Design of linear-phase IIR integrators with maximally-flat and Chebyshev magnitude responses

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Abstract

This paper proposes two methods for designing linear-phase infinite impulse response integrators. The first method, referred to as the maximally-flat one, imposes flatness conditions on the frequency response error function, leading to a system of linear equations that have to be solved to determine unknown coefficients. Furthermore, a relation is established between the proposed maximally-flat integrators and existing integer-order linear-phase integrators derived using the algebraic polynomial-based quadrature rules, demonstrating that the latter represent special cases of the proposed integrators. The second method, referred to as the optimal one, minimizes the complex frequency response error function in the weighted Chebyshev sense, which is achieved by an efficient exchange algorithm that exhibits rapid convergence. The proposed linear-phase integrators are also compared with several existing linear- and nearly linear-phase integrators.

Keywords: digital integrators, linear-phase, maximally-flat design, optimal design

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1. Introduction

Digital integrators are used in various engineering applications where the computation of the time-integral of an input signal from its discrete-time sam-

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ples is required. These applications include voltage and current measurement devices [18], acceleration sensor data processing [14], and magnetic field measurements [7], among others. Additionally, digital integrators are essential building blocks for Prism signal processing [13]. Based on the duration of the impulse response, digital integrators can be classified into finite impulse response (FIR) and infinite impulse response (IIR) filters. On the other hand, since numerical integration can be interpreted as the recursive digital filtering [11], it comes as no surprise that the majority of existing digital integrators are of IIR type.

There are two main approaches to the IIR integrator design. Methods of the first approach [6, 23, 25, 20, 24, 8, 4, 5, 17] employ numerical quadrature rules to derive IIR integrators transfer functions. For instance, design of IIR integrators using Newton-Cotes, Gauss-Legendre, and Romberg integration rules is consider in [20, 23, 24]. While the integer-order Newton-Cotes IIR integrators from [23], including the well-known rectangular, trapezoidal, Simpson 1/3, Simpson 3/8 and Boole's integrators [11], are linear-phase filters, the other integrators from [23, 24] utilize fractional delay elements, whose design have to be treated separately. In [8], modified Newton-Cotes rules are used to derive integer-order linear-phase IIR integrators with improved magnitude responses. To address the inherent limitation of integrators obtained by the algebraic polynomial-based quadrature rules, which is the increasing behavior of the magnitude response error, the design of the linear-phase IIR integrators using numerical integration rules constructed to integrate trigonometric polynomials is proposed in [6]. Additionally, several nonlinear-phase IIR integrators with improved magnitude responses at midband and high frequencies have been obtained by interpolating trapezoidal integration rule with rectangular [4, 17], Simpson 1/3 [5], Simpson 3/8 [5], and Simpson 1/3 and Boole's rules [5].

Methods of the second approach formulate the IIR integrator design problem as the frequency domain constrained optimization problem, solved using either classical [21, 19, 16, 2] or metaheuristic [1, 10, 3] optimization techniques. In [21] and [2], second-order [21] and arbitrary-order [2] linear-phase integrators are designed by minimizing the weighted Chebyshev norms of the magnitude response error over a specified frequency range using linear programming. On the other hand, nearly linear-phase IIR integrators [19, 16] are designed by minimizing the phase response linearity error subject to the constraints imposed on the magnitude response error function, while the second-order integrators [10] are obtained by minimizing the magnitude error subject to the constraints imposed on the phase response linearity error. In [3], second-, third-, and fourth-order IIR integrators are obtained by inverting transfer functions of differentiators that were designed by minimizing the L_1 norm of the complex frequency response error function. Another optimization-based design method is discussed in [1], where nearly linear-phase is indirectly achieved by minimizing the varied L_k norm of the absolute magnitude response error.

In this paper, starting from the suitably chosen expression for the linear-phase IIR transfer function and its frequency response, two new linear-phase IIR integrator design methods are proposed. Since the highest accuracy around a particular frequency ω_0 is achieved through a maximally-flat design, this method is discussed first. Additionally, it will be shown that magnitude responses of integer-order linear-phase IIR integrators from [8, 23] are maximally flat at $\omega = 0$, meaning that these integrators are special cases of the proposed ones. For the convenience of the filter designers, closed form expressions or exact coefficients values (in the case of $\omega_0 = 0$) are tabulated for various filter orders. The second proposed design method minimizes the Chebyshev norm of the complex frequency response error over the frequency range of interest. This is achieved using an iterative algorithm that typically converges in a few iterations. Therefore, the integrators designed in this way can be regarded as optimal, with the second-order IIR integrators from [21], including the well-known Tick's integrator [12], representing special cases.

The rest of the paper is structured as follows. Linear-phase IIR integrator design problem is formulated in Sec. 2, while the maximally-flat and optimal design methods are described in Sec. 3. A relation between the proposed and existing maximally-flat linear-phase integrators (although not explicitly treated as such in the available literature) is established in Sec. 4, along with the derivation of a closed-form expression for the coefficients of the proposed integrators when $\omega_0 = 0$. Design examples and comparison with the existing linear- and nearly linear-phase integrators, both of IIR and FIR types, are given in Sec. 5, while concluding remarks are provided in Sec. 6.

2. Problem formulation

Phase response of digital integrator, whose frequency response in ideal case is given by

$$H_{\mathrm{d}}\left(\mathrm{e}^{\mathrm{j}\omega}\right) = \frac{1}{\mathrm{j}\omega}\mathrm{e}^{-\mathrm{j}\omega\tau}, \tag{1}$$

where τ is group delay, exhibits a discontinuity of π radians at $\omega = 0$, as it jumps between $\pi/2$ an $-\pi/2$. This behavior can be achieved by a transfer function with a pole placed at $z = \mathrm{e}^{\mathrm{j}0} = 1$, which, since on the unit circle, does not compromise the linearity of the filter's phase response. Therefore, one possible form of the transfer function of the linear-phase integrator is

$$H(z) = \frac{B(z)}{1 - z^{-K}}, \tag{2}$$

where $K \geq 1$ is the feedback delay, and $B(z)$ is a linear-phase finite impulse response transfer function whose roots are not canceled by the roots of $\left(1 - z^{-K}\right)$. Therefore, $B(z)$ has to be a type I or II transfer function, and if $B(z)$ is type II transfer function (with a zero inherently placed at $z = \mathrm{e}^{\mathrm{j}\pi} = -1$), K has to be odd. Additionally, for $K > 1$, magnitude response of integrator tends to infinity at frequencies $2k\pi/K$, for $k = 1, 2, \ldots, K-1$. Note that, without loss of generality, the sampling frequency is assumed to be equal to 1 Hz in (1).

Denoting the length of the linear-phase FIR filter $B(z)$ by L, and its impulse response samples by b_k, $k = 0, 1, \ldots, L-1$, frequency response $B\left(\mathrm{e}^{\mathrm{j}\omega}\right)$ can be expressed as

$$B\left(\mathrm{e}^{\mathrm{j}\omega}\right) = 2\mathrm{e}^{-j\widetilde{\tau}\omega}\mathbf{c}(\omega)\cdot\mathbf{g}, \tag{3}$$

where

$$\widetilde{\tau} = \frac{L-1}{2} \tag{4}$$

is the group delay of $B(z)$, while vectors $\mathbf{c}(\omega)$ and $\mathbf{g}$ are

$$\mathbf{c}(\omega)=\begin{bmatrix}\cos(\widetilde{\tau}\omega) & \cos((\widetilde{\tau}-1)\omega) & \dots & \cos((\widetilde{\tau}-\lfloor\widetilde{\tau}\rfloor)\omega)\end{bmatrix}, \tag{5}$$

$$\mathbf{g}=\begin{cases}\begin{bmatrix}b_0 & b_1 & \dots & b_{\widetilde{\tau}-1} & \frac{1}{2}b_{\widetilde{\tau}}\end{bmatrix}^{\mathrm{T}}, & L \text{ odd}\\ \begin{bmatrix}b_0 & b_1 & \dots & b_{\widetilde{\tau}-1/2}\end{bmatrix}^{\mathrm{T}}, & L \text{ even}\end{cases}. \tag{6}$$

The superscript T in defined vectors denotes the transposition operation. Note that for L even, $\lfloor\widetilde{\tau}\rfloor=\widetilde{\tau}-1/2$, while for L odd, $\lfloor\widetilde{\tau}\rfloor=\widetilde{\tau}$.

Substituting $z=\mathrm{e}^{\mathrm{j}\omega}$ and (3) in (2), frequency response of proposed linear-phase integrators can be formulated as

$$H\left(\mathrm{e}^{\mathrm{j}\omega}\right)=\mathrm{e}^{-\mathrm{j}(\widetilde{\tau}-K/2)\omega}\frac{\mathbf{c}(\omega)\cdot\mathbf{g}}{\mathrm{j}\cdot\sin\left(\frac{K\omega}{2}\right)}, \tag{7}$$

i.e. the group delay of $H(z)$ is constant and equal to

$$\tau=\widetilde{\tau}-\frac{K}{2}. \tag{8}$$

Therefore, if $B(z)$ is type I (II) linear-phase FIR transfer function, and K is even (odd), τ is an integer. On the other hand, if τ is integer, magnitude response of $H(z)$ equals zero (L even) or tends to infinity (K even) at $\omega=\pi$.

In this paper, we propose methods to derive the unknown vector $\mathbf{g}$, and consequently coefficients of the linear-phase FIR transfer function $B(z)$, such that frequency response error function of the corresponding linear-phase IIR integrator

$$\varepsilon(\omega)=H\left(\mathrm{e}^{\mathrm{j}\omega}\right)-H_{\mathrm{d}}\left(\mathrm{e}^{\mathrm{j}\omega}\right) \tag{9}$$

is either maximally-flat at some frequency ω_0, or minimized in the weighted Chebyshev sense. Substituting (1) and (7) in (9), complex frequency response error function $\varepsilon(\omega)$ can be rewritten as

$$\varepsilon(\omega)=\frac{\mathrm{e}^{-\mathrm{j}\omega\tau}}{\mathrm{j}\omega}\left[\frac{2\mathbf{c}(\omega)\cdot\mathbf{g}}{K\,\mathrm{sinc}\left(\frac{K\omega}{2}\right)}-1\right]=\frac{\mathrm{e}^{-\mathrm{j}\omega\tau}}{\mathrm{j}\omega}\varepsilon_{\mathrm{r}}(\omega), \tag{10}$$

where

$$\mathrm{sinc}(x)=\begin{cases}\sin(x)/x, & x\neq 0\\ 1, & x=0\end{cases}.$$

Notably, $\varepsilon_{\mathrm{r}}(\omega)$, as defined in the previous equation, cannot have an infinite value, in contrast to $\varepsilon(\omega)$, and it represents the relative magnitude response error of the proposed linear-phase IIR integrators if $K \leq 2$.

3. Design methods

This section provides an explanation of the proposed methods for the design of linear-phase IIR integrators, whose transfer function is given by (2). These methods are referred to as the *maximally-flat* and *optimal*.

3.1. Maximally-flat method

Maximally-flat design method starts from the maximal linearity constraints of frequency response error function

$$\left.\frac{\mathrm{d}^n \varepsilon(\omega)}{\mathrm{d}\omega^n}\right|_{\omega=\omega_0} = 0, \tag{11}$$

for $n = 0, 1, \ldots, N-1$, where N is degree of flatness. Now, since utilization of Leibniz derivative rule gives

$$\frac{\mathrm{d}^n \varepsilon_{\mathrm{r}}(\omega)}{\mathrm{d}\omega^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{\mathrm{d}^k \left(\mathrm{j}\omega \mathrm{e}^{\mathrm{j}\omega\tau}\right)}{\mathrm{d}\omega^k} \frac{\mathrm{d}^{n-k} \varepsilon(\omega)}{\mathrm{d}\omega^{n-k}}, \tag{12}$$

while

$$\frac{\mathrm{d}^k \left(\mathrm{j}\omega \mathrm{e}^{\mathrm{j}\omega\tau}\right)}{\mathrm{d}\omega^k} = \begin{cases} \mathrm{j}\omega \mathrm{e}^{\mathrm{j}\omega\tau}, & k = 0 \\ \mathrm{j}\,(\mathrm{j}\tau)^{k-1}\,(k + \mathrm{j}\omega\tau)\,\mathrm{e}^{\mathrm{j}\omega\tau}, & k \neq 0 \end{cases}, \tag{13}$$

flatness conditions given by (11) can be rewritten as

$$\left.\frac{\mathrm{d}^n \varepsilon_{\mathrm{r}}(\omega)}{\mathrm{d}\omega^n}\right|_{\omega=\omega_0} = 0, \tag{14}$$

for $n = 0, 1, \ldots, N-1$.

In the next step $\varepsilon_{\mathrm{r}}(\omega)$, defined by (10), is reformulated as

$$K \operatorname{sinc}\left(\frac{K\omega}{2}\right) \varepsilon_{\mathrm{r}}(\omega) = 2\mathbf{c}(\omega) \cdot \mathbf{g} - K \operatorname{sinc}\left(\frac{K\omega}{2}\right), \tag{15}$$

and left- and right-hand sides of this relation are differentiated n times

$$K \frac{\mathrm{d}^n}{\mathrm{d}\omega^n}\left[\operatorname{sinc}\left(\frac{K\omega}{2}\right) \varepsilon_{\mathrm{r}}(\omega)\right] = 2\frac{\mathrm{d}^n \mathbf{c}(\omega)}{\mathrm{d}\omega^n} \cdot \mathbf{g} - K \frac{\mathrm{d}^n \operatorname{sinc}\left(\frac{K\omega}{2}\right)}{\mathrm{d}\omega^n}. \tag{16}$$

Again, application of Leibniz derivative rule on the left-hand side of (16) gives

$$K\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^{k}\varepsilon_{\mathrm{r}}\left(\omega\right)}{\mathrm{d}\omega^{k}}\frac{\mathrm{d}^{n-k}\,\mathrm{sinc}\left(\frac{K\omega}{2}\right)}{\mathrm{d}\omega^{n-k}}=2\frac{\mathrm{d}^{n}\mathbf{c}\left(\omega\right)}{\mathrm{d}\omega^{n}}\cdot\mathbf{g}-K\frac{\mathrm{d}^{n}\,\mathrm{sinc}\left(\frac{K\omega}{2}\right)}{\mathrm{d}\omega^{n}},\tag{17}$$

leading to the following formulation of the flatness conditions, note (14),

$$\left.\frac{\mathrm{d}^{n}\mathbf{c}\left(\omega\right)}{\mathrm{d}\omega^{n}}\right|_{\omega=\omega_0}\cdot\mathbf{g}=\frac{K}{2}\left.\frac{\mathrm{d}^{n}\,\mathrm{sinc}\left(\frac{K\omega}{2}\right)}{\mathrm{d}\omega^{n}}\right|_{\omega=\omega_0},\tag{18}$$

for $n=0,\,1,\,\ldots,\,N-1$.

Having in mind that

$$\frac{\mathrm{d}^{n}\cos\left(k\omega\right)}{\mathrm{d}\omega^{n}}=\begin{cases}\left(-1\right)^{n/2}k^{n}\cos\left(k\omega\right), & n\text{ even}\\ \left(-1\right)^{(n+1)/2}k^{n}\sin\left(k\omega\right), & n\text{ odd}\end{cases},\tag{19}$$

$n-$th derivative of $\mathbf{c}\left(\omega\right)$, defined by (5), can be expressed as $1\times(\lfloor\widetilde{\tau}\rfloor+1)$ vector $\mathbf{s}_{n}\left(\omega\right)=\left[s_{i}^{(n)}\left(\omega\right)\right]$ with elements

$$s_{i}^{(n)}\left(\omega\right)=\begin{cases}\left(-1\right)^{n/2}\left(\widetilde{\tau}-i+1\right)^{n}\cos\left(\left(\widetilde{\tau}-i+1\right)\omega\right), & n\text{ even}\\ \left(-1\right)^{(n+1)/2}\left(\widetilde{\tau}-i+1\right)^{n}\sin\left(\left(\widetilde{\tau}-i+1\right)\omega\right), & n\text{ odd}\end{cases}.\tag{20}$$

Therefore, unknown vector $\mathbf{g}$ can be determined as a solution to a system of linear equations given by (18). However, as will be discussed in the following subsections, the cases when $\omega_0=0$ and $\omega_0\neq0$ have to be treated separately, as (18) is always satisfied for $\omega_0=0$ and n odd, regardless the values of FIR filter $B\left(z\right)$ coefficients, i.e. vector $\mathbf{g}$.

3.1.1. The case when $\omega_0=0$

As $n-$th derivative of $\mathrm{sinc}\left(\frac{K\omega}{2}\right)$ at $\omega=0$ equals

$$\left.\frac{\mathrm{d}^{n}\,\mathrm{sinc}\left(\frac{K\omega}{2}\right)}{\mathrm{d}\omega^{n}}\right|_{\omega=0}=\begin{cases}\left(\frac{K}{2}\right)^{n}\frac{(-1)^{n/2}}{n+1}, & n\text{ even}\\ 0, & n\text{ odd}\end{cases},\tag{21}$$

while $n-$th derivative of $\mathbf{c}\left(\omega\right)$ at $\omega=0$ becomes zero vector for n odd, note (20), flatness conditions given by (18), in case of $\omega_0=0$, read

$$\left.\frac{\mathrm{d}^{2k}\mathbf{c}\left(\omega\right)}{\mathrm{d}\omega^{2k}}\right|_{\omega=0}\cdot\mathbf{g}=\frac{\left(-1\right)^{k}}{2k+1}\left(\frac{K}{2}\right)^{2k+1},\tag{22}$$

for $k = 0, 1, \ldots, \lfloor \widetilde{\tau} \rfloor$. Therefore, when $\omega_0 = 0$, degree of flatness N equals $2 \lfloor \widetilde{\tau} \rfloor + 1$.

Using (20), previous system of linear equations reduces to

$$\begin{bmatrix} \widetilde{\tau}^{2k} & (\widetilde{\tau}-1)^{2k} & \ldots & (\widetilde{\tau}-\lfloor \widetilde{\tau} \rfloor)^{2k} \end{bmatrix} \cdot \mathbf{g} = \frac{1}{2k+1} \left(\frac{K}{2}\right)^{2k+1}, \tag{23}$$

for $k = 0, 1, \ldots, \lfloor \widetilde{\tau} \rfloor$, or alternatively

$$\mathbf{V} \cdot \mathbf{g} = \mathbf{d}, \tag{24}$$

where $\mathbf{V} = \begin{bmatrix} v_{ki} = \lambda_i^{k-1} \end{bmatrix}$ is $(\lfloor \widetilde{\tau} \rfloor + 1) \times (\lfloor \widetilde{\tau} \rfloor + 1)$ Vandermonde matrix, and $\mathbf{d} = \begin{bmatrix} d_k \end{bmatrix}$ is $(\lfloor \widetilde{\tau} \rfloor + 1) \times 1$ vector, with elements λ_i and d_k equal to

$$\lambda_i = (\widetilde{\tau} - i + 1)^2, \quad d_k = \frac{1}{2k-1} \left(\frac{K}{2}\right)^{2k-1}.$$

Note that Vandermonde system of linear equations (24) can be solved using various methods, e.g. the ones discussed in [9, 22]. However, as only IIR integrators of relatively low orders are of practical interest, the exact solution for the coefficients vector can be easily obtained by matrix inversion and using rational number arithmetic supported by various programming languages. Therefore, coefficients of the linear-phase integrators designed using the proposed maximally-flat method for $\omega_0 = 0$, along with their performance metrics, can be systematically cataloged.

Coefficients of $B(z)$ for L up to 8, and $K \leq 2$, are given in Table 1. As the case when $K = 2$ and $B(z)$ is type II transfer function reduces to the case when $K = 1$ and $B(z)$ is type I transfer function, corresponding cells in Table 1 are left empty. From Table 1 it can be concluded that proposed IIR integrators designed with $K = 1$ and $L \leq 8$, for $\omega_0 = 0$, are the same as the ones proposed in [8], while for $L = K = 1$, $(L, K) = (2, 1)$, and $(L, K) = (3, 2)$, backward Euler (rectangular), trapezoidal and Simpson integrators are obtained, respectively. Moreover, as will be shown in Section 4, arbitrary-order linear-phase IIR integrators from [8, 23] are just special cases of the proposed integrators.

Table 1: Coefficients $b_0, b_1, \ldots, b_{\lfloor\widetilde{\tau}\rfloor}$ of the linear-phase FIR transfer function $B(z)$ when $\omega_0 = 0$, for $K = 1, 2$.

$L \backslash K$	1	2
1	1	2
2	$\frac{1}{2}$	
3	$\frac{1}{24}, \frac{11}{12}$	$\frac{1}{3}, \frac{4}{3}$
4	$-\frac{1}{24}, \frac{13}{24}$	
5	$-\frac{17}{5760}, \frac{77}{1440}, \frac{863}{960}$	$-\frac{1}{90}, \frac{17}{45}, \frac{19}{15}$
6	$\frac{11}{1440}, -\frac{31}{480}, \frac{401}{720}$	
7	$\frac{367}{967680}, -\frac{281}{53760}, \frac{6361}{107520}, \frac{215641}{241920}$	$\frac{1}{756}, -\frac{2}{105}, \frac{167}{420}, \frac{1172}{945}$
8	$-\frac{191}{120960}, \frac{1879}{120960}, -\frac{353}{4480}, \frac{68323}{120960}$	

3.1.2. The case when $\omega_0 \neq 0$

For $\omega_0 \neq 0$, unknown vector $\mathbf{g}$ can be determined from a system of linear equations given by (18) for $n = 0, 1, \ldots, \lfloor\widetilde{\tau}\rfloor$, where degree of flatness N obviously equals $N = \lfloor\widetilde{\tau}\rfloor+1$. By replacing $\operatorname{sinc}\left(\frac{K\omega}{2}\right)$ term in (18) with $\frac{2}{K\omega}\sin\left(\frac{K\omega}{2}\right)$, its $n-$th derivative can be expressed using Leibniz derivative rule as

$$\frac{\mathrm{d}^n \operatorname{sinc}\left(\frac{K\omega}{2}\right)}{\mathrm{d}\omega^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{\mathrm{d}^k\left(\frac{2}{K\omega}\right)}{\mathrm{d}\omega^k} \frac{\mathrm{d}^{n-k}\sin\left(\frac{K\omega}{2}\right)}{\mathrm{d}\omega^{n-k}}. \tag{25}$$

Now, since

$$\frac{\mathrm{d}^k\left(\frac{2}{K\omega}\right)}{\mathrm{d}\omega^k} = \frac{2}{K}\frac{(-1)^k k!}{\omega^{k+1}} \tag{26}$$

and

$$\frac{\mathrm{d}^{n-k}\sin\left(\frac{K\omega}{2}\right)}{\mathrm{d}\omega^{n-k}} = \begin{cases} (-1)^{(n-k)/2}\left(\frac{K}{2}\right)^{n-k}\sin\left(\frac{K\omega}{2}\right), & n-k \text{ even} \\ (-1)^{(n-k-1)/2}\left(\frac{K}{2}\right)^{n-k}\cos\left(\frac{K\omega}{2}\right), & n-k \text{ odd} \end{cases}, \tag{27}$$

after some mathematical manipulations, flatness conditions given by (18) are rewritten in matrix form as

$$\mathbf{A}\cdot\mathbf{g} = \mathbf{e}, \tag{28}$$

where $\mathbf{A} = \left[a_{ki}\right]$ is $(\lfloor\widetilde{\tau}\rfloor+1)\times(\lfloor\widetilde{\tau}\rfloor+1)$ square matrix, and $\mathbf{e} = \left[e_k\right]$ is

$(\lfloor\widetilde{\tau}\rfloor+1)\times 1$ vector with elements

$$a_{ki}=\begin{cases}(\widetilde{\tau}-i+1)^{k-1}\cos\left((\widetilde{\tau}-i+1)\,\omega_0\right), & k \text{ odd}\\ (\widetilde{\tau}-i+1)^{k-1}\sin\left((\widetilde{\tau}-i+1)\,\omega_0\right), & k \text{ even}\end{cases}, \tag{29}$$

$$e_k=\begin{cases}k^{-1}\left[\left(\frac{K}{2}\right)^k\cos\left(\frac{K\omega_0}{2}\right)+\gamma_k\right], & k \text{ odd}\\ \omega_0^{-1}\gamma_{k-1}, & k \text{ even}\end{cases}, \tag{30}$$

while γ_k is defined by

$$\gamma_k=\frac{k!}{\omega_0^k}\sum_{m=0}^{\lfloor k/2\rfloor}(-1)^m\,\beta_{k-2m}, \tag{31}$$

$$\beta_k=\frac{1}{k!}\left(\frac{K\omega_0}{2}\right)^k\left[k\,\mathrm{sinc}\left(\frac{K\omega_0}{2}\right)-\cos\left(\frac{K\omega_0}{2}\right)\right]. \tag{32}$$

By closely observing (30), it follows that only values of γ_k (and consequently β_k) with odd indices need to be determined. Additionally, these values do not depend on length L of the FIR transfer function $B(z)$. For example, it is sufficient to determine $\gamma_1=\beta_1/\omega_0$, $\gamma_3=6(\beta_3-\beta_1)/\omega_0^3$, and $\gamma_5=120(\beta_5-\beta_3+\beta_1)/\omega_0^5$ for $L=10$.

Closed form expressions for coefficients of $B(z)$ for L up to 4, and $K\le 2$ are given in Table 2. Again, since the case when $K=2$ and L is even reduces to the case when $K=1$ and $B(z)$ is type I transfer function, certain number of cells in Table 2 are left empty.

3.2. Optimal method

In order to minimize the complex frequency response error function $\varepsilon(\omega)$, given by (10), in the weighted Chebyshev sense, an appropriate weighting function has to be defined first. In the paper, we choose the following function

$$W(\omega)=\begin{cases}1, & \omega\in\left[\omega_{\mathrm{p}_1},\omega_{\mathrm{p}_2}\right]\\ 0, & \text{otherwise}\end{cases}, \tag{33}$$

where ω_{p_1} and ω_{p_2} are lower and upper boundaries of the frequency range of interest. Therefore, having in mind the linearity of $\arg\{\varepsilon(\omega)\}$, optimization

Table 2: Coefficients of the linear-phase FIR transfer function $B(z)$ for $L \leq 4$, $K \leq 2$, and $\omega_0 \neq 0$.

$L \backslash K$	1	2
1	$b_0 = \operatorname{sinc}\left(\frac{\omega_0}{2}\right)$	$b_0 = 2\operatorname{sinc}(\omega_0)$
2	$b_0 = \dfrac{\operatorname{sinc}\left(\frac{\omega_0}{2}\right)}{2\cos\left(\frac{\omega_0}{2}\right)}$	
3	$b_0 = \dfrac{\operatorname{sinc}\left(\frac{\omega_0}{2}\right) - \cos\left(\frac{\omega_0}{2}\right)}{2\omega_0 \sin(\omega_0)}$ $b_1 = \operatorname{sinc}\left(\frac{\omega_0}{2}\right) - 2b_0\cos(\omega_0)$	$b_0 = \dfrac{\operatorname{sinc}(\omega_0) - \cos(\omega_0)}{\omega_0 \sin(\omega_0)}$ $b_1 = \dfrac{2\left(1 - \operatorname{sinc}(2\omega_0)\right)}{\omega_0 \sin(\omega_0)}$
4	$b_0 = \dfrac{\operatorname{sinc}(\omega_0) - 1}{2\omega_0\left(1 + \cos(\omega_0)\right)\sin(\omega_0)}$ $b_1 = \dfrac{\operatorname{sinc}\left(\frac{\omega_0}{2}\right) - 2b_0\cos\left(\frac{3\omega_0}{2}\right)}{2\cos\left(\frac{\omega_0}{2}\right)}$	

problem that characterizes the proposed optimal method can be formulated as

$$\begin{array}{ll} \underset{\delta,\,\mathbf{g}}{\text{minimize}} & \delta \\ \text{subject to:} & -\delta \leq \dfrac{\varepsilon_{\mathrm{r}}(\omega,\,\mathbf{g})}{\omega} \leq \delta,\ \omega \in \left[\omega_{\mathrm{p}_1},\,\omega_{\mathrm{p}_2}\right] \end{array}, \tag{34}$$

where notation $\varepsilon_{\mathrm{r}}(\omega,\,\mathbf{g})$ is used to emphasize dependence of $\varepsilon_{\mathrm{r}}(\omega)$ on unknown coefficients vector $\mathbf{g}$, while δ is weighted Chebyshev norm.

Since $\varepsilon_{\mathrm{r}}(\omega,\,\mathbf{g})$ is linear in $\mathbf{g}$, (34) can be formulated as the ordinary linear programming optimization problem in $N = \lfloor\widetilde{\tau}\rfloor + 2 = \lfloor\frac{L+3}{2}\rfloor$ unknowns if semi-infinite inequality constraints are replaced by their evaluations at M discrete frequency points from the closed interval $\left[\omega_{\mathrm{p}_1},\,\omega_{\mathrm{p}_2}\right]$. Assuming linear spacing, M should be chosen to satisfy $M \geq 15N$ [26], i.e. the number of constraints is at least $15(L+3)$. On the other hand, it shows that computationally less intensive exchange algorithm can be used for coefficients vector $\mathbf{g}$ determination, instead of the linear programming, while the cases when $\omega_{\mathrm{p}_1} = 0$ and $\omega_{\mathrm{p}_1} \neq 0$ have to be treated separately, note the ratio $\varepsilon_{\mathrm{r}}(\omega,\,\mathbf{g})/\omega$ in (34). Note that when the width of the frequency range of interest, i.e. $\omega_{\mathrm{p}_2} - \omega_{\mathrm{p}_1}$, approaches zero, optimal method reduces to the maximally-flat one.

It should be noted here that method presented in [21] also formulates the linear-phase second-order IIR integrator design problem as the linear programming one given by (34). Therefore, integrators from [21], including the well-known Tick's integrator [12], are the special cases of the proposed ones, and they are obtained when $L = 3$ and $K = 2$.

3.2.1. The case when $\omega_{\mathrm{p}_1} = 0$

For $\omega = 0$, the ratio $\varepsilon_{\mathrm{r}}(\omega, \mathbf{g})/\omega$ is finite if $\varepsilon_{\mathrm{r}}(0, \mathbf{g}) = 0$, i.e. if

$$\mathbf{c}(0) \cdot \mathbf{g} = \frac{K}{2}, \tag{35}$$

note (10), where $\mathbf{c}(0)$ is a vector of ones, see (5). On the hand, for $\varepsilon_{\mathrm{r}}(0, \mathbf{g}) = 0$ one has

$$\lim_{\omega \to 0} \frac{\varepsilon_{\mathrm{r}}(\omega, \mathbf{g})}{\omega} = \left. \frac{\mathrm{d}\varepsilon_{\mathrm{r}}(\omega, \mathbf{g})}{\mathrm{d}\omega} \right|_{\omega=0} = 0, \tag{36}$$

regardless the value of coefficients vector $\mathbf{g}$, note (14), (18) and (21). Therefore, the first equation that needs to be considered when $\omega_{\mathrm{p}_1} = 0$ is given by (35).

Since the proposed linear-phase IIR integrators have magnitude responses approximating the ideal one in the weighted Chebyshev sense, additional $\lfloor \widetilde{\tau} \rfloor + 1$ equations are given by

$$\varepsilon_{\mathrm{r}}(\omega_k, \mathbf{g}) = (-1)^{k+p} \omega_k \cdot \delta, \tag{37}$$

for $k = 1, 2, \ldots, \lfloor \widetilde{\tau} \rfloor + 1$, note (34), or alternatively

$$\frac{2\mathbf{c}(\omega_k)}{K \operatorname{sinc}\left(\frac{K\omega_k}{2}\right)} \cdot \mathbf{g} - 1 = (-1)^{k+p} \omega_k \cdot \delta, \tag{38}$$

for $k = 1, 2, \ldots, \lfloor \widetilde{\tau} \rfloor + 1$, note (10), where $\varepsilon_{\mathrm{r}}(\omega, \mathbf{g})$ exhibits $\lfloor \widetilde{\tau} \rfloor + 1$ sign-alternating extremal values at frequencies ω_k, $k = 1, 2, \ldots, \lfloor \widetilde{\tau} \rfloor + 1$, satisfying $0 < \omega_1 < \omega_2 < \cdots < \omega_{\lfloor \widetilde{\tau} \rfloor + 1} \leq \omega_{\mathrm{p}_2}$, while p denotes whether first extremal value is minimum ($p = 0$) or maximum ($p = 1$).

If the extremal frequencies positions ω_k, for $k = 1, 2, \ldots, \lfloor \widetilde{\tau} \rfloor + 1$, were known in advance, then (35) and (38) would form a square system in $\lfloor \widetilde{\tau} \rfloor + 2$ unknowns: coefficients vector $\mathbf{g}$ and weighted Chebyshev norm δ. However, as the extremal frequencies positions are not known initially, following exchange algorithm is proposed:

1. Set $t = 0$. Determine initial coefficients vector $\mathbf{g}_0$ such that $\varepsilon_{\mathrm{r}}(\omega, \mathbf{g}_0)/\omega$ exhibits $\lfloor \widetilde{\tau} \rfloor + 1$ extremal values in interval $\left(0, \omega_{\mathrm{p}_2}\right]$. Additionally, (35)

should be satisfied, i.e. $\varepsilon_{\mathrm{r}}(0, \mathbf{g}_0) = 0$. Mentioned can be achieved by setting $\varepsilon_{\mathrm{r}}(\omega, \mathbf{g}_0)$ to zero at $\lfloor \tau \rfloor + 1$ distinct frequencies

$$\omega'_k = \frac{k\pi}{\alpha\widetilde{\tau}}, \tag{39}$$

for $k = 0, 1, \ldots, \lfloor \widetilde{\tau} \rfloor$, where parameter α should be chosen such that $\omega'_{\lfloor \widetilde{\tau} \rfloor} < \omega_{\mathrm{p}_2}$, i.e. $\alpha > \pi/\omega_{\mathrm{p}_2}$. In other words, initial coefficients vector can be determined by solving following system of linear equations

$$\mathbf{c}(\omega'_k) \cdot \mathbf{g}_0 = \frac{K}{2}\,\mathrm{sinc}\left(\frac{K\omega'_k}{2}\right), \tag{40}$$

for $k = 0, 1, \ldots, \lfloor \widetilde{\tau} \rfloor$, note (10).

Plots of $\varepsilon_{\mathrm{r}}(\omega, \mathbf{g}_0)/\omega$ for $L = 10$, $K = 1$, and $\omega_{\mathrm{p}_2} = 0.8\pi$ is shown in Fig. 1, where a sufficient number of extremal points can be observed.

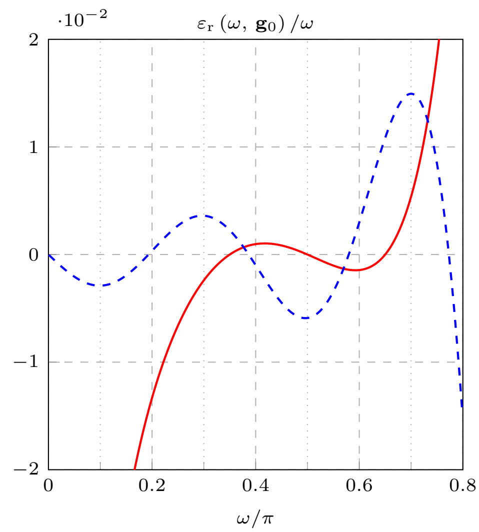

Figure 1: Ratio $\varepsilon_{\mathrm{r}}(\omega)/\omega$ that corresponds to the initial solution of the optimal design method, for $L = 10$, $K = 1$, $\alpha = 1.15$, $\omega_{\mathrm{p}_1} = 0$, $\omega_{\mathrm{p}_2} = 0.8\pi$ (dashed line), and $L = 5$, $K = 2$, $\omega_{\mathrm{p}_1} = 0.2\pi$, $\omega_{\mathrm{p}_2} = 0.8\pi$ (solid line). Note that for $K \leq 2$, $\varepsilon_{\mathrm{r}}(\omega)/\omega$ is absolute magnitude response error of proposed linear-phase IIR integrators.

Note that initial linear-phase IIR integrator $H(z, \mathbf{g}_0)$ of the proposed optimal method correspond to the linear-phase IIR integrator based on trigonometric quadrature rules [6], whose coefficients can be expressed in closed form as function of α [6, equation (11)].

2. Update current iteration, $t = t + 1$. Based on known coefficients vector $\mathbf{g}_{t-1}$ determine frequencies $\omega_k^{(t-1)}$, $1 \leq k \leq \lfloor \widetilde{\tau} \rfloor + 1$, where extremal values of $\varepsilon_{\mathrm{r}}(\omega,\, \mathbf{g}_{t-1})/\omega$ occur in $\left(0,\, \omega_{\mathrm{p}_2}\right]$.
3. Determine new coefficients vector $\mathbf{g}_t$ by solving square system of linear equations

$$\begin{aligned} \mathbf{c}(0) \cdot \mathbf{g}_t &= \frac{K}{2}, \\ \frac{2\mathbf{c}\left(\omega_k^{(t-1)}\right)}{K \operatorname{sinc}\left(\frac{K\omega_k^{(t-1)}}{2}\right)} \cdot \mathbf{g}_t - (-1)^{k+p}\, \omega_k^{(t-1)} \cdot \delta &= 1, \end{aligned} \tag{41}$$

for $k = 1,\, 2,\, \ldots,\, \lfloor \widetilde{\tau} \rfloor + 1$, note (35) and (38).
4. If $\max\{|\mathbf{g}_t - \mathbf{g}_{t-1}|\} \leq \Delta_{\mathrm{tol}}$, where Δ_{tol} is the prescribed tolerance, unknown coefficients vector is $\mathbf{g} = \mathbf{g}_t$, otherwise proceed from the step 2.

The main step within the considered iterative algorithm is solving the square system of $\lfloor \widetilde{\tau} \rfloor + 2 = \lfloor \frac{L+3}{2} \rfloor$ linear equations. Since the linear solver has the polynomial-time complexity of the third-order with respect to the number of unknown variables, and the proposed exchange algorithm is iterative, it may be concluded that the overall complexity of the proposed optimal algorithm is high. However, since only linear-phase IIR integrators of relatively low orders (and consequently low L) are of practical interest, while the proposed algorithm exhibits rapid convergence (as will be discussed in Sec. 5), the computational complexity of the optimal method does not present a practical limitation. For example, the square system of linear equations (41) consists of only 6 equations for $L = 10$. Additionally, the proposed optimal method is significantly computationally less intensive compared to the linear programming solvers that can be used to solve (34).

3.2.2. The case when $\omega_{\mathrm{p}_1} \neq 0$

Since for $\omega_{\mathrm{p}_1} \neq 0$, function $\varepsilon_{\mathrm{r}}(\omega,\, \mathbf{g})/\omega$ is finite in closed interval $\left[\omega_{\mathrm{p}_1},\, \omega_{\mathrm{p}_2}\right]$, proposed linear-phase IIR integrators have magnitude responses approximating the ideal one in the weighted Chebyshev sense if

$$\varepsilon_{\mathrm{r}}(\omega_k,\, \mathbf{g}) = (-1)^{k+p}\, \omega_k \cdot \delta, \tag{42}$$

for $k = 1, 2, \ldots, \lfloor\widetilde{\tau}\rfloor + 2$, where $\varepsilon_{\mathrm{r}}(\omega, \mathbf{g})$ exhibits $\lfloor\widetilde{\tau}\rfloor + 2$ sign-alternating extremal values at frequencies ω_k, $k = 1, 2, \ldots, \lfloor\widetilde{\tau}\rfloor + 2$, satisfying $\omega_{\mathrm{p}_1} \leq \omega_1 < \omega_2 < \cdots < \omega_{\lfloor\widetilde{\tau}\rfloor+2} \leq \omega_{\mathrm{p}_2}$.

Again, similar to the case when $\omega_{\mathrm{p}_1} = 0$, as the extremal frequencies positions are not known in advance, following exchange algorithm is proposed:

1. Set $t = 0$. Determine initial coefficients vector $\mathbf{g}_0$ such that $\varepsilon_{\mathrm{r}}(\omega, \mathbf{g}_0)/\omega$ exhibits $\lfloor\widetilde{\tau}\rfloor + 2$ extremal values in closed interval $\left[\omega_{\mathrm{p}_1}, \omega_{\mathrm{p}_2}\right]$, which can be achieved by setting $\varepsilon_{\mathrm{r}}(\omega, \mathbf{g}_0)$ to zero at $\lfloor\tau\rfloor + 1$ equidistant frequencies

$$\omega_k'' = \omega_{\mathrm{p}_1} + k \cdot \frac{\omega_{\mathrm{p}_2} - \omega_{\mathrm{p}_1}}{\lfloor\widetilde{\tau}\rfloor + 2}, \tag{43}$$

for $k = 1, 2, \ldots, \lfloor\widetilde{\tau}\rfloor + 1$. Consequently, initial coefficients vector can be determined by solving following system of linear equations

$$\mathbf{c}(\omega_k'') \cdot \mathbf{g}_0 = \frac{K}{2}\operatorname{sinc}\left(\frac{K\omega_k''}{2}\right), \tag{44}$$

for $k = 1, 2, \ldots, \lfloor\widetilde{\tau}\rfloor + 1$, note (10).

Plot of $\varepsilon_{\mathrm{r}}(\omega, \mathbf{g}_0)$ for $L = 5$, $K = 2$, $\omega_{\mathrm{p}_1} = 0.2\pi$, and $\omega_{\mathrm{p}_2} = 0.8\pi$ is shown in Fig. 1.

2. Update current iteration, $t = t + 1$. Based on known coefficients vector $\mathbf{g}_{t-1}$ determine frequencies $\omega_k^{(t-1)}$, $1 \leq k \leq \lfloor\widetilde{\tau}\rfloor + 2$, where extremal values of $\varepsilon_{\mathrm{r}}(\omega, \mathbf{g}_{t-1})/\omega$ occur in $\left[\omega_{\mathrm{p}_1}, \omega_{\mathrm{p}_2}\right]$.

3. Determine new coefficients vector $\mathbf{g}_t$ by solving square system of linear equations

$$\frac{2\mathbf{c}\left(\omega_k^{(t-1)}\right)}{K\operatorname{sinc}\left(\frac{K\omega_k^{(t-1)}}{2}\right)} \cdot \mathbf{g}_t - (-1)^{k+p}\,\omega_k^{(t-1)} \cdot \delta = 1, \tag{45}$$

for $k = 1, 2, \ldots, \lfloor\widetilde{\tau}\rfloor + 2$, note (42).

4. If $\max\{|\mathbf{g}_t - \mathbf{g}_{t-1}|\} \leq \Delta_{\mathrm{tol}}$, where Δ_{tol} is the prescribed tolerance, unknown coefficients vector is $\mathbf{g} = \mathbf{g}_t$, otherwise proceed from the step 2.

Regarding the computational complexity of the optimal method for $\omega_{\mathrm{p}_1} \neq 0$, the same conclusions apply as in the case when $\omega_{\mathrm{p}_1} = 0$.

4. Relation between the proposed and existing maximally-flat linear-phase integrators

In this section, we argue that arbitrary-order linear-phase Newton-Cotes integrators [23] and integrators from [8] are maximally flat at $\omega = 0$, i.e. that they are special cases of proposed maximally-flat integrators. In order to establish a relation between these integrators, let us consider difference equation

$$y[n] - y[n-K] = \sum_{k=0}^{L-1} b_k x[n-k] \tag{46}$$

that corresponds to the transfer function given by (2), where $x[n]$ and $y[n]$ are nth sample at input and output, respectively. As the group delay of $H(z)$ is constant and equal to $\tau = (L-1-K)/2$, note (8), in ideal case, $y[n]$ and $y[n-K]$ correspond to the areas up to the $(n-\tau)$th and $(n-\tau-K)$th input sample, respectively. Therefore, since, without loss of generality, the sampling frequency can be set to 1 Hz, coefficients of the FIR filter $B(z)$ should be determined such that

$$\int\limits_{n-\tau-K}^{n-\tau} f_n(t)\,\mathrm{d}t = \sum_{k=0}^{L-1} b_k x[n-k], \tag{47}$$

where $f_n(t)$ is interpolating function satisfying $f_n(m) = x[m]$, for $m = n,\ n-1,\ \ldots,\ n-(L-1) = n-2\widetilde{\tau}$.

Now, if $f_n(t)$ is $(L-1)$th degree Lagrange interpolating polynomial, unknown coefficients can be obtained from (47) as [11, 9]

$$b_{L-1-k} = b_k = \int\limits_{\widetilde{\tau}-K/2}^{\widetilde{\tau}+K/2} \prod_{i=0,\, i\neq k}^{L-1} \frac{(u-i)}{(k-i)}\mathrm{d}u, \tag{48}$$

for $k = 0, 1, 2, \ldots, L-1$. Note that coefficients of the linear-phase Newton-Cotes integrators (including trapezoidal, Simpson's 1/3, Simpson's 3/8 and Boole's integrators) [23] and integrators from [8] can be obtained using (48) if $K = L-1$ and $K = 1$, respectively. On the other hand, since the area between $(-K/2)$th and $(K/2)$th input sample can be expressed as

$$\int_{-K/2}^{K/2} f_{\widetilde{\tau}}(t)\,\mathrm{d}t = \sum_{k=0}^{L-1} b_k f_{\widetilde{\tau}}(\widetilde{\tau} - k), \tag{49}$$

note (47), coefficients of the FIR transfer function $B(z)$ satisfy

$$\sum_{k=0}^{L-1} b_k (\widetilde{\tau} - k)^m = \frac{1}{m+1}\left[\left(\frac{K}{2}\right)^{m+1} - \left(-\frac{K}{2}\right)^{m+1}\right], \tag{50}$$

for $m = 0, 1, \ldots, L-1$. Due to the symmetry of the coefficients $b_{2\widetilde{\tau}-k} = b_k$, previous equation is always satisfied for m odd, and it can be reformulated as

$$\sum_{k=0}^{L-1} b_k (\widetilde{\tau} - k)^{2i} = \frac{2}{2i+1}\left(\frac{K}{2}\right)^{2i+1}, \tag{51}$$

for $i = 0, 1, \ldots, \lfloor \widetilde{\tau} \rfloor$, which is essentially (23).

Therefore, it can be concluded that linear-phase integrators from [8, 23] are special cases of proposed integrators designed with $\omega_0 = 0$, i.e. their magnitude responses are maximally flat at $\omega = 0$. Additionally, it follows that coefficients of the proposed integrators can be also calculated using the closed form expression given by (48).

5. Design examples and comparison with the existing integrators

In this section, proposed linear-phase IIR integrators are compared among themselves, as well as with the linear- and nearly linear-phase integrators from [1, 6, 10, 15]. Comparison with the existing linear-phase maximally-flat IIR integrators from [23, 8], among which are rectangular, trapezoidal, Simpson's 1/3, Simpson's 3/8 and Boole's integrators, is unnecessary as these integrators are just special cases of proposed maximally-flat integrators. The same applies to the second-order linear-phase integrators from [21], including Tick's integrator [12, 21], as they can be obtained by the proposed optimal method.

The comparison among integrators is based on several performance metrics: (average) group delay τ, maximum phase response linearity error in degrees

$$\eta = \frac{180}{\pi}\max_{\omega}\left|\arg\left\{H\left(\mathrm{e}^{\mathrm{j}\omega}\right)\right\} - \left(-\frac{\pi}{2} - \tau\omega\right)\right|, \tag{52}$$

and required number of multiplications and delay elements for direct form implementation. Additionally, maximally-flat integrators are also compared in terms of the width of the frequency band about frequency ω_0, denoted by $\Delta\omega$, where absolute magnitude response error is below 40 dB, while optimal integrators are compared in terms of weighted Chebyshev norm δ.

We adopt $K \leq 2$ in all examples, as, otherwise, amplification of proposed integrators would be infinite at frequency $2\pi/K < \pi$. Therefore, $\varepsilon_\mathrm{r}(\omega)$ can be treated as the relative magnitude response error, while

$$20\log_{10}|\varepsilon(\omega)| = 20\log_{10}\left|\frac{\varepsilon_\mathrm{r}(\omega)}{\omega}\right|$$

becomes absolute magnitude response error in dB. In other words, magnitude response of optimal linear-phase IIR integrators is minimized in the weighted Chebyshev sense if $K \leq 2$. In considered examples, optimal integrators are designed using $\Delta_\mathrm{tol} = 10^{-8}$.

In the first example, we consider the maximally-flat linear-phase IIR integrators obtained for $2 \leq L \leq 8$, $K \leq 2$, and $\omega_0 \in \{0,\, 0.45\pi,\, 0.6\pi,\, 0.75\pi\}$. From the results given in Table 3 and Figs. 2 and 3, it follows that even-order integrators with non-integer delays ($K = 1$, L odd) outperform odd-order integrators of orders higher by one, which is somewhat expected as magnitude response of type II transfer function $B(z)$ equals 0 at Nyquist frequency, regardless its coefficients values, Fig. 2. Additionally, mentioned even-order integrators also outperform the even-order integer delay integrators ($K = 2$, L even) that inherently have infinite amplification at Nyquist frequency, Fig. 3. On the other hand, the even-order integrators with integer delays outperform the odd-order integrators of orders higher by one, but their applicability is limited to the cases when spectrum of input signal does not contain high frequency components. From Table 3 it can be also concluded that with the increase of ω_0, the width of the frequency band where absolute magnitude response error is below 40 dB, $\Delta\omega$, decreases.

In the second example, optimal linear-phase IIR integrators are designed for $2 \leq L \leq 8$, $K \leq 2$, $\omega_{\mathrm{p}_1} = 0$, and various ω_{p_2}. Initial solutions were

Table 3: Example 1, maximally-flat method: design results

				$\omega_0 = 0$	$\omega_0 = 0.45\pi$	$\omega_0 = 0.6\pi$	$\omega_0 = 0.75\pi$
L	K	mult.(delays)	τ	$\Delta\omega/\pi$	$\Delta\omega/\pi$	$\Delta\omega/\pi$	$\Delta\omega/\pi$
2	1	1(1)	0	0.038	0.03	0.023	0.015
3	1	2(2)	0.5	0.477	0.428	0.343	0.259
4	1	2(3)	1	0.280	0.172	0.120	0.073
5	1	3(4)	1.5	0.636	0.638	0.480	0.330
6	1	3(5)	2	0.417	0.301	0.200	0.120
7	1	4(6)	2.5	0.717	0.751	0.547	0.358
8	1	4(7)	3	0.499	0.397	0.259	0.153
3	2	2(2)	0	0.366	0.283	0.202	0.126
5	2	3(4)	1	0.523	0.491	0.336	0.202
7	2	4(6)	2	0.603	0.616	0.411	0.240

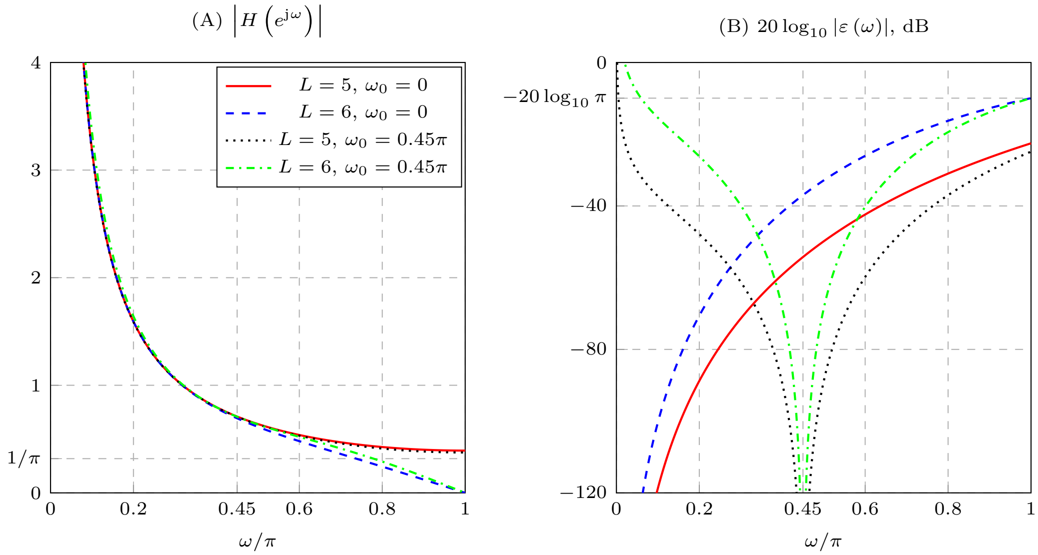

Figure 2: Example 1, $K = 1$. (A) Magnitude responses, and (B) absolute magnitude response error functions in dB.

obtained using $\alpha = 1.1\pi/\omega_{\mathrm{p}_2}$, and proposed exchange algorithm converged to the solution in no more than 5 iterations. From the design results given in Tab. 4 and Fig. 4, conclusions similar to those from the previous example can be drawn. Namely, even-order integrators with non-integer delays outperform both odd- and even-order integrators with integer group delays, while even-order integrators with integer delay outperform odd-order ones. Furthermore, with the increase of ω_{p_2}, δ also increases, and $K = 2$ cannot be used if $\omega_{\mathrm{p}_2} = \pi$, due to the infinite amplification at Nyquist frequency. Results given in Table 4 also

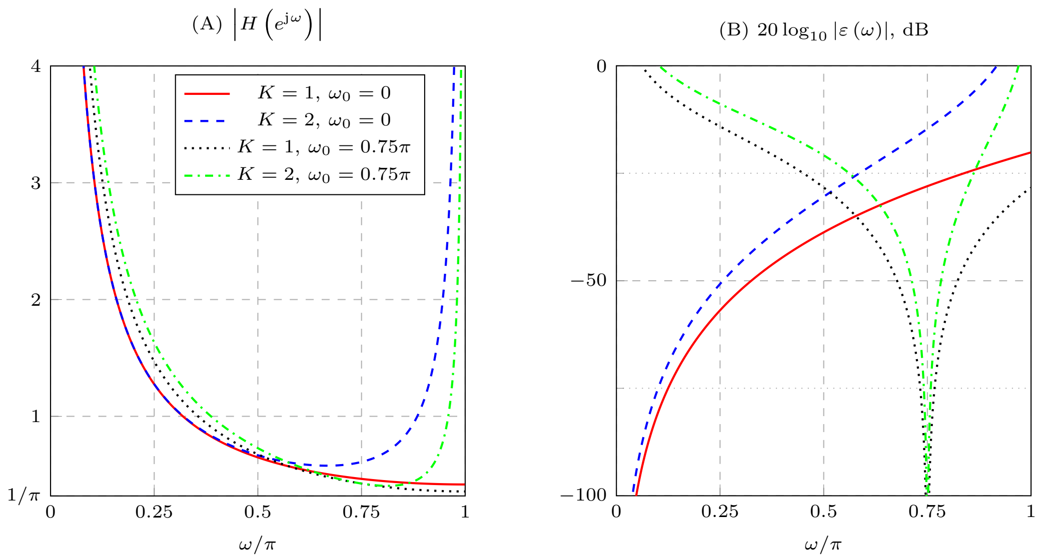

Figure 3: Example 1, $L = 3$. (A) Magnitude responses, and (B) absolute magnitude response error functions in dB.

suggests that with the increase of ω_{p_2}, δ increases, while odd-order integrators with integer group delays have absolute magnitude response error at Nyquist frequency equal to $-20\log_{10}\pi = -9.94$ dB, which is expected since type II FIR filter $B(z)$ exhibits a zero at $z = e^{j\pi}$.

Table 4: Example 2, optimal method, $\omega_{p_1} = 0$: design results

				$\omega_{p_2} = \pi/4$	$\omega_{p_2} = \pi/2$	$\omega_{p_2} = 3\pi/4$	$\omega_{p_2} = \pi$
L	K	mult.(delays)	τ	δ, dB	δ, dB	δ, dB	δ, dB
2	1	1(1)	0	−23.59	−17.29	−13.26	−9.94
3	1	2(2)	0.5	−68.74	−50.09	−38.48	−29.38
4	1	2(3)	1	−54.27	−34.96	−21.96	−9.94
5	1	3(4)	1.5	−102.62	−71.36	−51.62	−35.56
6	1	3(5)	2	−83.98	−51.95	−30.52	−9.94
7	1	4(6)	2.5	−134.68	−90.75	−62.76	−39.67
8	1	4(7)	3	−113.24	−68.47	−38.70	−9.94
3	2	2(2)	0	−62.97	−43.36	−29.55	
5	2	3(4)	1	−97.49	−65.18	−42.85	
7	2	4(6)	2	−129.81	−84.78	−54	

In the third example, odd-length FIR maximally-flat integrators for midband frequencies [15] are compared to the proposed maximally flat integrators whose orders are lower by one and $K = 1$. In this way, both FIR integrators [15] and their IIR counterparts have integer group delays, and require the same number

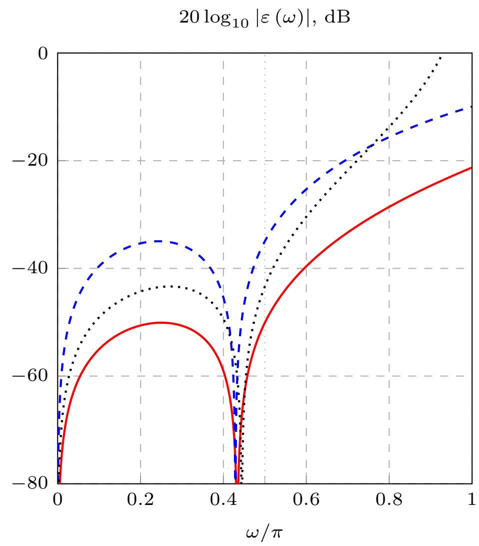

Figure 4: Example 2, $\omega_{\mathrm{p}_1} = 0$, $\omega_{\mathrm{p}_2} = \pi/2$. Absolute magnitude response errors in dB of the proposed optimal linear-phase IIR integrators obtained for $L = 3$, $K = 1$ (solid line), $L = 4$, $K = 1$ (dashed line), $L = 3$, $K = 2$ (dotted line).

of multiplications for direct form implementation. Widths of the frequency band where absolute magnitude response error is below 40 dB, i.e. $\Delta\omega$, as functions of ω_0, for two lengths of FIR integrators [15] and $\omega_0 \in [0.2\pi,\, 0.8\pi]$, are shown in Fig. 5(A), while absolute magnitude response error functions of several proposed integrators and their FIR counterparts [15] are presented in Fig. 5(B). General conclusion is that proposed maximally-flat IIR integrators are significantly better compared to FIR counterparts [15] for low ω_0. On the other hand, values of $\Delta\omega$ become comparable as ω_0 increases. However, compared to the FIR counterparts [15], proposed integrators have group delay values lower by one.

In the fourth example, proposed optimal linear-phase integrators are compared to the FIR compensator-based integrators from [2]. It shows that these integrators can be obtained by modification of the optimal method by adopting weighting function $W(\omega) = K\omega\,\mathrm{sinc}\,(K\omega/2)$ in the frequency range of interest $\left[0,\, \omega_{\mathrm{p}_2}\right]$. Results of comparison between two optimal fourth-order integrators

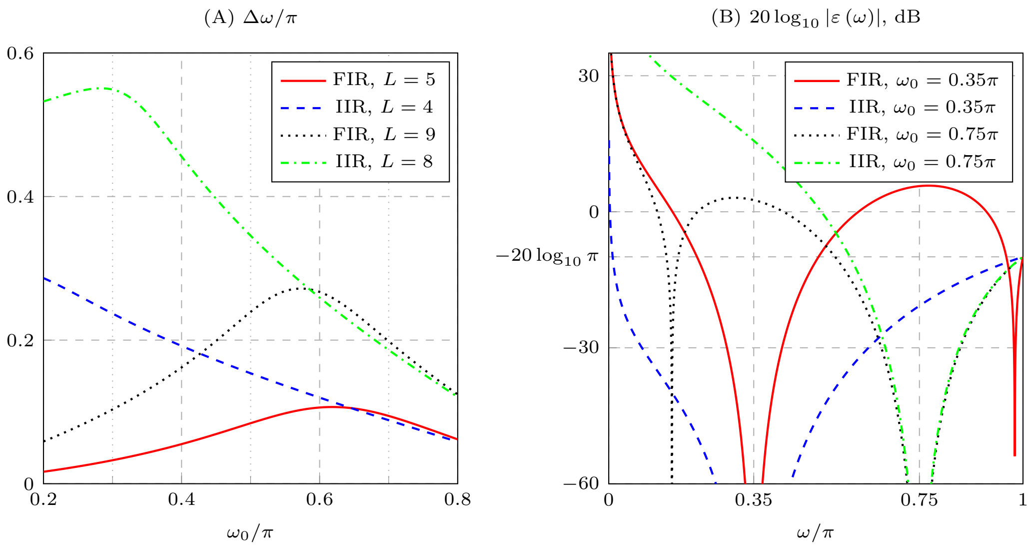

Figure 5: Example 3, $K=1$. (A) $\Delta\omega$ as function of ω_0, for $\omega_0 \in [0.2\pi,\ 0.8\pi]$, and (B) absolute magnitude response error functions in dB of the proposed maximally-flat IIR integrators obtained for $L=6$, and their FIR counterparts [15] of lengths 7.

designed using $K=1(2)$, $\omega_{\mathrm{p}_1}=0$ and $\omega_{\mathrm{p}_2}=3\pi/4$,

$$H_1(z)=\frac{-0.0076\left(1+z^{-4}\right)+0.0662\left(z^{-1}+z^{-3}\right)+0.8828z^{-2}}{1-z^{-1}}, \tag{53}$$

$$H_2(z)=\frac{-0.0297\left(1+z^{-4}\right)+0.4244\left(z^{-1}+z^{-3}\right)+1.2106z^{-2}}{1-z^{-2}}, \tag{54}$$

and corresponding integrators from [2]

$$H_{\mathrm{Abed}}^{(1)}(z)=\frac{-0.0085\left(1+z^{-4}\right)+0.0672\left(z^{-1}+z^{-3}\right)+0.8827z^{-2}}{1-z^{-1}}, \tag{55}$$

$$H_{\mathrm{Abed}}^{(2)}(z)=\frac{-0.0298\left(1+z^{-4}\right)+0.4240\left(z^{-1}+z^{-3}\right)+1.2116z^{-2}}{1-z^{-2}}, \tag{56}$$

are summarized in Table 5, while their magnitude response error functions are shown in Fig. 6. Obviously, while requiring the same number of multiplications and delay elements, proposed integrators outperform existing ones from [2] in terms of minimum absolute magnitude response error.

As the fifth example, second-order integrator from [6]

$$H_{\mathrm{Ali}}(z)=\frac{0.3366\left(1+z^{-2}\right)+1.3268z^{-1}}{1-z^{-2}}, \tag{57}$$

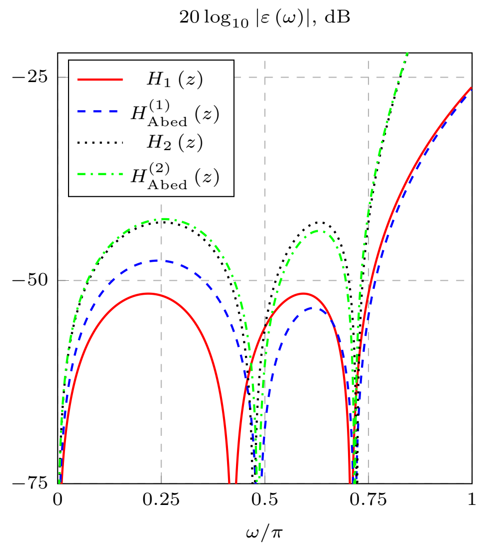

Figure 6: Example 4, $L = 5$, $\omega_{\mathrm{p}_1} = 0$, $\omega_{\mathrm{p}_2} = 3\pi/4$. Absolute magnitude response errors in dB of optimal and linear-phase integrators from [2].

designed to integrate the input signal limited to the digital frequency range of $[\pi/128, 3\pi/16]$, is compared to the proposed optimal IIR integrator designed using $L = 3$, $K = 2$, $\omega_{\mathrm{p}_1} = \pi/128$, and $\omega_{\mathrm{p}_2} = 3\pi/16$,

$$H_3(z) = \frac{0.3364\left(1 + z^{-2}\right) + 1.3273z^{-1}}{1 - z^{-2}}. \tag{58}$$

Results of the comparison are given in Table 5. Both integrators have the group delay equal to 1, and require two multiplications and two delay elements. On the other hand, the proposed integrator outperforms the existing one in terms of minimum absolute magnitude response error in dB (−71.06 vs. −69.29), which is expected result, as the absolute magnitude response error of the proposed optimal integrators is minimized in the Chebyshev sense.

In the sixth example, second-order nearly linear-phase IIR integrator from [10]

$$H_{\mathrm{Garg}}(z) = 0.05578\frac{1 + 15.5885z^{-1} + 8.4692z^{-2}}{1 - 0.5361z^{-1} - 0.4639z^{-2}} \tag{59}$$

is compared to two optimal integrators designed using $L = 5(7)$, $K = 1$, $\omega_{\mathrm{p}_1} =$

$0(0.22\pi)$, and $\omega_{\mathrm{p}_2} = \pi$,

$$H_4\left(z\right) = \frac{-0.0177\left(1+z^{-4}\right)+0.0825\left(z^{-1}+z^{-3}\right)+0.8704z^{-2}}{1-z^{-1}}, \tag{60}$$

$$H_5\left(z\right) = \frac{\left\{\begin{array}{c} 0.0149\left(1+z^{-6}\right)-0.0138\left(z^{-1}+z^{-5}\right) \\ +0.0828\left(z^{-2}+z^{-4}\right)+0.8787z^{-3} \end{array}\right\}}{1-z^{-1}}. \tag{61}$$

Results of comparison are summarized in Table 5, while phase response linearity error of existing integrator from [10] and absolute magnitude response error functions are shown in Figs. 7(A) and 7(B), respectively. Obviously, due to the linear-phase restriction and a pole placed at $z = e^{\mathrm{j}0}$, proposed competing integrators are of higher orders (and consequently with higher group delays), however, they require less multiplications for direct form implementation. In Fig. 7(B) it can be seen that curve of the sixth-order integrator magnitude response error function is below that of existing integrator from [10], almost up to the Nyquist frequency.

Table 5: Examples 4, 5, 6, and 7: design results.

	mult.(delays)	τ	η, deg.	δ, dB	$\lvert\varepsilon\left(\pi\right)\rvert$, dB
$H_{\mathrm{Abed}}^{(1)}\left(z\right)$	3(4)	1.5	0	-47.55	
$H_1\left(z\right)$	3(4)	1.5	0	-51.62	
$H_{\mathrm{Abed}}^{(2)}\left(z\right)$	3(4)	1	0	-41.60	
$H_2\left(z\right)$	3(4)	1	0	-42.85	
$H_{\mathrm{Ali}}\left(z\right)$	2(2)	1	0	-69.29	
$H_3\left(z\right)$	2(2)	1	0	-70.06	
$H_{\mathrm{Garg}}\left(z\right)$	5(2)	0.5	5.67		-88
$H_4\left(z\right)$	3(4)	1.5	0	-35.56	-35.56
$H_5\left(z\right)$	4(6)	2.5	0	-40.38	-40.38
$H_{\mathrm{Abab}}\left(z\right)$	5(2)	0.5	4.83		-35.03
$H_6\left(z\right)$	4(6)	2.5	0	-86.23	-25.82

Finally, in the seventh example, we consider the second-order nearly linear-phase IIR integrator from [1]

$$H_{\mathrm{Abab}}\left(z\right) = \frac{0.098+1.5024z^{-1}+0.6582z^{-2}}{1.6844-1.1103z^{-1}-0.5741z^{-2}}, \tag{62}$$

and compare it with two optimal integrators designed using $L = 5(7)$, $K = 1$,

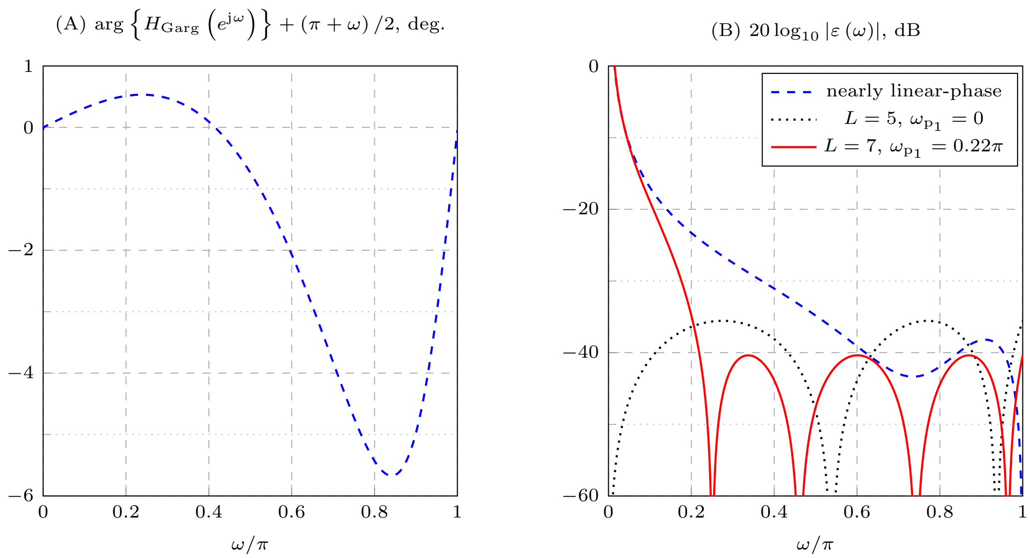

Figure 7: Example 6, $K=1$, $\omega_{\mathrm{p}_2}=\pi$. (A) Phase response linearity error of $H_{\mathrm{Garg}}(z)$, and (B) absolute magnitude response error functions in dB.

$\omega_{\mathrm{p}_1}=0\,(0.085\pi)$, and $\omega_{\mathrm{p}_2}=\pi(0.55\pi)$, i.e. with $H_4(z)$ and

$$H_6(z)=\frac{\left\{\begin{matrix}0.001\left(1+z^{-6}\right)-0.0077\left(z^{-1}+z^{-5}\right)\\ +0.0643\left(z^{-2}+z^{-4}\right)+0.8849z^{-3}\end{matrix}\right\}}{1-z^{-1}}. \tag{63}$$

Phase response linearity error of integrator from [1], and absolute magnitude response error functions of all considered integrators are shown in Figs. 8(A) and 8(B), while results of comparison are given in Table 5. From Table 5 and Fig. 8(B) it can be concluded that Chebyshev norm of the proposed optimal fourth-order integrator is lower than that of the existing nearly linear-phase integrator [1], while the opposite is true for the magnitude response error function values over almost the entire frequency range. On the other hand, magnitude response error function of the proposed sixth-order optimal integrator $H_6(z)$ is below that of the existing integrator [1] up to 0.58π. Regarding the computational complexity of considered integrators, the orders (i.e. numbers of required delay elements) of the linear-phase IIR integrators are higher compared to the existing nearly linear-phase IIR integrator [1], while required number of multi-

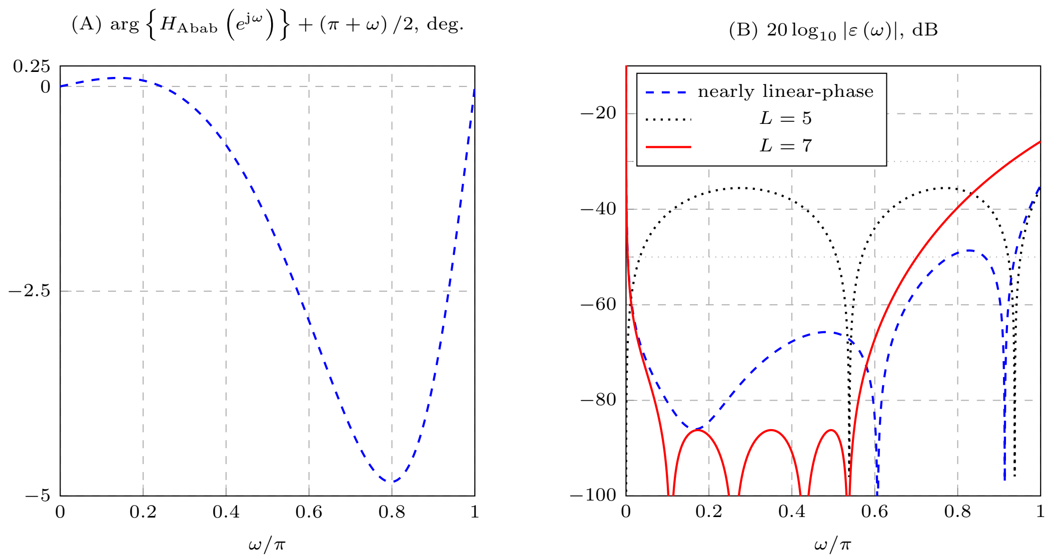

Figure 8: Example 7. (A) Phase response linearity error of $H_{\text{Abab}}(z)$, and (B) absolute magnitude response error functions in dB.

plications is lower due to the coefficients symmetry.

6. Conclusion

In this paper, starting from the frequency response of the linear-phase IIR integrator, two design methods are proposed. While integrators obtained using the noniterative maximally-flat method (where coefficients can be obtained as a solution to a system of linear equations) are suitable for processing narrowband signals, true optimality (in the Chebyshev sense) is achieved by utilization of the iterative optimal design method which typically converges in a few iterations. Furthermore, existing integer-order linear-phase IIR integrators [23, 8], among which are the well-known rectangular, trapezoidal, Simpson 1/3, Simpson 3/8, and Boole's integrators, are shown to be maximally flat at $\omega = 0$, i.e. just a special cases of the proposed maximally-flat integrators. The similar conclusion can be drawn for the second-order integrators from [21], including the well-known Tick's integrator [12], as they can be obtained using the proposed optimal method.

Coefficients of the proposed maximally-flat integrators, either their exact values if $\omega_0 = 0$ or closed-form expressions if $\omega_0 \neq 0$, are tabulated for various orders to facilitate filter designers. Results of comparison between proposed and existing linear-phase integrators show that proposed maximally-flat integrators can serve as an effective alternative to the FIR midband integrators. Additionally, the proposed optimal integrators outperform those from [6], which can be obtained as the initial solution in the proposed exchange algorithm. Finally, compared to the nearly linear-phase IIR integrators, the proposed integrators, due to the linear-phase restriction and a pole placed at $z = e^{\mathrm{j}0} = 1$, have to be of higher orders, i.e. their group delays are higher. On the other hand, they require fewer multiplications for realization. One possible direction of the future research is the utilization of the proposed linear-phase integrators in practical signal processing applications.

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