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Research Article

The impact of discovery-based learning with physical manipulatives in teaching the area of triangles and quadrilaterals on students' achievement

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Discovery-based learning is characterized by a learning environment where students are not passive observers but actively construct new knowledge, which is expected to contribute to their cognitive development. Since it is desirable for students to reflect on their physical actions during this process and considering the positive effects of using physical manipulatives in the teaching process, it naturally follows that discovery-based learning with manipulatives in mathematics education could potentially have a positive impact on students' knowledge acquisition and skill development. The aim of this research is to examine the impact of discovery-based learning with manipulatives, specifically tangram models made from cardboard, on sixth-grade students' achievements in learning the area of triangles and quadrilaterals. The sample consisted of students from four classes at an elementary school in Vranje, Serbia. The research results indicate that discovery-based learning with physical manipulatives leads to better student performance in solving higher-complexity mathematical problems, specifically those that require applying the key property of finite additivity. Also, results support previously observed characteristics in literature - the positive impact of the heuristic approach to teaching on the retention of acquired mathematical knowledge.

Keywords: Area of triangles and quadrilaterals; Discovery-based learning; Heuristic approach in teaching; Manipulatives; Tangram;

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1. Introduction

The goal of mathematics education should not be limited to providing students with the opportunity to memorize and reproduce rules, concepts, and formulas without understanding them. Instead, students should acquire the planned content with understanding while making connections with previously learnt material, thereby forming a foundation for the later acquisition of new concepts, rules, and procedures. The learning theory that many researchers believe holds great potential for improving the quality of students' knowledge and achievements is constructivism (Eby et al., 2005; Von Glasersfeld, 1995). Constructivist learning is characterized by students' cognitive processes that enable the active construction of knowledge and understanding

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of conceptual connections. One instructional approach grounded in constructivism is discovery learning. Discovery learning represents an innovative pedagogical approach aimed at enabling and further encouraging active student participation, increasing motivation for learning, and fostering their intellectual development, independence, and creativity. This, in turn, leads to more meaningful and long-lasting knowledge acquisition (Diano Jr. et al., 2021; Hoon et al., 2013). On the other hand, research results indicate that the introduction of physical manipulatives has a positive impact on students' academic achievement in mathematics (Bower et al., 2020).

One of the topics covered in the sixth grade of elementary school in Serbia is *The Area of Triangles and Quadrilaterals*. The foundation of studying the areas of parallelograms, triangles, trapezoids, quadrilaterals with perpendicular diagonals, and other quadrilaterals is the additivity of area as a measure of a part of the plane. However, this property is not sufficiently utilized in traditional teaching, even though it represents fundamental knowledge for students, which is later built upon when learning procedures for determining the area of other geometric figures in the plane and geometric solids (based on calculating the surface area of their nets). Considering this, as well as the positive results of discovery-based learning and the use of manipulatives in mathematics education, we came up with the idea to examine the impact of discovery-based learning using manipulatives in teaching this topic on students' achievements.

2. Theoretical Framework

2.1. Discovery-Based Learning

The term "heuristic" derives from the Greek word "heureksein" meaning "to discover" or "to find." According to the definition originally provided by Pólya (Pólya, 1954), heuristics is the study of methods and strategies for problem-solving and relates to experience-based techniques aimed at improving problem-solving abilities (Mousoulides & Sriraman, 2014).

In general, a heuristic strategy serves as a tool for problem-solving where the steps are sequential (Rosyada & Retnawati, 2021). According to Pólya, it consists of four phases: understanding the problem, devising/experimenting and selecting a solution strategy, exploring/solving, and reflecting/backtracking (Pólya, 1954). Discovery-based learning begins with the teacher presenting information or problems. Subsequently, students identify problems, gather, process, and analyze data from these problems. Based on their research findings, students formulate hypotheses. They then verify the validity of these hypotheses by consulting various reference sources and discussing them with other students or groups, ultimately reaching conclusions. After students draw conclusions, the teacher confirms their accuracy, provides explanations, and corrects any errors or misconceptions that may exist, ensuring that students have accurate representations of the concepts learnt (Kamaluddin & Widjajanti, 2019; Westwood, 2008).

Through the heuristic method in mathematics education, students learn through personal experience and discovery, which helps them develop cognitive, affective and psychomotor skills. Additionally, heuristic method fosters cognitive development and increases students' confidence in solving mathematical problems (Pugosa et al., 2024). Within this approach to mathematics education, students construct their understanding of mathematical concepts, making the primary responsibility of teaching not merely to impart knowledge from teacher to student but to create situations where students enhance their mental structures. In an environment where discovery learning occurs, students are not passive recipients of knowledge; rather, they construct new mathematical knowledge by reflecting on their physical and mental processes (Milenković & Dimitrijević, 2019).

Research has shown that discovery learning can significantly positively impact students' academic achievements. Discovery-based learning is associated with better understanding of concepts, longer retention of knowledge, and the development of critical thinking skills (Alfieri et al., 2011). A meta-analysis conducted by Hattie and Timperley indicated that students who learn through discovery achieve better outcomes across various subjects compared to those who learn through traditional methods (Hattie & Timperley, 2007).

In the context of mathematics learning, discovery learning is linked to better academic performance, especially in areas such as algebra and number theory (Abonyi & Umeh, 2014; Diano Jr. et al., 2021; Pugosa et al., 2024; Schoenfeld, 1985). Students who learn through discovery tend to develop a deeper understanding of mathematical concepts and a greater ability to transfer knowledge to new situations (Kirschner et al., 2006; Pugosa et al., 2024). Research findings support the idea that discovery learning has also increased students' interest in learning mathematics (Ramadhani et al., 2023). Discovery learning provides students with the opportunity to discover and explore various methods of solving a specific mathematical problem. This approach to teaching and learning encourages students to seek solutions to the given problem rather than simply memorizing rules and procedures (Diano Jr. et al., 2021).

Discovery learning, as a teaching method that emphasizes the active role of students in constructing new knowledge, has various advantages. Numerous studies provide empirical evidence that discovery learning positively affects cognitive aspects of mathematics learning, such as achievement, critical thinking skills, creative thinking skills, mathematical reasoning abilities, and communication. Furthermore, learning mathematics through discovery positively influences affective aspects, such as students' motivation and beliefs (Kamaluddin & Widjajanti, 2019). Each step in discovery learning plays a crucial role in learning mathematics. A study by Kamaluddin and Widjajanti (2019) showed that the step of problem formulation can enhance students' observation, motivation, and critical thinking skills, while the research phase can positively affect knowledge retention, conceptual understanding, the examination of different properties, analogy-making, and students' achievement, as well as their critical thinking, beliefs, self-regulation, observations, and skills in group discussions. Moreover, in the authors' view, the step of hypothesis formulation has great potential to enhance students' abilities in retention, analogy-making, and drawing conclusions, while the verification step can develop skills in making conclusions, communication, and group discussion, and the confirmation step can help students develop conceptual understanding and communication skills (Kamaluddin & Widjajanti, 2019).

Learning geometric content through discovery can significantly enhance students' understanding of spatial-visual concepts and the development of geometric skills (Clements & Battista, 1992). Research conducted by Hilbert et al. (2008) regarding the application of the heuristic method in proving geometric theorems shows that the skills of performing proofs and the conceptual knowledge of students are significantly better among those who learnt through discovery compared to those who did not. A heuristic approach can significantly improve the ability to solve non-standard geometric problems and influence changes in students' entrenched beliefs that problems must be solved exclusively by applying a specific formula (Hoon et al., 2013). Generally, students engaged in discovery-based learning activities where they manipulate geometric objects demonstrate better understanding and application of geometric principles in problem-solving (Battista, 1999).

2.2. The Use of Manipulatives in Mathematics Education

Manipulatives are physical objects used in mathematics education to help students visualize and understand abstract concepts. More precisely, Damjanović states that manipulatives represent real (physical, tangible) or virtual (computer/software programs, packages) objects that students work with (or on) using their sensorimotor abilities. Through empirical means, they verify and practice (confirm) their acquired knowledge, skills, and abilities; observe new circumstances and facts, connect and conclude, thereby constructing a new body of knowledge, skills, and competencies (Damjanović, 2008). The application of manipulatives allows students to explore and discover mathematical relationships in a concrete manner, which can enhance their understanding and retention of knowledge (Moyer, 2001).

Results from various studies support the idea that the use of manipulatives in mathematics education positively impacts students' achievements and their motivation to learn (Carbonneau et al., 2012; Kontaş, 2016; Ojose & Sexton, 2009; Sowell, 1989). Carbonneau (2012) investigated the effect of using physical manipulatives in mathematics instruction compared to a traditional teaching approach where abstract concepts are represented using symbolic representations of concepts and found a small to moderate effect favoring instructional strategies that involve the use of manipulatives. Furthermore, they demonstrated that mathematics instruction using manipulatives led to moderate or significant effects regarding students' retention of knowledge and small effects when higher-level outcomes, such as problem-solving, knowledge transfer, and argumentation, were considered (Carbonneau et al., 2012).

In geometry instruction, manipulatives play a key role in developing students' spatial-visual skills. The use of manipulatives such as geometric models, blocks, and other tools enable students to explore and visualize geometric concepts in a concrete manner (Clements, 1999). Research shows that students who use manipulatives in geometry instruction have a better understanding and apply geometric concepts with greater accuracy (Clements & Battista, 1992). The introduction of innovations and the improvement of instruction through optimal and skillful use of concrete manipulatives not only motivates students by alleviating mathematical anxiety but also enhances achievement in learning geometry (Gurung & Chaudhary, 2022).

Research conducted by Clements and Sarama aimed at examining the effects of using manipulatives in early mathematics education indicates that students who used manipulatives during learning developed a deeper understanding of geometric concepts and problem-solving skills compared to children who learnt without manipulatives (Sarama & Clements, 2009).

2.3. The Aim

The objective of the research is to examine the impact of discovery-based learning, using physical manipulatives, on student achievement in the topic of triangle and quadrilateral areas, and to compare their achievements with those of students who learnt about triangle and quadrilateral areas in a traditional manner.

3. Method

3.1. Research Design and Participants

This study employed a quasi-experimental design with a pre-test–post-test control group structure to investigate the effects of discovery-based learning using physical manipulatives. The intervention was implemented in natural classroom settings, comparing the experimental group exposed to tangram-based activities with a control group receiving traditional instruction.

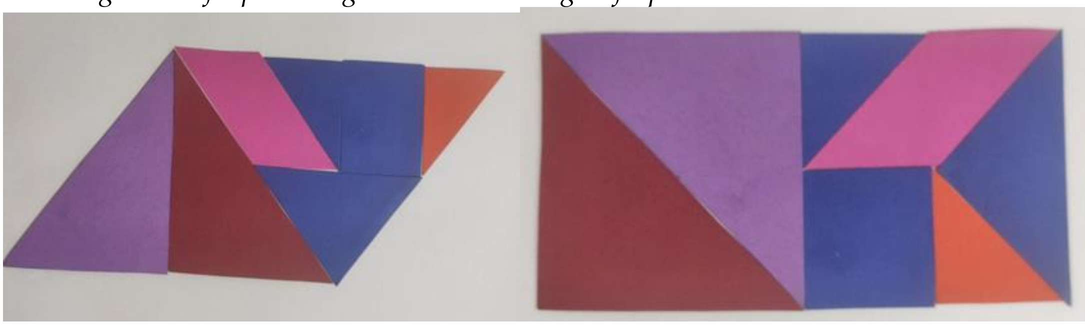
The research sample consisted of 98 students from four classes of the elementary school "Svetozar Marković" in Vranje, Serbia who were in the sixth grade during the 2021-2022 school year when the research was conducted. Two classes, with a total of 46 students, made up the control group. The remaining two classes, consisting of 52 students, formed the experimental group. In terms of gender distribution, there were 40 boys and 58 girls in the sample.

3.2. Procedure

The topic *Area of Triangles and Quadrilaterals* covered over 12 lessons, held in May and June 2022, including four lessons on new material, seven lessons on practice, and one lesson for knowledge assessment. In the first lesson, the students from the experimental group were introduced to the manipulatives they would use throughout all lessons on new material – cardboard models based on tangram shapes (each student was able to independently use their own tangram cardboard models during class). These manipulatives were used to teach the following units: area of a parallelogram, area of a triangle, area of a trapezoid, and the area of a quadrilateral with perpendicular diagonals. At the beginning of the topic, students were asked to assemble different shapes and determine the area of the combined figure. The goal of this exercise was for students to independently discover the property of additivity of area as a measure – the concept that if a figure is divided into multiple parts and these parts are rearranged through translations and rotations in the plane without leaving gaps or overlaps, the area remains unchanged. Additivity of area is the key concept that students need to grasp since the formulas for calculating areas of all quadrilaterals and triangles are derived by rearranging the figure into one whose area can be calculated.

The lessons on new material followed this structure: students used the tangram pieces to assemble the figure whose area they needed to calculate, then tried to rearrange the figure to form a shape with a known area formula. Figure 1 shows a parallelogram that students rearranged into a rectangle, observing that the sides of the rectangle correspond to the length of the parallelogram's base and its corresponding height.

Figure 1
Rearrangement of a parallelogram into a rectangle of equal area



In Figure 2, a trapezoid is shown that the students rearranged into a rectangle. In Figure 3, the rearrangement of a triangle is shown. During the lesson on the Area of a Triangle, students were also tasked with cutting out a model of a parallelogram from paper, then dividing it along one diagonal into two triangles (see Figure 4). They observed that these two triangles are congruent, proved it, and then derived a conclusion on how to calculate the area of any triangle.

Figure 2
Rearrangement of an isosceles trapezoid into a rectangle of equal area

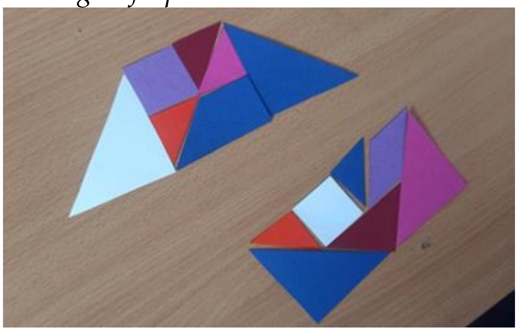


Figure 3
Procedure for rearranging a triangle into a rectangle

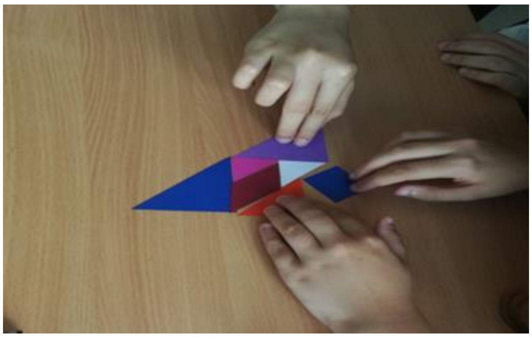
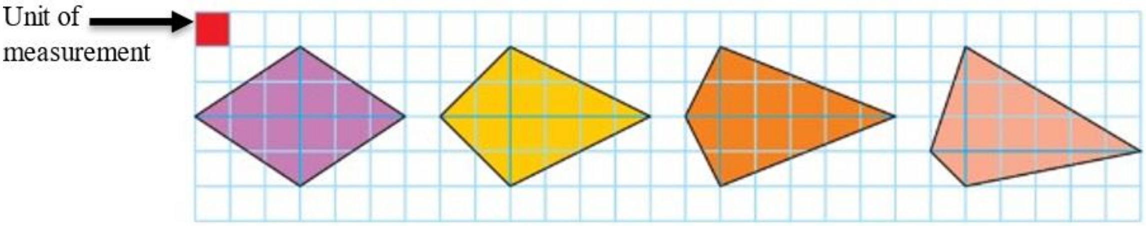


Figure 4
Division of a parallelogram along a diagonal into two congruent triangles

When it came to the last lesson on this topic, the work with the students in the experimental group began differently - by asking them to calculate the areas of figures represented on a unit grid. Students with above-average achievements surprisingly quickly recognized an appropriate approach - dividing each figure into four right triangles and then determining the area of those triangles, a method they had previously learnt (Figure 5). Soon, students with average achievements came to the same conclusions, after which, with the help of their peers and the teacher, even below-average students understood the given idea. The idea was then illustrated and applied using previously used manipulatives that the students were accustomed to.

Figure 5
Motivational task for calculating the area of a quadrilateral with perpendicular diagonals (Ikodinović & Dimitrijević, 2022)



The instruction sessions with the two classes that formed the control group were conducted in a traditional manner. The teacher used a frontal approach with lecture-based teaching method, along with drawing appropriate figures, dividing figures on the board, proving the congruence of certain triangles, deriving formulas, and then solving tasks to apply the concepts through adequate examples.

Thus, the method of delivering new content significantly differed in both approach and the activities of the teacher and students, as well as their relationship. In the control groups, the teacher was at the center of the teaching process, whereas in the experimental groups, the teacher played more of an organizer and moderator role, providing moderate guidance for students to discover new knowledge themselves. However, the practice lessons were not different between the two groups of students. In all four classes, the practice sessions were conducted in the same way—some lessons were frontal, some group-based, and others involved work in pairs.

3.3. Instruments

Three instruments were created for the purposes of this research: a pre-test, a final test, and a retest. The pre-test (see Appendix 1) consisted of five tasks. The requirements in these tasks involved determining the area of squares and rectangles (content related to the areas of figures that students had learnt during the first cycle of primary education), as well as the properties of triangles, parallelograms, trapezoids, and deltoids, which they were learning in the sixth grade, i.e., the figures whose areas they would calculate within the teaching topic Areas of Triangles and Quadrilaterals. The students took the pre-test immediately before the start of the Areas of Triangles and Quadrilaterals topic. The final test (see Appendix 2) consisted of four tasks that the students solved immediately after completing the teaching and review of all units of the topic. The requirements in these tasks involved calculating the areas of a parallelogram, triangle, deltoid, and trapezoid. The retest (Appendix 3) contained five tasks, requiring students to calculate the areas of a triangle, parallelogram, trapezoid, and rhombus, as well as to determine the area of a complex figure that needed to be divided into several smaller, familiar figures. In addition, the students were required to write down the formulas for calculating the areas of triangles and different classes of quadrilaterals in the retest (see Appendix 3).

3.4. Data Analysis

The collected data was processed using the IMB SPSS Statistics 20 software package. The following statistical measures and procedures were used: arithmetic mean, statistical tests for determining the normality of numerical data distribution, and the non-parametric Mann-Whitney test for comparing variable distributions in two independent groups.

4. Results

4.1. Results of the Pre-Test

Before the implementation of the instructional unit on the *Area of Triangles and Quadrilaterals*, students' knowledge about the area of squares and rectangles was assessed, as was their understanding of the topics related to triangles and quadrilaterals. The test consisted of five tasks, each scored with a maximum of 10 points. Since the total score on the initial test did not have a normal distribution, students' achievements were compared using the Mann-Whitney test (Table 1). The results of the test indicated that the prior knowledge of students in the experimental group ($M = 35.57$, $SD = 9.16$) and the control group ($M = 33.92$, $SD = 8.56$) did not differ significantly ($Z = -1.312$, $p = .190$), thus fulfilling the prerequisite to proceed with further research.

Table 1
Results of the Mann-Whitney test for the Pre-Test

Group	*N*	*Mean*	*Median*	*Man-Whitney U test*	
				Z	*p (2-tailed)*
Experimental	52	35.57	30	−1.312	.190
Control	46	33.92	30		

4.2. Results of the Final Test

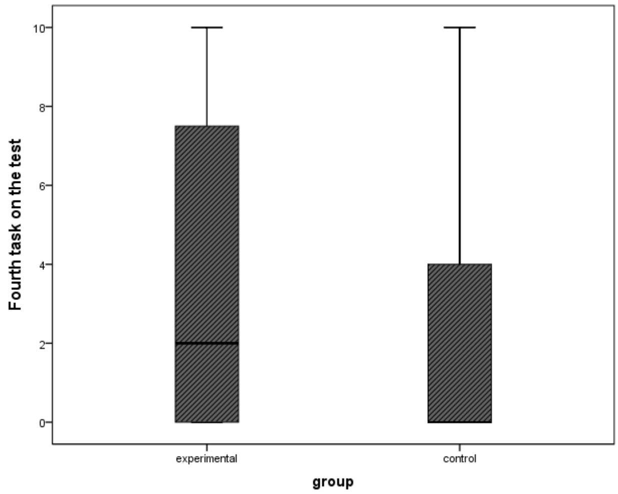
Regarding the achievements of students who followed the lessons on the instructional unit on the *Area of Triangles and Quadrilaterals* through traditional teaching methods with a frontal approach, compared to those who learnt through heuristic approach using manipulatives, the achievements were analyzed based on the students' performance on identical tasks in the final test, conducted after completing the topic. Students solved four tasks, with the first task relating to determining the area of a parallelogram when the length of one side and the height corresponding to that side are known. The second task required determining the height of a triangle that corresponds to a known side length when the area of that triangle is known. The third task asked students to calculate the lengths of the diagonals of a quadrilateral with normal diagonals and a known area. In the final task, students were asked to calculate the area of an isosceles trapezoid if the lengths of the height and the longer base of the trapezoid were known, and the acute angle of the trapezoid was 45°. Each task was scored out of ten points (a total of forty), and the tasks were partially graded.

The number of points scored by students on the test did not have a normal distribution for any of the four tasks, nor for the total points scored. For this reason, the results achieved by the students were analyzed using the Mann-Whitney test. The results of the Mann-Whitney test (see Table 2) indicate that the differences in the total number of points scored by students in the control group and the experimental group are not statistically significant ($Z = -1.082$, $p = .279$), although the median total score for the control group students was 20, while the median score for the experimental group students on the test was 28 points. In addition to the total number of points, we also analyzed the scores obtained by the two groups of students for each task. The results show that there are no statistically significant differences in the number of points scored by students in the experimental and control groups for the first task ($Z = -0.547$, $p = .584$), the second task ($Z = -1.154$, $p = .249$), and the third task ($Z = -0.557$, $p = .578$). However, there is a statistically significant difference in the number of points scored by the experimental group students compared to the control group students when observing the fourth task on the test ($Z = -2.143$, $p = .032$). When analyzing the requirements of the fourth task, it was found to be the most challenging on the test that the students completed. It is particularly interesting that students were required to divide the given isosceles trapezoid into a rectangle and two right triangles and then utilize the characteristics of the isosceles right triangle that they were familiar with to successfully solve the task. These differences suggest that students who used manipulatives and learnt through discovery in math classes significantly better understood the relationships among the elements of the isosceles trapezoid (see Figure 6).

Table 2
Results of the Mann-Whitney test for the Final Test

Task	*Group*	*N*	*Mean*	*Median*	*Man-Whitney U test*	
					Z	*p (2-tailed)*
First	Experimental	52	9.50	10.00	−0.547	.584
	Control	46	9.63	10.00		
Second	Experimental	52	7.02	8.00	−1.154	.249
	Control	46	6.37	6.00		
Third	Experimental	52	5.44	7.00	−0.557	.578
	Control	46	4.88	5.00		
Fourth	Experimental	52	3.75	4.00	−2.143	.032
	Control	46	2.33	0.00		
Total	Experimental	52	25.71	28.00	−1.082	.279
	Control	46	23.21	20.00		

Figure 6
The number of points achieved by students in completing the fourth task



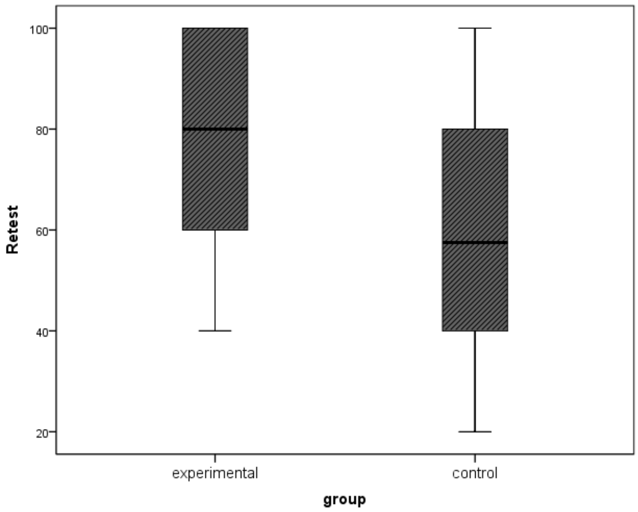
4.4. Results of the Retest

In the retest, as mentioned earlier, students had five tasks in addition to the initial requirement to write formulas for calculating the area of parallelograms, rhombuses, triangles, trapezoids, and deltoids. The number of points they scored on each task individually, as well as the total score on the entire test, did not follow a normal distribution in either the control or experimental group, so the Mann-Whitney test was again used to compare the achievements of the two groups of students (see Table 3). Regarding the knowledge of formulas needed to calculate the areas of triangles and quadrilaterals, it turned out that the differences in the achievements of students from the two groups were not statistically significant ($Z = -1.242$, $p = .214$). However, at the level of student performance in solving the five tasks on the retest (Figure 7), the differences in the achievements of the two groups of students were statistically significant in favor of those who learnt about the area of triangles and quadrilaterals through discovery ($Z = -2.705$, $p = .007$).

Table 3
Results of the Mann-Whitney test for the Final Test

Task	Group	N	Mean	Median	Man-Whitney U test	
					Z	*p (2-tailed)*
First	Experimental	52	19.66	20.00	−1.00	.317
	Control	46	18.75	20.00		
Second	Experimental	52	18.75	20.00	−1.519	.129
	Control	46	17.16	20.00		
Third	Experimental	52	14.09	15.00	−3.033	.002
	Control	46	7.73	5.00		
Fourth	Experimental	52	13.30	15.00	−0.842	.400
	Control	46	11.25	15.00		
Fifth	Experimental	52	9.32	5.00	−2.073	.038
	Control	46	5.00	0.00		
Total	Experimental	52	75.45	80.00		
	Control	46	60.80	57.50	−2.705	.007

Figure 7
The number of points scored by students on the retest



By examining Table 3, it can be observed that the differences in achievements between the two groups of students are not statistically significant in the first, second, and fourth tasks. Analyzing the content of the tasks, we see that in the first task, students are asked only to apply the formula for calculating the area of a triangle when the lengths of the sides and the corresponding height are given. In the second task, they are required to calculate the height of a parallelogram based on the values of the area and the lengths of the sides (again, a simple application of the formula). In the fourth task, students are asked to express the area of a rhombus in two ways, and then, after calculating the area of the rhombus, determine the length of a side (and subsequently the perimeter) based on the length of the height of the rhombus. Thus, solving these three tasks boils down to knowing the formulas, applying those formulas, and performing simpler calculations.

On the other hand, the differences in achievements between the two groups of students are statistically significant in the third and fifth tasks. Like the results of the final test, the differences are statistically significant in the third task, where it is necessary to divide a right trapezoid into an isosceles right triangle and a rectangle, determine the lengths of the sides based on the properties of these two figures, and then apply the formulas to determine their areas. Among all five tasks, the differences in achievements between the experimental and control groups are the greatest at this task level. Additionally, the differences are statistically significant (see Table 3) in the success of students in solving the fifth task. Specifically, with knowledge of the properties of axis-symmetric figures (which facilitates the problem-solving process but is not a necessary condition), students need to divide the given complex figure into triangles and rectangles and use the property of area as a measure of parts of a plane, which states that the area of the figure is equal to the sum of the areas of the parts into which that figure is divided.

5. Discussion

Our results are in line with the findings of the study conducted by In'am and Hajar (2017), which indicate that the implementation of discovery-based learning enhances mathematics learning outcomes compared to traditional teaching methods. This approach has been shown in previous studies to be particularly effective for students with high self-efficacy, who tend to have better achievements in mathematics compared to students with moderate or low self-efficacy (Ramadhani et al., 2023).

The results of the retest are consistent with previous research indicating that discovery-based learning is associated with a better understanding of concepts and longer retention of knowledge (Alfieri et al., 2011). The achievements of the two groups of students on the retest differ in tasks where students are required not only to apply formulas but also to demonstrate theoretical and conceptual knowledge and then apply the formulas, they are familiar with for calculating the areas of triangles and quadrilaterals. Interestingly, students who learnt about the areas of triangles and quadrilaterals through discovery showed a higher level of success and knowledge on the final test and retest in tasks requiring a deeper, fundamental understanding of area as a measure. This aligns with findings from studies that suggest discovery-based learning develops a deeper understanding of mathematical concepts and the ability to apply knowledge in new situations (Kirschner et al., 2006; Pugosa et al., 2024). Students who acquired geometric concepts through discovery achieved a higher level of conceptual knowledge compared to their peers who did not have that opportunity (Hilbert et al., 2008).

Moreover, discovery-based learning fosters development of reasoning skills in mathematics, more precisely in geometry (Khairunnisa & Juandi, 2022). These skills can be highly beneficial for students, considering that this topic serves as a foundation for more advanced geometric concepts they will encounter in their future mathematical education. Maarif and Soebagyo (2024) argue that, beyond its impact on student achievement, discovery-based learning also encourages student interaction among themselves and the expression of their ideas, which is essential for the development of mathematical competence.

When interpreting the differences in success between students who learnt about the areas of triangles and quadrilaterals through traditional teaching methods and those who learnt through discovery, we must not overlook the fact that only students in the latter group used physical manipulatives during their learning process. Students who used manipulatives demonstrated significantly better results in mathematics compared to those who did not (Baruiz & Dioso, 2023). In this context, our results are consistent with earlier findings that suggest discovery-based learning combined with the manipulation of physical geometric objects leads to better understanding and successful application of geometric concepts in solving tasks (Battista, 1999) as well as fostering a better understanding of geometric concepts and problem-solving skills among students (Sarama & Clements, 2009). Thus, the use of tangrams for rearranging figure parts had a positive impact on students' geometry knowledge (Ponte et al., 2023). More precisely, the results of our study are consistent with a previously conducted study (Baki et al. 2011), which demonstrated that groups learning geometric concepts with the help of physical manipulatives achieved significantly better performance compared to groups that learned geometric concepts using traditional methods.

The combination of using manipulatives during lessons introducing new concepts, while practice and systematization lessons were conducted using traditional methods, supports the notion that students who adopted this approach achieved better academic results. Similar findings were previously reported by Kablan (2016), who showed that students who did not use physical manipulatives did not perform at the same level as their peers who did. However, Kablan (2016) also found that the amount of time students spent with manipulatives was not positively correlated with their mathematics achievement.

Furthermore, based on the results of the total number of points that students achieved on the retest, the application of physical manipulatives in the form of tangrams and discovery-based learning had a medium effect size (r = 0.27) on knowledge retention (Pallant, 2020). This finding is largely consistent with the results of a meta-study by Carbonneau (2013), which indicated that the effects of learning with manipulatives have a moderate to large impact on knowledge retention.

6. Conclusion

Although discovery-based learning has been the subject of numerous studies worldwide for some time, this is not the case when it comes to research examining the impact of the heuristic method in Serbia (especially in geometric content). Along these lines, considering the positive results of using manipulatives in mathematics instruction and the specific nature of the topic *Area of Triangles and Quadrilaterals*, which is fundamentally based on the additivity of area (a concept that should be applied when determining the area of increasingly complex geometric figures), discovery-based learning of this content emerges as a natural choice of teaching and learning approach. The results of this study confirm this assumption. More specifically, the achievements on the entire final test of the two groups of students did not differ significantly, while the differences in students' achievements on the retest were statistically significant. This result supports the previously observed and empirically confirmed characteristics of discovery-based learning, namely, a deeper understanding of learnt content and longer retention of knowledge. Additionally, by examining the tasks in which statistically significant differences were found on both the final test and the retest, we conclude that students who learnt the content from *Area of Triangles and Quadrilaterals* through discovery-based learning with the use of manipulatives had a significantly better grasp of the fundamental concept of area as a measure of a part of a plane and applied it in more complex tasks. Based on the results obtained, we can conclude that discovery-based learning with the use of manipulatives in teaching this geometric content leads to better mathematics achievement among sixth grade (13 years old) students.

Recommendations for future research could include repeating this study on a larger sample of students from different schools across multiple cities in Serbia and wider, as well as conducting a longitudinal study that would involve analyzing students' achievements in determining the areas of figures in plane in the sixth grade (when students learn about the areas of triangles and quadrilaterals) and solid figures in the eighth grade (when students study the areas of prisms, pyramids, cylinders, and cones).

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Data availability: The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Ethical statement: All participants who took part in the study provided their consent for participation, data collection, and analysis of the collected data. All personal information that could lead to the identification of participants has been removed. No additional ethical approval was required.

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Appendix 1. Pre-Test

1. a) Calculate the area of a square with side length a = 9 cm.
b) Calculate the area of a rectangle with side lengths a = 10 cm and b = 8 cm.

2. If two interior angles of a triangle are given as α = 45° and β = 72°, then the measure of the third angle γ is:
a) 83°; b) 73°; c) 63°; d) 53°; e) 43°.

3. Which of the following statements are true?
1) The diagonals of a parallelogram are equal.
2) The diagonals of a parallelogram bisect each other.
3) The diagonals of a parallelogram are perpendicular to each other.
4) Opposite angles of a parallelogram are equal.
5) Adjacent angles of a parallelogram are equal.
a) 1), 2), and 4); b) 2), 3), and 4); c) 2) and 4); d) 2) and 5); e) 1) and 4).

4. The measures of the opposite angles of a trapezoid are 56° and 111°. The other angles of the trapezoid are:
a) 56° and 111°; b) 124° and 69°; c) 48° and 131°; d) 156° and 34°.

5. The angles between the equal sides of a deltoid are 84° and 72°. The measures of the other angles of the deltoid are:
a) 102°; b) 204°; c) 51°; d) 84°; e) 72°.

Appendix 2. Final Test
1. Calculate the area of a parallelogram if side a = 12 cm and the height h_a = 7 cm.

2. The area of a triangle is 102 cm² and the side a = 12 cm. Calculate the height corresponding to side a.

3. Determine the lengths of the diagonals of a deltoid if one diagonal is four times longer than the other, and the area of the deltoid is 18 cm².

4. In an isosceles trapezoid, the larger base is 16 cm, the height is 6 cm, and the acute angles are 45°. Calculate the area of this trapezoid.

Appendix 3. Retest
Write the formula for calculating the area of:
- Parallelogram,
- Rhombus,
- Triangle,
- Trapezoid,
- Deltoid.

1. Calculate the area of a triangle if the side of the triangle is a = 8 cm and the height h_a = 10 cm.

2. What is the height of the parallelogram that corresponds to the side of the parallelogram with a length of 15 cm if the area of the parallelogram is 60 cm²?

3. Calculate the area of a right trapezoid whose base lengths are 18 cm and 8 cm, and the acute angle is 45°.

4. The diagonals of a rhombus are 12 cm and 16 cm, and the height of the rhombus is 9.6 cm. Calculate the perimeter of the rhombus.

5. Calculate the area of the figure shown in the picture.