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On the product of periodic distributions. Product in shift-invariant spaces

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Abstract. We connect through the Fourier transform shift-invariant Sobolev type spaces $V_s \subset H^s$, $s \in \mathbb{R}$, and the spaces of periodic distributions and analyze the properties of elements in such spaces with respect to the product. If the series expansions of two periodic distributions have compatible coefficient estimates, then their product is a periodic tempered distribution. We connect product of tempered distributions with the product of shift-invariant elements of V_s. The idea for the analysis of products comes from the Hörmander's description of the Sobolev type wave front in connection with the product of distributions. Coefficient compatibility for the product of f and g in the case of "good" position of their Sobolev type wave fronts is proved in the 2-dimensional case. For larger dimension it is an open problem because of the difficulties on the description of the intersection of cones in dimension $d \geqslant 3$.

1. Introduction

Our main interest is the analysis of the product of distributions f and g defined in a neighbourhood of a point x_0 belonging to shift-invariant spaces V_s, $s \in \mathbb{R}$. We show that, locally, this product is also an element of a shift-invariant space V_{s_0}, for some $s_0 \in \mathbb{R}$. For this purpose we use the idea of Hörmander's wave front set (cf. [17, 18]) and the fact that the product exists if the wave fronts are in an appropriate position. Actually, for periodic distributions and shift-invariant distributions we introduce the corresponding notion of compatible coefficient estimates which imply the existence of the product. The results for periodic distributions are transferred to the results for the product in shift-invariant spaces of distributions and vice versa.

Following the range function approach used in Bownik [11]-[13] (cf. [9], [10], [16], [22]), we investigated in [4] the structure of shift-invariant subspaces of Sobolev spaces $H^s = H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, denoted by V_s, generated by at most countable family of generators (cf. [1] for Sobolev spaces). In this paper we consider V_s generated by a finite set of generators, elements of $\mathcal{A}_{s,r} = \{\varphi_1, \dots, \varphi_r\} \subset H^s$; V_s is the closure of the span

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of integer translations of functions in $\mathcal{A}_{s,r}$, $s \in \mathbb{R}$. We will use the notation $V_s(\varphi_1, \dots, \varphi_r)$ when we want to underline the generators of this space. In the case $s = 0$, Bownik [11] gave a comprehensive analysis of the space V ($V = V_0$, $\mathcal{A}_r = \mathcal{A}_{0,r}$). A deep extension of results in [11] was obtained in [2], [3], [22], [25]. Our investigation goes towards the multiplication in spaces V_s for which we need to extend some results concerning the product of periodic distributions.

Let $E_s(\mathcal{A}_{s,r}) = \{\varphi_i(\cdot + k) : k \in \mathbb{Z}^d, i = 1, \dots, r\}$ be a frame of V_s (cf. [15] for frames). An $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to V_s if and only if its Fourier transform has the form $\widehat{f} = \sum_{i=1}^r f_i g_i$, $f_i = \widehat{\varphi_i} \in L^2_s(\mathbb{R}^d)$ and $g_i = \sum_{k\in\mathbb{Z}^d} a^i_k e^{2\pi\sqrt{-1}\langle \cdot, k\rangle}$ with $(a^i_k)_{k\in\mathbb{Z}^d} \in \ell^2$, $i = 1, \dots, r$. This is shown in [4]. Note that, the products $f_i g_i$, $i = 1, \dots, r$, exist in $L^2_s(\mathbb{R}^d) = \mathcal{F}(H^s)$.

Another approach, with the frames consisting of the finite set of generators $\mathcal{A}_r \subset L^2(\mathbb{R}^d)$ and expansions with coefficients in ℓ^2-sequence space, was developed in [5], [6], [19]. The spaces with the sequences of coefficients in ℓ^2_s were treated in [21], where the weights are $(1 + |k|^2)^{s/2}$, $k \in \mathbb{Z}^d$, $s \geqslant 0$, and the finite set of generators are subsets of $L^2_s = \mathcal{F}(H^s)$, $s \geqslant 0$. Actually, in [5] and [6], ℓ^p_s, $p \geqslant 1$, were considered, but here we restrict ourselves to the case $p = 2$. Moreover, connecting two different approaches to shift-invariant spaces V_s and $\mathcal{V}^2_s$, $s > 0$, under the assumption that the generators φ_i, $i = 1, \dots, r$, belong to $H^s \cap L^2_s$, we have given the characterization of elements in V_s through the expansions with coefficients in ℓ^2_s. The corresponding assertions hold for the intersections of all such spaces and their duals in the case when the generators are elements of $\mathcal{S}(\mathbb{R}^d)$ (see [4]).

Our framework in this paper is the space of periodic distributions, see for example [20] and [24], where the authors studied wave fronts through the analysis of Fourier expansions of periodic distributions. See also [14] and [23], where the authors studied generalized functions on the d-dimensional torus $\mathbb{T}^d$ and discrete wave fronts.

The paper is organized as follows. After recalling in Section 2 the basic facts about periodic distributions and shift-invariant spaces, we repeat in Section 3 our results of [4] concerning different approaches in [5], [6], [11], [21], connecting shift-invariant spaces with the subspaces of periodic distributions. In Section 4, after recalling results for the multiplication of periodic distributions f_1 and f_2 belonging to spaces $\mathscr{P}^{1,s_1}$, $\mathscr{P}^{2,s_2}$ respectively, we give our main result, Theorem 4.3 related to periodic ultradistributions which (according to Definition 4.4) have compatible coefficient estimates. Then, in Theorem 4.5 we transfer this result to the product of elements of finitely generated shift-invariant spaces V_s. Since our approach was motivated by Hörmander notion of Sobolev's wave fronts, we devote Section 5 to the product of periodic distributions and of shift-invariant distributions (and vice versa) relating the wave front sets with the compatibility coefficient condition in the expansions of f_1 and f_2.

2. Notation

Let $x = (x_1, \dots, x_d) \in \mathbb{K}^d$, where $\mathbb{K}^d \in \{\mathbb{R}^d, \mathbb{Z}^d\}$. We use notations $|x| = \sqrt{x_1^2 + \cdots + x_d^2}$ and $\langle x\rangle^s = (1+|x|^2)^{s/2}$, $s \in \mathbb{R}$. Obviously, $|x| \leqslant \langle x\rangle$. Let $0 < \eta \leqslant 1$. As in [20], we use the notation

$$\mathbb{T}_{\eta,x} = \prod_{j=1}^d \left(x_j - \frac{\eta}{2}, x_j + \frac{\eta}{2}\right) \text{ and } \mathbb{T}_\eta := \mathbb{T}_{\eta,0},\ \mathbb{T} = \mathbb{T}_1.$$

Define the Fourier transform $\widehat{f}$ of an integrable function f by $\mathcal{F}f(t) = \widehat{f}(t) = \int_{\mathbb{R}^d} f(x) e^{-2\pi\sqrt{-1}\langle x,t\rangle}\,dx$, $t \in \mathbb{R}^d$ ($\mathcal{F}^{-1}f(t) = \widehat{f}(-t)$), where $\langle x, t\rangle = \sum_{i=1}^d x_i t_i$, $x, t \in \mathbb{R}^d$. Further on,

$$\ell^p_s = \ell^p_s(\mathbb{Z}^d) = \left\{(c_k)_{k\in\mathbb{Z}^d} : \sum_{k\in\mathbb{Z}^d} |c_k|^p \langle k\rangle^{p\cdot s} < +\infty\right\}, \quad s \in \mathbb{R},\ p \geqslant 1.$$

We will consider the case $p = 2$. Then, the scalar product is given by $\langle (c_k)_{k\in\mathbb{Z}^d}, (d_k)_{k\in\mathbb{Z}^d}\rangle_{\ell^2_s} = \sum_{k\in\mathbb{Z}^d} c_k \overline{d}_k \langle k\rangle^{2s}$.

In the sequel we will denote by C constants which are not the same in general; from the context will be clear that in various inequalities they are different.

2.1. Periodic distributions

Our framework is the space of functions and distributions on $\mathbb{R}^d$ which are periodic of period 1 in each variable, i.e. $T_n f(x) = f(x-n) = f(x)$, $x \in \mathbb{R}^d, n \in \mathbb{Z}^d$. We refer to the next literature [7], [8], [24], [26]. Let $x, y \in \mathbb{K}^d$. We use notation $e_y(x) = \mathrm{e}^{2\pi\sqrt{-1}\langle y,x\rangle}$ $(\langle y,x\rangle = \sum_{i=1}^d y_i \bar{x}_i)$. The space of periodic test functions $\mathscr{P} = \mathscr{P}(\mathbb{R}^d)$ consists of smooth periodic functions of the form $\varphi = \sum_{n\in\mathbb{Z}^d} \varphi_n e_n$ such that $\sum_{n\in\mathbb{Z}^d} |\varphi_n|^2 \langle n\rangle^{2k} < +\infty$ for every $k \in \mathbb{Z}$ ($\varphi_n = \int_{\mathbb{T}} \varphi(x) e_{-n}(x)\,\mathrm{d}x$, $n \in \mathbb{Z}^d$); its topology is given via the sequence of norms $\|\varphi\|_k = \sup_{x\in\mathbb{T}, |\alpha|\leqslant k} |\varphi^{(\alpha)}(x)|$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The dual space of $\mathscr{P}$, the space of periodic distributions, is denoted by $\mathscr{P}'$. One has: $f = \sum_{n\in\mathbb{Z}^d} f_n e_n \in \mathscr{P}'$ if and only if $\sum_{n\in\mathbb{Z}^d} |f_n|^2 \langle n\rangle^{-2k_0} < +\infty$, for some $k_0 \in \mathbb{N}$. We use notation $\mathscr{P}'^{k_0}$ when this holds. If $f = \sum_{n\in\mathbb{Z}^d} f_n e_n \in \mathscr{P}'$ and $\varphi = \sum_{n\in\mathbb{Z}^d} \varphi_n e_n \in \mathscr{P}$, then their dual pairing is given by $\langle f, \varphi\rangle = \sum_{n\in\mathbb{Z}^d} f_n \varphi_n$.

Denote by $\mathscr{P}^{p,s}$, $p \geqslant 1$, $s \in \mathbb{R}$, the space of elements $h \in \mathcal{D}'(\mathbb{R}^d)$ with the property that $h = \sum_{n\in\mathbb{Z}^d} a_n e_n$, where $(a_n)_{n\in\mathbb{Z}^d} \in \ell^p_s$. These spaces are subspaces of $\mathscr{P}'$ for $s \leqslant 0$. Note, $\bigcap_{s\geqslant 0} \mathscr{P}^{p,s} = \mathscr{P}$ and $\bigcup_{s\leqslant 0} \mathscr{P}^{p,s} = \mathscr{P}'$.

Let $x_0 \in \mathbb{R}^d$, $\psi \in \mathcal{D}(\mathbb{T}_{\eta,x_0})$ and $f \in \mathcal{D}'(\mathbb{R}^d)$. Then $(f\psi)_{per}$ is defined as the periodic extension, by $(f\psi)_{per}(t) = (f\psi)(x)$, where $t + k = x \in \mathbb{T}_{\eta,x_0}$, $k \in \mathbb{Z}^d$ (this k is unique). So,

$$(f\psi)_{per}(t) = \sum_{k\in\mathbb{Z}^d} a_k e_k(t), \quad t \in \mathbb{R}^d,$$

where $a_k = \int_{\mathbb{T}_{\eta,x_0}} (f\psi)(t) e_{-k}(t)\,\mathrm{d}t$, $k \in \mathbb{Z}^d$. We denoted by $\mathscr{P}^{p,s}_{loc}$ the local space which contains distributions $f \in \mathcal{D}'(\mathbb{R}^d)$ such that $(f\psi)_{per} \in \mathscr{P}^{p,s}$, for all $x_0 \in \mathbb{R}^d$ and $\psi \in \mathcal{D}(\mathbb{T}_{1,x_0})$. In particular, we consider the cases $p = 1, 2$.

2.2. Shift-invariant spaces

Recall ([4]), the Hilbert space $H(\mathbb{T}, \ell^2_s)$ consists of all vector valued measurable square integrable functions $F : \mathbb{T} \to \ell^2_s$ with the norm $\|F\|_{H(\mathbb{T},\ell^2_s)} = \left(\int_{\mathbb{T}} \|F(t)\|^2_{\ell^2_s}\,\mathrm{d}t\right)^{1/2} < +\infty$. In the case $s = 0$, it is denoted by $L^2(\mathbb{T}, \ell^2)$. If $\mathcal{A}_r \subset L^2(\mathbb{R}^d)$, then $\mathcal{A}_{s,r} = \{\varphi \in \mathcal{S}'(\mathbb{R}^d) : \widehat{\varphi} = \widehat{\psi}\langle\cdot\rangle^{-s}$ for some $\psi \in \mathcal{A}_r\}$.

Note that any space $W \subset H^s$ is called shift-invariant if $\varphi \in W$ implies $T_k\varphi \in W$, for any $k \in \mathbb{Z}^d$. We define $V_s = \overline{\mathrm{span}}\{(1 - \frac{\Delta}{4\pi^2})^{-s/2} T_k\psi : \psi \in \mathcal{A}_r, k \in \mathbb{Z}^d\}$, where Δ is the Laplacian. It is a shift-invariant space.

Following the definition of the mapping $\mathcal{T} : L^2 \to L^2(\mathbb{T}, \ell^2)$ ([11]), we define in [4], $\mathcal{T}_s : H^s \to H(\mathbb{T}, \ell^2_s)$ ($\mathcal{T} = \mathcal{T}_s$, for $s = 0$) by

$$\mathcal{T}_s\varphi(t) = \left(\frac{\widehat{\psi}(t+k)}{\langle k\rangle^s}\right)_{k\in\mathbb{Z}^d}, \quad t \in \mathbb{T},\ \varphi \in H^s,$$

where $\left(1 - \frac{\Delta}{4\pi^2}\right)^{s/2}\varphi = \psi (\in L^2(\mathbb{R}^d))$.

Lemma 2.1 ([4]). *Let* $s \in \mathbb{R}$.

a) $\mathcal{T}_s : H^s \to H(\mathbb{T}, \ell^2_s)$ *is an isometric isomorphism.*

b) The following diagram of isometries commutes

$$\begin{array}{ccc} L^2 & \xrightarrow{\mathcal{T}} & L^2(\mathbb{T}, \ell^2) \\ \downarrow \alpha_s & & \downarrow \beta_s \\ H^s & \xrightarrow{\mathcal{T}_s} & H(\mathbb{T}, \ell^2_s), \end{array}$$

where $\alpha_s(g) = \mathcal{F}^{-1}\left(\frac{\widehat{g}(\cdot)}{\langle\cdot\rangle^s}\right)$ *and* $\beta_s\left((f_k(\cdot))_{k\in\mathbb{Z}^d}\right) = \left(\frac{f_k(\cdot)}{\langle k\rangle^s}\right)_{k\in\mathbb{Z}^d}$*; in particular,* $\beta_s\left((\widehat{g}(\cdot+k))_{k\in\mathbb{Z}^d}\right) = \left(\frac{\widehat{g}(\cdot+k)}{\langle k\rangle^s}\right)_{k\in\mathbb{Z}^d}$.

c) Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$*. Then* $\mathcal{T}_s T_j\varphi(\cdot) = e_{-j}(\cdot)\mathcal{T}_s\varphi(\cdot)$, $j \in \mathbb{Z}^d$.

3. Structural theorems

We introduce the following assumptions on generators ψ^i, $i = 1,\dots,r$, of $V_s(\psi^1,\dots,\psi^r)$, in order to have that their linear combinations determine subspaces of H^s and of L^2_s:

$$\psi^i \in H^s \cap L^2_s \cap \mathcal{L}^\infty, \quad i = 1,\dots,r. \tag{1}$$

Recall [6], the Wiener amalgam type space, denoted by $\mathcal{L}^\infty$ is defined by

$$\mathcal{L}^\infty = \Big\{\psi : \|\psi\|_{\mathcal{L}^\infty} = \sup_{t\in\mathbb{T}} \sum_{j\in\mathbb{Z}^d} |\psi(t+j)| < +\infty\Big\}$$

and following this paper in the case $p = 2$, in [21] is defined:

$$\mathcal{V}^2_s = \Big\{f : f = \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} c^i_k T_{-k}\psi^i,\ (c^i_k)_{k\in\mathbb{Z}^d} \in \ell^2_s,\ i = 1,\dots,r\Big\}. \tag{2}$$

Theorem 3.1 ([4]). *Let $s \geqslant 0$, and* (1) *hold.*

a) Assume that $\mathcal{V}^2_s$ and $\mathcal{F}(\mathcal{V}^2_s)$ are closed in L^2_s. Then, $\mathcal{V}^2_s \subset H^s$ and $\mathcal{V}^2_s = V_s(\psi^1,\dots,\psi^r)$. In particular, any element $f \in V_s(\psi^1,\dots,\psi^r)$ has the frame expansion as in (2).

b) Assume that $s > \frac{1}{2}$ and that $\mathcal{V}^2_s$ is closed in L^2_s. Then, $\mathcal{F}(\mathcal{V}^2_s)$ is closed in L^2_s and both assertions in a) hold true.

c) Assume that the conditions of assertion a) or conditions of assertion b) hold. Then in (both cases),

(i) $(\mathcal{V}^2_s)' = \mathcal{V}^2_{-s}$, where $\mathcal{V}^2_{-s}$ is the space of formal series of the form

$$F(\cdot) = \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} b^i_k \psi^i(\cdot + k) \quad \text{such that} \quad \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} |b^i_k|^2 \langle k\rangle^{-2s} < +\infty,$$

with the dual pairing $\langle F, f\rangle = \sum_{i=1}^r \sum_{k\in\mathbb{Z}^d} b^i_k c^i_k$, ($f$ is of the form given in (2)).

(ii) $\mathcal{V}^2_{-s} = V_{-s}$.

Theorem 3.2 ([4]). *Assume $\psi^i \in \mathcal{S}(\mathbb{R}^d)$, $i = 1,\dots,r$. Then, $\bigcap_{s\geqslant 0} \mathcal{V}^2_s = \bigcap_{s\geqslant 0} V_s$ and the expansion for their elements has the form as in* (2) *with*

$$\sup_{k\in\mathbb{Z}^d} |c^i_k||k|^s < +\infty, \quad i = 1,\dots,r, \text{ for every } s > 0.$$

Moreover, $\mathcal{F}\Big(\bigcap_{s\geqslant 0} \mathcal{V}^2_s\Big) = \Big\{\sum_{i=1}^r \widehat{\psi^i}(\cdot)\Phi_i(\cdot) : \Phi_i \in \mathscr{P}\Big\}$, where $\Phi_i(\cdot) = \sum_{k\in\mathbb{Z}^d} c^i_k e_k(\cdot)$, $(c^i_k)_{k\in\mathbb{Z}^d} \in \ell^2_s$ for every $s \geqslant 0$, $i = 1,\dots,r$, and $V'_s = \mathcal{V}^2_{-s}$, $\bigcup_{s\geqslant 0} V'_s = \bigcup_{s\geqslant 0} \mathcal{V}^2_{-s}$. Also, $\mathcal{F}\Big(\bigcup_{s\leqslant 0} \mathcal{V}^2_s\Big) = \Big\{\sum_{i=1}^r \widehat{\psi^i}(\cdot)F_i(\cdot) : F_i \in \mathscr{P}'\Big\}$, where $F_i(\cdot) = \sum_{k\in\mathbb{Z}^d} c^i_k e_k(\cdot)$, $(c^i_k)_{k\in\mathbb{Z}^n} \in \ell^2_s$ for some $s \leqslant 0$, $i = 1,\dots,r$.

4. Multiplication

Let $f_1 = \sum_{n\in\mathbb{Z}^d} f_{1,n}e_n \in \mathscr{P}^{1,s}$ and $f_2 = \sum_{n\in\mathbb{Z}^d} f_{2,n}e_n \in \mathscr{P}^{2,s}$. Their product is defined as

$$f = f_1 f_2 := \sum_{n\in\mathbb{Z}^d} f_n e_n, \text{ where } f_n = \sum_{j\in\mathbb{Z}^d} f_{1,n-j}f_{2,j},\ n \in \mathbb{Z}^d.$$

Then ([20]), $f \in \mathscr{P}^{2,s}$ and the mapping

$$\mathscr{P}^{1,s} \times \mathscr{P}^{2,s} \ni (f_1, f_2) \mapsto f_1 f_2 \in \mathscr{P}^{2,s}$$

is continuous. If $s, s_1, s_2 \in \mathbb{R}$ satisfy $s_1 + s_2 \geqslant 0$ and $s \leqslant \min\{s_1, s_2\}$, then the mapping

$$\mathscr{P}^{1,s_1} \times \mathscr{P}^{2,s_2} \ni (f_1, f_2) \mapsto f_1 f_2 \in \mathscr{P}^{2,s} \tag{3}$$

is continuous.

This implies the following assertion.

Proposition 4.1. *Let $f_1(\cdot) = \sum_{k\in\mathbb{Z}^d} f_{1,k}\varphi(\cdot + k)$, so that $(f_{1,k})_{k\in\mathbb{Z}^d} \in \ell^1_{s_1}$, $f_2(\cdot) = \sum_{k\in\mathbb{Z}^d} f_{2,k}\phi(\cdot + k)$, so that $(f_{2,k})_{k\in\mathbb{Z}^d} \in \ell^2_{s_2}$, where $\varphi \in H^{s_1}$, $\phi \in H^{s_2} \cap \mathcal{F}^{-1}\big(L^\infty(\mathbb{R}^d)\big)$ and $s_1 + s_2 \geqslant 0$. Then $f = f_1 * f_2 \in V_s$, where $s \leqslant \min\{s_1, s_2\}$, is generated by $\varphi * \phi \in H^s$, i.e. $f(\cdot) = \sum_{n\in\mathbb{Z}^d} f_n(\varphi * \phi)(\cdot + n)$, where $(f_n)_{n\in\mathbb{Z}^d} \in \ell^2_s$.*

Proof. By the assumptions, $\widehat{\varphi}\widehat{\phi} \in L^2_s(\mathbb{R}^d)$. Since

$$\widehat{f_1}\widehat{f_2} = \widehat{\varphi}\widehat{\phi}\sum_{n\in\mathbb{Z}^d} f_n e_n,$$

where $f_n = \sum_{j\in\mathbb{Z}^d} f_{1,n-j}f_{2,j}$, $n \in \mathbb{Z}^d$ belongs to ℓ^2_s, by (3) one has that

$$f(t) = (f_1 * f_2)(t) = (\varphi * \phi)(t) * \sum_{n\in\mathbb{Z}^d} f_n\delta(t + n) = \sum_{n\in\mathbb{Z}^d} f_n(\varphi * \phi)(t + n), \quad t \in \mathbb{R}^d,$$

whence the assertion follows. □

The previous considerations allow us to introduce multiplication in the local versions of these spaces. Let $f_1 \in \mathscr{P}^{1,s}_{loc}$ and $f_2 \in \mathscr{P}^{2,s}_{loc}$. To define their product $f = f_1 f_2$, we proceed locally. Let $x_0 \in \mathbb{R}^d$ and $0 < \eta < 1$. Let $\phi \in \mathscr{D}(\mathbb{T}_{1,x_0})$ be such that $\phi(x) = 1$ for $x \in \mathbb{T}_{\varepsilon,x_0}$, $\varepsilon < \eta$. We define f_{η,x_0} as the restriction to $\mathbb{T}_{\eta,x_0}$ of the product $(\phi f_1)_{per}(\phi f_2)_{per}$. So, $f_{\eta,x_0} \in \mathscr{D}'(\mathbb{T}_{\eta,x_0})$. By the use of the partition of unity the authors of [20] have the next assertion.

Corollary 4.2 ([20]). *The product $f = f_1 f_2$ of $f_1 \in \mathscr{P}^{1,s_1}_{loc}$ and $f_2 \in \mathscr{P}^{2,s_2}_{loc}$ is an element of $\mathscr{P}^{2,s}_{loc}$, where $s_1 + s_2 \geqslant 0$ and $s \leqslant \min\{s_1, s_2\}$. Moreover, the mapping*

$$\mathscr{P}^{1,s_1}_{loc} \times \mathscr{P}^{2,s_2}_{loc} \ni (f_1, f_2) \mapsto f_1 f_2 = f \in \mathscr{P}^{2,s}_{loc}$$

is continuous.

Now we consider the product of two periodic distributions.

Theorem 4.3. *Let $f_1, f_2 \in \mathscr{P}'$, i.e.*

$$f_1 = \sum_{i=1}^{l_1}\sum_{k\in\mathbb{Z}^d} a^i_{1,k}e_k, \quad f_2 = \sum_{j=1}^{l_2}\sum_{k\in\mathbb{Z}^d} a^j_{2,k}e_k,$$

such that there exist sets Λ^1_i, $i = 1, \dots, l_1$, and Λ^2_j, $j = 1, \dots, l_2$, subsets of $\mathbb{Z}^d$ so that

$$\sum_{k\in\Lambda^1_i} |a^i_{1,k}|^2\langle k\rangle^{-2\alpha_1} < +\infty, \qquad \sum_{k\in\mathbb{Z}^d\setminus\Lambda^1_i} |a^i_{1,k}|^2\langle k\rangle^{2\beta_1} < +\infty, \quad i = 1, \dots, l_1, \tag{4}$$

$$\sum_{m\in\Lambda^2_j} |a^j_{2,m}|^2\langle m\rangle^{-2\alpha_2} < +\infty, \qquad \sum_{m\in\mathbb{Z}^d\setminus\Lambda^2_j} |a^j_{2,m}|^2\langle m\rangle^{2\beta_2} < +\infty, \quad j = 1, \dots, l_2, \tag{5}$$

for some $\beta_1 \geqslant \alpha_2 \geqslant 0$, $\beta_2 \geqslant \alpha_1 \geqslant 0$, and $\Lambda^1_i \cap (-\Lambda^2_j) = \emptyset$, $i = 1, \dots, l_1$, $j = 1, \dots, l_2$. Moreover, we assume that for every $i = 1, \dots, l_1$, for every $j = 1, \dots, l_2$ and every $n \in \mathbb{Z}^d$, there exist $C > 0$ and $\gamma \geqslant 1$ such that

$$c^1_{i,j}(n) = \operatorname{card}\{k \in \mathbb{Z}^d \,:\, n - k \in \Lambda^2_j \wedge k \in \Lambda^1_i\} \leqslant C|n|^\gamma. \tag{6}$$

Then there exists $\tau \in \mathbb{R}$ such that $f_1 f_2 \in \mathscr{P}'^{\tau}$.

We give the proof of Theorem 4.3 immediately after the following definition, which seems reasonable.

Definition 4.4. *It is said that f_1, $f_2 \in \mathscr{P}'$ have compatible coefficient estimates if* (4)-(6) *hold. We say that f_1, $f_2 \in \mathcal{D}'(\mathbb{R}^d)$ have compatible coefficient estimates in a neighborhood of x_0 if for some $\varphi \in \mathcal{D}(\mathbb{T}_{\eta,x_0})$, $(f_1\varphi)_{per}$ and $(f_2\varphi)_{per}$ have Fourier expansions so that* (4)-(6) *hold. The sequences $(a^i_{1,k})_{k\in\mathbb{Z}^d}$, $i = 1,\dots,l_1$, and $(a^j_{2,k})_{k\in\mathbb{Z}^d}$, $j = 1,\dots,l_2$, are compatible if* (4)-(6) *hold.*

Proof. [Proof of Theorem 4.3] First, we note that if $c^2_{j,i}(n) = \operatorname{card}\{k \in \mathbb{Z}^d : n-k \in \Lambda^1_i \wedge k \in \Lambda^2_j\}$, then $c^1_{i,j}(n) = c^2_{j,i}(n)$.

The proof will be given for $l_1 = l_2 = 1$. The transfer to the general case is just repetition of arguments which are to follow. So, we will cancel indexes i and j. Thus, we have

$$f_1 f_2 = \left(\sum_{k\in\Lambda^1} + \sum_{k\in\mathbb{Z}^d\setminus\Lambda^1}\right) a_{1,k}e_k \cdot \left(\sum_{m\in\Lambda^2} + \sum_{m\in\mathbb{Z}^d\setminus\Lambda^2}\right) a_{2,m}e_m = f_1^1 f_2^1 + f_1^1 f_2^2 + f_1^2 f_2^1 + f_1^2 f_2^2,$$

and assume that

$$2\tau \geqslant \max\left\{4\gamma(\alpha_1+\alpha_2) + 2\gamma + d + 1, 2\alpha_1 + d + 1, 2\alpha_2 + d + 1\right\}.$$

We will estimate separately all the summaries. We have

$$f_1^1 f_2^1 = \sum_{n\in\mathbb{Z}^d} a_n^{11} e_n, \quad \text{where} \quad a_n^{11} = \sum_{\substack{n-k\in\Lambda^1\\ k\in\Lambda^2}} a_{1,n-k}a_{2,k},\ n\in\mathbb{Z}^d.$$

There holds,

$$\sum_{n\in\mathbb{Z}^d} |a_n^{11}|^2 \langle n\rangle^{-2\tau} \leqslant \sum_{n\in\mathbb{Z}^d}\left(\sum_{\substack{n-k\in\Lambda^1\\ k\in\Lambda^2}} |a_{1,n-k}|\langle n-k\rangle^{-\alpha_1}|a_{2,k}|\langle k\rangle^{-\alpha_2}\cdot\langle n-k\rangle^{\alpha_1}\langle k\rangle^{\alpha_2}\right)^2 \langle n\rangle^{-2\tau}.$$

By (6), for $k\in\Lambda^2$ and $(n-k)\in\Lambda^1$,

$$\langle n-k\rangle \leqslant \left\langle (n_1 + |n|^\gamma, \dots, n_d + |n|^\gamma)\right\rangle \leqslant C\langle n\rangle^{2\gamma}, \quad \langle k\rangle \leqslant C\langle n\rangle^{2\gamma}.$$

We continue,

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d} |a_n^{11}|^2\langle n\rangle^{-2\tau} &\leqslant C\sum_{n\in\mathbb{Z}^d}\left(\sum_{\substack{n-k\in\Lambda^1\\ k\in\Lambda^2}} |a_{1,n-k}|\langle n-k\rangle^{-\alpha_1}|a_{2,k}|\langle k\rangle^{-\alpha_2}\right)^2 \frac{\langle n\rangle^{4\gamma(\alpha_1+\alpha_2)+2\gamma}}{\langle n\rangle^{2\tau}}\\ &\leqslant C\sum_{n\in\mathbb{Z}^d}\left(\sum_{\substack{n-k\in\Lambda^1\\ k\in\Lambda^2}} |a_{1,n-k}|^2\langle n-k\rangle^{-2\alpha_1}\right)\left(\sum_{\substack{n-k\in\Lambda^1\\ k\in\Lambda^2}} |a_{2,k}|^2\langle k\rangle^{-2\alpha_2}\right)\frac{1}{\langle n\rangle^{d+1}} \leqslant C\sum_{n\in\mathbb{Z}^d}\frac{1}{\langle n\rangle^{d+1}} < +\infty.\end{aligned}$$

Let us estimate

$$f_1^1 f_2^2 = \sum_{n\in\mathbb{Z}^d} a_n^{12} e_n, \quad \text{where} \quad a_n^{12} = \sum_{\substack{n-k\in\Lambda^1\\ k\in\mathbb{Z}^d\setminus\Lambda^2}} a_{1,n-k}a_{2,k}, \quad n\in\mathbb{Z}^d.$$

There holds,

$$\sum_{n\in\mathbb{Z}^d} |a_n^{12}|^2\langle n\rangle^{-2\tau} \leqslant \sum_{n\in\mathbb{Z}^d}\left(\sum_{\substack{n-k\in\Lambda^1\\ k\in\mathbb{Z}^d\setminus\Lambda^2}} |a_{1,n-k}|\langle n-k\rangle^{-\alpha_1}|a_{2,k}|\langle k\rangle^{\beta_2}\cdot\frac{\langle n-k\rangle^{\alpha_1}}{\langle k\rangle^{\beta_2}}\right)^2\langle n\rangle^{-2\tau}.$$

Since, $\langle n-k\rangle^{\alpha_1} \leqslant C\langle n\rangle^{\alpha_1}\langle k\rangle^{\alpha_1}$ and $\beta_2 \geqslant \alpha_1 \geqslant 0$, we have

$$\sum_{n\in\mathbb{Z}^d} |a_n^{12}|^2\langle n\rangle^{-2\tau} \leqslant C\sum_{n\in\mathbb{Z}^d}\frac{1}{\langle n\rangle^{2\tau-2\alpha_1}} < +\infty.$$

Further, we will use the inequality

$$\langle y\rangle^r \leqslant C\langle x\rangle^r \langle y-x\rangle^{|r|}, \quad x,y\in\mathbb{R}^d,\ r\in\mathbb{R}, \tag{7}$$

which we now show holds. Indeed, since $(1+t)^2 = 1+2t+t^2 \leqslant 2(1+t^2)$ for every $t \geqslant 0$, if we choose $t = |y-x|$ then we get

$$\langle y\rangle \leqslant \langle x\rangle + |y-x| \leqslant \langle x\rangle(1+|y-x|) \leqslant 2^{1/2}\langle x\rangle\langle y-x\rangle.$$

Thus, for $r \geqslant 0$ inequality (7) holds. For $r < 0$ we have

$$\frac{\langle y\rangle^r}{\langle x\rangle^r} = \frac{\langle x\rangle^{|r|}}{\langle y\rangle^{|r|}} \leqslant \frac{C\langle y\rangle^{|r|}\langle y-x\rangle^{|r|}}{\langle y\rangle^{|r|}} = C\langle y-x\rangle^{|r|}.$$

Hence, the inequality (7) holds for every $r\in\mathbb{R}$.

Now, by (7) for $k,n\in\mathbb{Z}^d$ and $\alpha_2 \geqslant 0$, we have that $\langle k\rangle^{\alpha_2} \leqslant C\langle k-n\rangle^{\alpha_2}\langle n\rangle^{\alpha_2}$ holds. Using the last inequality and $\beta_1 \geqslant \alpha_2 \geqslant 0$, the estimate for $f_1^2 f_2^1$ simply follows:

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d} |a_n^{21}|^2\langle n\rangle^{-2\tau} &\leqslant \sum_{n\in\mathbb{Z}^d}\Bigg(\sum_{\substack{n-k\in\mathbb{Z}^d\setminus\Lambda^1\\ k\in\Lambda^2}} |a_{1,n-k}|\langle n-k\rangle^{\beta_1}|a_{2,k}|\langle k\rangle^{-\alpha_2}\cdot\frac{\langle k\rangle^{\alpha_2}}{\langle n-k\rangle^{\beta_1}}\Bigg)^2\langle n\rangle^{-2\tau}\\ &\leqslant C\sum_{n\in\mathbb{Z}^d}\Bigg(\sum_{\substack{n-k\in\mathbb{Z}^d\setminus\Lambda^1\\ k\in\Lambda^2}}|a_{1,n-k}|^2\langle n-k\rangle^{2\beta_1}\Bigg)\Bigg(\sum_{\substack{n-k\in\mathbb{Z}^d\setminus\Lambda^1\\ k\in\Lambda^2}}|a_{2,k}|^2\langle k\rangle^{-2\alpha_2}\Bigg)\frac{\langle n\rangle^{2\alpha_2}}{\langle n\rangle^{2\tau}} \leqslant C\sum_{n\in\mathbb{Z}^d}\frac{1}{\langle n\rangle^{d+1}} < +\infty.\end{aligned}$$

The estimate for $f_1^2 f_2^2$ is proved in a similar way. Hence, $f_1 f_2 \in \mathscr{P}'^{\tau}$. □

Theorem 4.5. *Let $g_1 \in V_{s_1}(\varphi_1^1,\dots,\varphi_1^{l_1})$, $g_2\in V_{s_2}(\varphi_2^1,\dots,\varphi_2^{l_2})$, $s_1,s_2\geqslant 0$, so that*

$$g_1(\cdot) = \sum_{i=1}^{l_1}\sum_{k\in\mathbb{Z}^d} a_{1,k}^i\varphi_1^i(\cdot+k),\quad g_2(\cdot) = \sum_{j=1}^{l_2}\sum_{k\in\mathbb{Z}^d} a_{2,k}^j\varphi_2^j(\cdot+k),$$

and that there exist sets Λ_i^1, $i=1,\dots,l_1$, and Λ_j^2, $j=1,\dots,l_2$, subsets of $\mathbb{Z}^d$ such that $\Lambda_i^1\cap(-\Lambda_j^2)=\emptyset$, $i=1,\dots,l_1$, $j=1,\dots,l_2$. Moreover, assume that (4) *and* (5) *hold and that $c_{i,j}^1(n)$ ($c_{j,i}^2(n)$), $i=1,\dots,l_1$, $j=1,\dots,l_2$, satisfy* (6)*. Then, there exists $s\in\mathbb{R}$ such that for $\varphi_1^i,\varphi_2^j\in V_s\cap\mathcal{V}_s^2$, $i=1,\dots,l_1$, $j=1,\dots,l_2$, we have*

$$g_1 * g_2 \in V_s\Big(\varphi_1^i*\varphi_2^j,\ i=1,\dots,l_1,\ j=1,\dots,l_2\Big).$$

More precisely,

$$(g_1*g_2)(\cdot) = \sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\sum_{n\in\mathbb{Z}^d}\sum_{n-k\in\mathbb{Z}^d} a_{1,n-k}^i a_{2,k}^j(\varphi_1^i*\varphi_2^j)(\cdot+n).$$

Proof. Again, we discuss only the case $l_1=l_2=1$ and cancel indices i and j. We have

$$\widehat{g_1}(x) = \widehat{\varphi_1}(x)f_1(x),\quad \widehat{g_2}(x)=\widehat{\varphi_2}(x)f_2(x),\quad x\in\mathbb{R}^d,$$

where f_1 and f_2 are the same as in Theorem 4.3. Thus,

$$\widehat{g_1}(x)\widehat{g_2}(x) = \widehat{\varphi_1}(x)\widehat{\varphi_2}(x)\sum_{n\in\mathbb{Z}^d} a_n e_n(x),\quad x\in\mathbb{R}^d,$$

and coefficients a_n, $n\in\mathbb{Z}^d$ satisfy

$$\sum_{n\in\mathbb{Z}^d}|a_n|^2\langle n\rangle^{-2\tau} < +\infty,$$

by Theorem 4.3. This implies

$$(g_1 * g_2)(t) = (\varphi_1 * \varphi_2)(t) * \sum_{n\in\mathbb{Z}^d} a_n\delta(t+n) = \sum_{n\in\mathbb{Z}^d} a_n(\varphi_1 * \varphi_2)(t+n), \quad t \in \mathbb{R}^d.$$

Let $s = -\tau$. Hence, $g_1 * g_2 \in V_s(\varphi_1 * \varphi_2)$.

The general case is again the repetition of the given proof but with much more complex notation which we skip. □

5. Wave front characterizations

We analyze the product of distributions considering them in the space of periodic distributions and then we transfer the obtained results to the shift-invariant spaces V_s. We recall Hörmander's definition [18].

Definition 5.1 ([18]). *Let $f \in \mathscr{D}'(\mathbb{R}^d)$, $(x_0,\xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{\mathbf{0}\})$, and $s \in \mathbb{R}$. We say that f is Sobolev microlocally regular at (x_0,ξ_0) of order s, that is $(x_0,\xi_0) \notin WF_s(f)$, if there exist an open cone Γ around ξ_0 and $\psi \in \mathscr{D}(\mathbb{R}^d)$ with $\psi \equiv 1$ in a neighborhood of x_0 such that*

$$\int_\Gamma |\widehat{\psi f}(\xi)|^2 \langle\xi\rangle^{2s}\,\mathrm{d}\xi < +\infty.$$

The next theorem is the characterization through the localization and the representation through the Fourier coefficients.

Theorem 5.2 ([20]). *Let $f \in \mathscr{D}'(\mathbb{R}^d)$. The following two conditions are equivalent.*

a) There exist an open cone Γ around ξ_0, $\phi \in \mathscr{D}(\mathbb{T}_{\eta,x_0})$ with $\eta \in (0,1)$ and $\phi \equiv 1$ in a neighborhood of x_0, such that

$$\sum_{n\in\Gamma\cap\mathbb{Z}^d} |a_n|^2\langle n\rangle^{2s} < +\infty, \quad \textit{where } (f\phi)_{per} = \sum_{n\in\mathbb{Z}^d} a_n e_n.$$

b) $(x_0,\xi_0) \notin WF_s(f)$.

The next assertion is interesting in itself.

Theorem 5.3. *Let Γ be an open convex cone in $\mathbb{R}^d \setminus \{\mathbf{0}\}$ and $f = \sum_{n\in\mathbb{Z}^d} a_n e_n \in \mathscr{P}'$, so that $\sum_{n\in\Gamma\cap\mathbb{Z}^d} |a_n|^2\langle n\rangle^{2s} < +\infty$. Then $(x_0,\xi_0) \notin WF_s(f)$ for any $x_0 \in \mathbb{R}^d$ and $\xi_0 \in \Gamma$.*

Proof. Let $\varphi \in \mathcal{D}(\mathbb{T}_{\eta,x_0})$, $\varphi \equiv 1$ in $\mathbb{T}_{\varepsilon,x_0}$, $0 < \varepsilon < \eta$. We know that $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$. Let $\Gamma_{\xi_0} \subset \Gamma$ and $\Gamma_1 \subset\subset \Gamma_{\xi_0}$ (that is, $\Gamma_1 \cap \mathbb{S}^{d-1}$ is a compact subset of $\Gamma_{\xi_0} \cap \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the unit sphere). Then there exists $C > 0$ such that

$$\xi \in \Gamma_1 \quad \wedge \quad n \in \mathbb{Z}^d \cap \Big((\mathbb{R}^d \setminus \{\mathbf{0}\}) \setminus \Gamma_{\xi_0}\Big) \quad \Rightarrow \quad \langle \xi - n\rangle \geqslant C\langle n\rangle. \tag{8}$$

We have, by (7) (with $y = \xi$, $x = -n$, $r = 2s$),

$$\begin{aligned}\int_{\Gamma_1} \langle\xi\rangle^{2s}|\widehat{\varphi f}(\xi)|^2\,\mathrm{d}\xi &= \int_{\Gamma_1}\langle\xi\rangle^{2s}|(\widehat{\varphi} * \widehat{f})(\xi)|^2\,\mathrm{d}\xi = \int_{\Gamma_1}\langle\xi\rangle^{2s}\Big|\widehat{\varphi}(\xi) * \sum_{n\in\mathbb{Z}^d} a_n\delta(\xi+n)\Big|^2\,\mathrm{d}\xi \\ &= \int_{\Gamma_1}\langle\xi\rangle^{2s}\Big|\sum_{n\in\mathbb{Z}^d} a_n\widehat{\varphi}(\xi+n)\Big|^2\,\mathrm{d}\xi \leqslant \int_{\Gamma_1}\langle\xi\rangle^{2s}\Big(\sum_{n\in\mathbb{Z}^d}|a_n|^2|\widehat{\varphi}(\xi+n)|\Big)\Big(\sum_{n\in\mathbb{Z}^d}|\widehat{\varphi}(\xi+n)|\Big)\mathrm{d}\xi \\ &\leqslant C\int_{\Gamma_1}\Big(\sum_{n\in\mathbb{Z}^d}|a_n|^2\langle n\rangle^{2s}|\widehat{\varphi}(\xi+n)|\langle\xi+n\rangle^{2|s|}\Big)\mathrm{d}\xi = C\cdot I,\end{aligned}$$

where we have used that $\sum_{n\in\mathbb{Z}^d}|\widehat{\varphi}(\xi+n)| < +\infty$, $\xi \in \mathbb{R}^d$, because $\widehat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$. Further on,

$$I = \int_{\Gamma_1}\sum_{n\in\mathbb{Z}^d\cap\Gamma_{\xi_0}}|a_n|^2\langle n\rangle^{2s}|\widehat{\varphi}(\xi+n)|\langle\xi+n\rangle^{2|s|}\,\mathrm{d}\xi + \int_{\Gamma_1}\sum_{n\in\mathbb{Z}^d\setminus\Gamma_{\xi_0}}|a_n|^2\langle n\rangle^{2s}|\widehat{\varphi}(\xi+n)|\langle\xi+n\rangle^{2|s|}\,\mathrm{d}\xi = I_1 + I_2.$$

For I_1, we have

$$I_1 = \sum_{n\in\mathbb{Z}^d\cap\Gamma_{\xi_0}} |a_n|^2\langle n\rangle^{2s} \int_{\Gamma_1} |\widehat{\varphi}(\xi+n)|\langle \xi+n\rangle^{2|s|}\,\mathrm{d}\xi \leqslant C \sum_{n\in\mathbb{Z}^d\cap\Gamma_{\xi_0}} |a_n|^2\langle n\rangle^{2s} < +\infty,$$

since $\int_{\mathbb{R}^d} |\widehat{\varphi}(\xi+n)|\langle \xi+n\rangle^{2|s|}\,\mathrm{d}\xi \leqslant C$, $n\in\mathbb{Z}^d$, also because $\widehat{\varphi}\in\mathcal{S}(\mathbb{R}^d)$. Concerning I_2, we note first that $f\in\mathscr{P}'$ implies $\sum_{n\in\mathbb{Z}^d}|a_n|^2\langle n\rangle^{-2m} < +\infty$ for some $m\in\mathbb{N}$. By (8), we have

$$\frac{\langle n\rangle^{2(m+s)}}{\langle \xi+n\rangle^{2(m+s)}} \leqslant C, \quad \xi\in\Gamma_1,\ n\in\mathbb{Z}^d\setminus\Gamma_{\xi_0}.$$

So

$$\begin{aligned} I_2 &= \int_{\Gamma_1} \sum_{n\in\mathbb{Z}^d\setminus\Gamma_{\xi_0}} |a_n|^2\langle n\rangle^{-2m} \frac{\langle n\rangle^{2(m+s)}}{\langle \xi+n\rangle^{2(m+s)}} \langle \xi+n\rangle^{2(m+s+|s|)} |\widehat{\varphi}(\xi+n)|\,\mathrm{d}\xi \\ &\leqslant C \sum_{n\in\mathbb{Z}^d\setminus\Gamma_{\xi_0}} |a_n|^2\langle n\rangle^{-2m} \int_{\Gamma_1} \langle \xi+n\rangle^{2(m+s+|s|)}|\widehat{\varphi}(\xi+n)|\,\mathrm{d}\xi < +\infty, \end{aligned}$$

where we used that $\int_{\mathbb{R}^d}\langle \xi+n\rangle^{2(m+s+|s|)}|\widehat{\varphi}(\xi+n)|\,\mathrm{d}\xi < +\infty$. This completes the proof. □

We have the following corollary.

Corollary 5.4. *Let $\varphi\in\mathcal{D}(\mathbb{R}^d)$ and $h\in V_s(\varphi)$, so that*

$$h(\cdot) = \sum_{k\in\mathbb{Z}^d} a_k\varphi(\cdot+k) \quad \textit{and} \quad \sum_{k\in\mathbb{Z}^d\cap\Gamma} |a_k|^2\langle n\rangle^{2s} < +\infty$$

for an open cone $\Gamma\subset\mathbb{R}^d\setminus\{\mathbf{0}\}$. If $\sum_{k\in\mathbb{Z}^d}|a_k|^2\langle k\rangle^{-2s_0} < +\infty$ for some $s_0\in\mathbb{N}$, then for every $x\in\Omega$ ($\Omega\subset\mathbb{R}^d$ is open set) and $\xi\in\Gamma$, $(x,\xi)\notin WF_s(\widehat{h})$.

Proof. Let $h_0 = \sum_{k\in\mathbb{Z}^d} a_k e_k$. Then, $\widehat{h} = \widehat{\varphi}h_0$. We know by Theorem 5.3 that for any $x\in\mathbb{R}^d$ and $\xi\in\Gamma$, $(x,\xi)\notin WF_s(h_0)$. Since the multiplication by a function in $\mathcal{S}(\mathbb{R}^d)$ does not decrease the set of Sobolev microlocal regular points, we conclude that $(x,\xi)\notin WF_s(\widehat{h})$. □

Next, we consider the case when Lambdas are the sets of intersections of cone and $\mathbb{Z}^d$, where the cones are such that the projection of the wave front on the second variable is contained in them. This is an interesting case. Thus, we will take

$$\Lambda^1 = \Gamma_1\cap\mathbb{Z}^d \quad \text{and} \quad \Lambda^2 = \Gamma_2\cap\mathbb{Z}^d,$$

such that $pr_2\big(WF_{s_1}(f_1)\big)\subset\Gamma_1$ and $pr_2\big(WF_{s_2}(f_2)\big)\subset\Gamma_2$, where $pr_2(x,\xi)=\xi$, $x,\xi\in\mathbb{R}^d$.

Theorem 5.5. *Let $f_1, f_2\in\mathscr{P}'$ (i.e. $f_1\in\mathscr{P}'^{\tau_1}$, $f_2\in\mathscr{P}'^{\tau_2}$), Γ_1 and Γ_2 be cones of $\mathbb{R}^d$ so that $\Gamma_1\cap(-\Gamma_2)=\emptyset$ and that the following conditions be fulfilled.*

a) There exist $C>0$ and $\gamma\geqslant 1$ such that

$$\operatorname{card}\{k\in\mathbb{Z}^d : n-k\in\Gamma_1 \wedge k\in\Gamma_2\} \leqslant C|n|^{\gamma}, \quad n\in\mathbb{Z}^d.$$

b) Let $(x_0,\xi_0)\in\mathbb{R}^d\times(\mathbb{R}^d\setminus\{\mathbf{0}\})$ and let $\psi\in\mathcal{D}(\mathbb{T}_{\eta,x_0})$ with $\eta\in(0,1)$ and $\psi\equiv 1$ in $\mathbb{T}_{\varepsilon,x_0}$, $\varepsilon<\eta$, so that

$$pr_2\big(WF_{s_1}(f_1\psi)\big)\subset\Gamma_1, \quad pr_2\big(WF_{s_2}(f_2\psi)\big)\subset\Gamma_2,$$

where $s_1\geqslant\tau_2$ and $s_2\geqslant\tau_1$.

Then, $f = (f_1\psi)_{per}(f_2\psi)_{per}$ exists in $\mathcal{D}'(\mathbb{R}^d)$. Moreover, $f \in \mathscr{P}'$.

Proof. Let

$$(f_1\psi)_{per} = \sum_{k\in\mathbb{Z}^d} a_{1,k}e_k, \quad (f_2\psi)_{per} = \sum_{k\in\mathbb{Z}^d} a_{2,k}e_k.$$

Note, if $x \in \operatorname{supp}\psi$ and $\xi \in (\mathbb{R}^d \setminus \{\mathbf{0}\}) \setminus \mathbf{\Gamma_1}$, then $(x,\xi) \notin WF_{s_1}(f_1\psi)$. The same holds for $f_2\psi$. Since $f_1 \in \mathscr{P}'^{\tau_1}$ and $f_2 \in \mathscr{P}'^{\tau_2}$, we know that

$$\sum_{k\in\mathbb{Z}^d\cap\Gamma_1} |a_{1,k}|^2\langle k\rangle^{-2\tau_1} < +\infty, \qquad \sum_{k\in\mathbb{Z}^d\cap\Gamma_2} |a_{2,k}|^2\langle k\rangle^{-2\tau_2} < +\infty. \tag{9}$$

Now as in the proof of Theorem 4.3, we show that $f = (f_1\psi)_{per}(f_2\psi)_{per}$ exists and $f \in \mathscr{P}'$. □

Remark 5.6. *Theorem 5.5 can be easily transferred to the case when one has several cones Γ_1^i, $i = 1,\dots,l_1$ (related to f_1) and Γ_2^j, $j = 1,\dots,l_2$ (related to f_2) so that $\Gamma_1^i \cap \mathbb{Z}^d$ and $\Gamma_2^j \cap \mathbb{Z}^d$ contain index sets for f_1 and f_2, $i = 1,\dots,l_1$, $j = 1,\dots,l_2$ which are compatible index sets.*

Remark 5.7. *We will show below that under the assumption that $\Gamma_1 \cap (-\Gamma_2) = \emptyset$ then condition in a) of Theorem 5.5 holds in the case $d = 2$ with $\gamma = 2$. Our hypothesis is that condition a) also holds (with $\gamma = d$) for $d \geqslant 3$, but the structure of cones is more complex and we do not have the proof of this hypothesis for $d \geqslant 3$.*

Proof. [Proof of the assertion in the Remark 5.7 in the case $d = 2$] We can assume that cones are acute because if it is not the case, we divide them into finite sets of such cones. So, assume that cones Γ_1 and $-\Gamma_2$ are acute and have empty intersection. By translation with vector $\overrightarrow{(0,0),(n_1,n_2)}$, there are several different positions of cones so that they have different surface of the domain laying between them. It can be equal to zero but the optimal case (maximal number of points with integer coordinates inside the intersection) is when they intersect in four points. Let us explain this case. We present the simplest position of cones (by rotations, this is not the restriction)

$$\Gamma_1 = \{(t,s) : k_1t \geqslant s, t \geqslant 0\}, \quad -\Gamma_2 = \{(t,s) : k_2t \geqslant s, t \leqslant 0\}, \quad k_2 > k_1 > 0.$$

Now translating $-\Gamma_2$ so that the tip of the cone is (n_1,n_2), one can calculate the points of intersection of cones $\Gamma_1 \cap \mathbb{Z}^2$ and $\big((n_1,n_2) - \Gamma_2\big) \cap \mathbb{Z}^2$. Coordinates of the sets of intersections, set points A_1, A_2, A_3, A_4 are linear combinations of the form

$$(\alpha^i_{1,1}n_1 + \alpha^i_{1,2}n_2, \alpha^i_{2,1}n_1 + \alpha^i_{2,2}n_2), \quad i = 1,2,3,4,$$

where $\alpha^i_{j,l}$ depend on k_1 and k_2. Now, by calculating the surface of the area of the intersection of these cones, we conclude that the domain surface between two cones can be estimated by $C(n_1^2 + n_2^2)$, for some $C > 0$. □

Using Theorems 4.5 and 5.5, we obtain the following statement.

Corollary 5.8. *Let $\varphi_i \in H^{s_i}$, $i = 1,2$, and let Γ_1 and Γ_2 be cones so that $\Gamma_1 \cap (-\Gamma_2) = \emptyset$. Assume that assertion a) in Theorem 5.5 holds.*

a) Let $x_0 \in \mathbb{R}^d$, $f_1, f_2 \in \mathscr{D}'(\mathbb{R}^d)$ and $\psi \in \mathscr{D}(\mathbb{T}_{\eta,x_0})$ with $\eta \in (0,1)$ so that $\psi \equiv 1$ in $\mathbb{T}_{\varepsilon,x_0}$, $\varepsilon < \eta$. Assume that

$$\widehat{(f_1\psi)} = \widehat{\varphi_1}\sum_{k\in\mathbb{Z}^d} a_ke_k, \quad \widehat{(f_2\psi)} = \widehat{\varphi_2}\sum_{k\in\mathbb{Z}^d} b_ke_k,$$

and that (9) holds for $\sum_{k\in\mathbb{Z}^d} a_ke_k$ and $\sum_{k\in\mathbb{Z}^d} b_ke_k$. Moreover, assume that condition b) of Theorem 5.5 holds. Then, there exists $s \in \mathbb{R}$ so that

$$f_1\psi(\cdot) = \sum_{k\in\mathbb{Z}^d} a_k\varphi_1(\cdot + k), \quad f_2\psi(\cdot) = \sum_{k\in\mathbb{Z}^d} b_k\varphi_2(\cdot + k)$$

are elements of $V_s(\varphi_1)$, $V_s(\varphi_2)$, respectively, and their product $(f_1\psi) * (f_2\psi) \in V_s(\varphi_1 * \varphi_2)$.

b) Let $g_i \in V_{s_i}(\varphi_i)$, $i = 1, 2$, and let $(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{\mathbf{0}\})$. For $\widehat{g_i} = \widehat{\varphi_i} f_i$ we assume that $pr_2\Big(WF_{s_i}(f_i)\Big) \subset \Gamma_i$, $i = 1, 2$. Moreover, we assume that (9) *holds, where $s_1 \geqslant \tau_2 \geqslant 0$ and $s_2 \geqslant \tau_1 \geqslant 0$. Then, there exists $s \in \mathbb{R}$ so that*

$$g = g_1 * g_2 \in V_s(\varphi_1 * \varphi_2) \quad \textit{and} \quad g(\cdot) = \sum_{n \in \mathbb{Z}^d} a_n (\varphi_1 * \varphi_2)(\cdot + n).$$

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