c_k^i)_{k \in \mathbb{Z}^n} \subset \ell^2,\ i = 1, \dots, r \Big\}.$$

Theorem 5.2. *Assume that $\mathcal{A}_r = \{\psi^i : i = 1, \dots, r\} \subset L^2(\mathbb{R}^n) \cap \mathcal{L}^\infty$. Then,*

$$\mathcal{V}^2 = V = S(\mathcal{A}_r),$$

if $\mathcal{V}^2$ is closed in $L^2(\mathbb{R}^n)$.

Proof. Recall [5] that the closedness of $\mathcal{V}^2$ in $L^2(\mathbb{R}^n)$ is necessary and sufficient condition that $\mathcal{B} = \big\{ T_k \psi^i : k \in \mathbb{Z}^n, i = 1, \dots, r \big\}$ is a frame for $\mathcal{V}^2$. Since $E(\mathcal{A}_r) = \mathcal{B}$ and $S(\mathcal{A}_r)$ is closed in $L^2(\mathbb{R}^n)$ by the definition, it follows that the same frame determines both spaces so that $\mathcal{V}^2 = V$. □

Now, we consider weighted versions [17]. Let $s > 0$ be fixed. We will introduce several assumptions on generators ψ^i, $i = 1, \dots, r$, in order to have that their linear combinations determine subspaces of H^s and of L_s^2:

$$\psi^i \in H^s \cap L_s^2, \quad i = 1, \dots, r. \tag{5.2}$$

Moreover, in order to have the same assumptions as in [5] (and [17]), we assume, as in the previous assumption, that

$$\psi^i \in \mathcal{L}^\infty, \quad i = 1, \dots, r. \tag{5.3}$$

Recall [17],

$$\mathcal{V}_s^2 = \left\{ f : f = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^n} c_k^i T_k \psi^i,\ (c_k^i)_{k\in\mathbb{Z}^n} \in \ell_s^2,\ i = 1, \dots, r \right\}. \tag{5.4}$$

Theorem 5.3. *Assume that* $s > 0$, (5.2) *and* (5.3) *hold.*

a) Assume that

$$\mathcal{V}_s \text{ and } \mathcal{F}(\mathcal{V}_s^2) \text{ are closed in } L_s^2.$$

Then,

$$\mathcal{V}_s^2 \subset H^s \text{ and } \mathcal{V}_s^2 = V_s = S_s(\mathcal{A}_{r,s}).$$

In particular, any element $f \in V_s$ *has the frame expansion as in* (5.4)*.*

b) Assume that $s > 1/2$ *and that* $\mathcal{V}_s^2$ *is closed in* L_s^2*. Then,* $\mathcal{F}(\mathcal{V}_s^2)$ *is closed in* L_s^2 *and both assertions in a) hold true.*

Proof. a) Since $\psi^i \in H^s$, $i = 1, \dots, r$, consider

$$\mathcal{B}_s = \left\{ T_k \psi^i(t) : k \in \mathbb{Z}^n, t \in \mathbb{R}^n,\ i = 1, \dots, r \right\} \subset H^s \cap L_s^2.$$

By [17], $\mathcal{V}_s^2$ is closed in L_s^2 is equivalent with $\mathcal{B}_s$ is a frame for $\mathcal{V}_s^2$. We know that the Fourier transform is an isomorphism of H^s and L_s^2. Since $\mathcal{F}(\mathcal{V}_s^2)$ is closed in L_s^2, it follows that $\mathcal{F}^{-1}(\mathcal{F}\mathcal{V}_s^2) = \mathcal{V}_s^2$ is a closed subset of H^s. Both sets, V_s and $\mathcal{V}_s^2$ have the same dense subset consisting of compactly supported functions $\sum_{i=1}^r \sum_{k\in\mathbb{Z}^n} c_k^i \psi^i(\cdot - k)$, we have that they are equal. The particular part of the assertion now easily follows and any $f \in V_s$ has the expansion as in (5.4).

b) If $f \in \mathcal{V}_s^2$, then

$$f(\cdot) = \sum_{i=1}^{r} \sum_{k \in \mathbb{Z}^n} c_k^i \psi^i(\cdot - k) \text{ and } \widehat{f}(\cdot) = \sum_{i=1}^{r} \widehat{\psi}^i(\cdot) \sum_{k \in \mathbb{Z}^n} c_k^i \mathrm{e}^{-2\pi\sqrt{-1}\langle \cdot, k\rangle}.$$

Let

$$\widehat{f}_N(\cdot) = \sum_{i=1}^{r} \widehat{\psi}^i(\cdot) \sum_{|k|>N} c_k^i \mathrm{e}^{-2\pi\sqrt{-1}\langle \cdot, k\rangle}.$$

In order to show that $\widehat{f} \in L_s^2$ we will show that

$$\int_{\mathbb{R}^n} \widehat{f}_N(\xi) \overline{\widehat{f}_N(\xi)} (1 + |\xi|^2)^s \,\mathrm{d}\xi \to 0, \quad N \to +\infty.$$

In the product $\widehat{f_N}(\xi)\overline{\widehat{f_N}(\xi)}$ under the integral sign we have

$$\sum_{i_1,i_2=1}^{r} \widehat{\psi}^{i_1}(\xi)\overline{\widehat{\psi}^{i_2}}(\xi) \sum_{|k|>N} c_k^{i_1} \mathrm{e}^{-2\pi\sqrt{-1}\langle\xi,k\rangle} \sum_{|k|>N} \overline{c_k^{i_2}} \mathrm{e}^{2\pi\sqrt{-1}\langle\xi,k\rangle}$$
$$= \sum_{i_1,i_2=1}^{r} \widehat{\psi}^{i_1}(\xi)\overline{\widehat{\psi}^{i_2}}(\xi) I_{i_1,i_2,N}.$$

Since

$$\widehat{\psi}^{i_1}(\xi)\overline{\widehat{\psi}^{i_2}}(\xi)(1+|\xi|^2)^s \in L^2(\mathbb{R}^n),$$

if we prove that

$$|I_{i_1,i_2,N}| \leq \sup_{\xi\in\mathbb{R}^n} \Big| \sum_{|k|>N} c_k^{i_1} \mathrm{e}^{-2\pi\sqrt{-1}\langle\xi,k\rangle} \sum_{|k|>N} \overline{c_k^{i_2}} \mathrm{e}^{2\pi\sqrt{-1}\langle\xi,k\rangle} \Big| \to 0, \quad N\to+\infty,$$

we will have $\widehat{f_N} \to 0$, $N \to +\infty$ in L^2_s. We have

$$I_{i_1,i_2,N} \leq \sum_{|k|>N} |c_k^{i_1}| \sum_{|k|>N} |c_k^{i_2}| \leq$$
$$\sum_{|k|>N} |c_k^{i_1}|^2(1+|k|^2)^s \sum_{|k|>N} \frac{1}{(1+|k|^2)^s} \sum_{|k|>N} |c_k^{i_2}|^2(1+|k|^2)^s \sum_{|k|>N} \frac{1}{(1+|k|^2)^s}.$$

Since $(c_k^i)_k \in \ell^2_s$, $i=1,\dots,r$, we see that the last expression tends to zero as $N \to +\infty$. This proves the claim and the assertion b). □

Concerning the duality, we have the following assertion.

Theorem 5.4. *Assume that $s>0$, (5.2) and (5.3) hold. Moreover, assume that the conditions of assertion a) or conditions of assertion b) of Theorem 5.3 hold. Then in (both cases),*

a) $(\mathcal{V}^2_s)' = \mathcal{V}^2_{-s}$, where $\mathcal{V}^2_{-s}$ is the space of formal series of the form

$$F(\cdot) = \sum_{i=1}^{r}\sum_{k\in\mathbb{Z}^n} b_k^i \psi^i(\cdot - k), \quad \sum_{i=1}^{r}\sum_{k\in\mathbb{Z}^n} |b_k^i|^2(1+|k|^2)^{-s} < +\infty,$$

with the dual pairing

$$\langle F, f\rangle = \sum_{i=1}^{r}\sum_{k\in\mathbb{Z}^n} b_k^i c_k^i, \ \ (f \text{ is of the form given in } (5.4)).$$

b) $\mathcal{V}^2_{-s} = V_{-s}$.

Proof. Part a) is clear while the second part follows from the fact that the compactly supported elements of the form $\sum_{i=1}^{r}\sum_{k\in\mathbb{Z}^n} b_k^i\psi^i(\cdot-k)$ are dense in both spaces $\mathcal{V}_{-s}^2$ and V_{-s}, $s>0$.

$\square$

In order to consider the intersections of V_s, $s\geq 0$, instead of conditions (5.2) and (5.3), we assume

$$\psi^i\in\mathcal{S}(\mathbb{R}^n),\quad i=1,\ldots,r. \tag{5.5}$$

Theorem 5.5. *Assume that* (5.5) *holds. Then,*

$$\bigcap_{s\geq 0}\mathcal{V}_s^2=\bigcap_{s\geq 0}V_s,$$

and the expansion for their elements has the form as in (5.4) *with*

$$\sup_{k\in\mathbb{Z}^n}|c_k^i|k^s<+\infty,\quad i=1,\ldots,r,\ \textit{for every } s>0.$$

Recall that the space $\mathcal{P}(\mathbb{R}^n)=\mathcal{P}$ of periodic smooth test functions (with period one in any variable) is given by

$$\mathcal{P}=\Big\{\phi:\phi(\cdot)=\sum_{k\in\mathbb{Z}^n}a_k\mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle},\ (a_k)_{k\in\mathbb{Z}^n}\in\ell_s^2 \text{ for every } s\geq 0\Big\},$$

while its dual space $\mathcal{P}'(\mathbb{R}^n)=\mathcal{P}'$ is given by

$$\mathcal{P}'=\big\{\phi:\phi(\cdot)=\sum_{k\in\mathbb{Z}^n}a_k\mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle},\ (a_k)_{k\in\mathbb{Z}^n}\in\ell_{-s}^2 \text{ for some } s\geq 0\big\}.$$

A direct consequence part b) of Theorem 5.3 is the following assertion.

Corollary 5.1. *Assume that* (5.5) *holds. Then*

$$\mathcal{F}\Big(\bigcap_{s\geq 0}\mathcal{V}_s^2\Big)=$$

$$\Big\{\sum_{i=1}^{r}\widehat{\psi}^i(\cdot)\sum_{k\in\mathbb{Z}^n}c_k^i\mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle}:(c_k^i)_{k\in\mathbb{Z}^n}\in\ell_s^2, i=1,\ldots,r,\ \textit{for every}\ \ s\geq 0\Big\},$$

where $\Phi_i(\cdot)=\sum_{k\in\mathbb{Z}^n}c_k^i\mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle}\in\mathcal{P}$, $i=1,\ldots,r$.

Concerning the duality, by Theorem 5.4 we have:

Corollary 5.2. *Assume that* (5.5) *holds. Then* $V'_s = \mathcal{V}^2_{-s}$, $\cup_{s>0} V'_s = \cup_{s>0}\mathcal{V}^2_{-s}$ *and*

$$\mathcal{F}\Big(\bigcup_{s\leq 0}\mathcal{V}^2_s\Big) =$$

$$\Big\{\sum_{i=1}^{r}\widehat{\psi}^i(\cdot)\sum_{k\in\mathbb{Z}^n} c^i_k \mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle} : (c^i_k)_{k\in\mathbb{Z}^n}\in\ell^2_s, i=1,\dots,r, \textit{for some } s\leq 0\Big\},$$

where $F_i(\cdot) = \sum_{k\in\mathbb{Z}^n} c^i_k \mathrm{e}^{-2\pi\sqrt{-1}\langle\cdot,k\rangle}\in\mathcal{P}'$, $i=1,\dots,r$.

Note that the assumption $\psi^i\in\mathcal{S}(\mathbb{R}^n)$ implies a well defined product of a smooth function and a (periodic) Schwartz distribution.

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