University of Banja Luka
Faculty of Mechanical Engineering

DEMI 2025

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PROCEEDINGS

Banja Luka, 29–30 May 2025

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Impact of a Customized Shape Function on the Prediction of Vibrations in Composite Laminates Using Shear Deformation Theory

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Abstract *This paper investigates the free vibrations of composite laminates using shear deformation theory, with a particular focus on the application of an adapted shape function that accounts for transverse shear deformation. The developed mathematical model allows for accurate analysis of the dynamic behavior of laminate structures, improving precision compared to classical theories that neglect these effects. Using the Matlab software package, a numerical algorithm has been implemented to solve the differential equations describing the free vibrations of laminates, enabling a detailed analysis of their dynamic characteristics.*
The study considers various geometric and material parameters of the laminates to assess their impact on natural frequencies and vibration dynamics. The results obtained from the numerical processing indicate a significant influence of the shape function on the accuracy of vibration predictions. This approach provides valuable insights for improving the vibration models of composite laminates, as well as for optimizing material and structural design in engineering applications.

Keywords *free vibrations, composite laminates, shear deformation theory, shape functions*

1. INTRODUCTORY CONSIDERATIONS

Composite materials, particularly multilayer laminates, represent a class of modern engineering materials with mechanical properties that make them suitable for a wide range of high-performance applications. Their main advantages include a high strength-to-weight ratio, resistance to fatigue and corrosion, as well as the ability to optimize mechanical properties through fiber orientation and distribution within the matrix [1, 2].

Such design flexibility allows engineers to develop structures that meet specific requirements, whether for reducing weight, increasing energy efficiency, or enhancing load-bearing capacity.

The application of composite laminates is particularly prominent in highly demanding industries such as aerospace, automotive and railway industries, shipbuilding, energy (e.g., wind turbines), as well as in construction and sports technologies [3,4]. In structures exposed to variable or impact loads, reliably predicting the dynamic behavior of composite elements becomes crucial. Dynamic loads, including vibrations, harmonic excitations, and impact

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forces, can lead to the occurrence of resonance, progressive damage, and potential system failure if not properly analyzed and controlled.

Due to the complexity of the internal structure of composites, their mechanical and dynamic analysis requires the use of advanced models and theories that account for material inhomogeneity, anisotropy, and interlayer interactions.

Traditional bending theories, such as Classical Laminate Theory (CLT) and First-order Shear Deformation Theory (FSDT), provide the foundation for the analysis of composite laminates, but come with certain limitations [1]. Classical Laminate Theory assumes that normal to the mid-surface remain normal after deformation, completely neglecting the effect of transverse shear deformation. This limitation may be acceptable for thin laminate panels, but for thicker or multi-layered laminates, this approximation leads to significant errors in the calculation of stresses, strains, and dynamic response [1, 5].

The influence of transverse shear deformations on the bending of laminate plates was among the first to be analyzed by Midlin [6] and Reissner [17], and their theories are often considered the starting point for the development of FSDT. Based on these foundations, numerous authors have extended FSDT and adapted it to different types of laminates and boundary conditions. For example, Whitney and Pagano [8], Qi and Knight [7], as well as Sun and Whitney [10], made important contributions in the domain of theoretical formulations. Pryor and Barker [11] developed FSDT elements for symmetric and antisymmetric cross-ply plates, while Owen and Li Hinton [12], and Reddy and Chao [13] expanded its application to general laminates.

Turvey [14] proposed analytical and approximate solutions for rectangular and cross-ply plates based on FSDT. Within this theory, the introduction of shear correction factors is necessary to adequately represent shear deformation energy. These factors depend on the properties of the layers, fiber orientation, boundary conditions, and the specific application.

To improve the accuracy of predicting the mechanical and dynamic behavior of laminates, higher-order shear deformation theories (HSDT) [15] and various shape function-based theories explicitly modeling changes in shear deformation along the thickness have been developed [16, 17, 18, 19]. These theories enable more realistic modeling of composite laminate behavior as they eliminate the need for correction factors and provide better approximations of geometrically nonlinear effects, especially for thicker and heterogeneous layered materials [20, 21].

In contemporary research, special emphasis is placed on the development of adapted shape functions within higher-order theories, achieving significant improvements in precision without a substantial increase in computational resources. This creates an efficient balance between accuracy and numerical efficiency, which is crucial for engineering applications in practice [2, 9].

2. IMPLEMENTATION OF HSDT THEORIES BASED ON SHAPE FUNCTIONS

Higher-Order Shear Deformation Theories (HSDT) represent an advanced approach in modeling layered composite structures, especially in the context of dynamic analysis. Unlike classical bending theories, which neglect or oversimplify transverse shear deformation, HSDT allows for a more realistic representation of displacement distribution through the thickness of the laminate, without the need for empirical shear correction factors.

The procedure for implementing HSDT within the framework of linear elasticity assumes of higher-order displacement functions. The common form of the displacement expressions in the x, y, and z directions uses additional shape functions that describe the influence of transverse shear deformation:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\phi_z(x,y) + f(z)\psi_x(x,y) \\ v(x,y,z) &= v_0(x,y) + z\phi_y(x,y) + f(z)\psi_y(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{1}$$

Where are:

- $u_0(x,y), v_0(x,y), w_0(x,y)$ displacements in the mid-surface,
- $\phi_x(x,y), \phi_y(x,y)$ are shape functions that describe the displacement in the xx and y directions at different thickness levels of the laminate. These functions represent linear variations of displacement along the thickness of the laminate.
- $\psi_x(x,y), \psi_y(x,y)$ are higher-order functions that account for transverse shear deformation. These functions describe how shear deformation varies along the thickness of the laminate, depending on the specific properties of the

composite layers. These functions enable precise modeling of shear effects in laminates, especially for thicker plates.

- $f(z)$ is a shape function that depends on the laminate thickness z, and is used for interpolating shear deformations, while satisfying the boundary conditions.

These kinematic assumptions enable precise modeling of the behavior of laminate structures through their thickness, accounting for complex shear deformation effects that are not covered in classical theories.

Based on such kinematic assumptions, strain relations from linear elasticity theory (small displacements) are used, which also include higher-order terms with respect to the thickness coordinate:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2}$$

Constitutive relations are established for each layer individually, in accordance with the characteristics of orthotropic materials, and then standard stiffness matrix transformations are applied depending on the fiber orientation angle. In this step, Hooke's law in the stress plane forms $(\sigma = Q\varepsilon)$ the basis for linking the stress and strain states.

The total resultant forces in the plane(N_x, N_y, N_{xy}), total resultant bending moments (M_x, M_y, M_{xy}), resultant moments associated with the transverse shear deformation function (P_x, P_y, P_{xy}) and the resultant transverse shear forces (R_x, R_y) can be calculated as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} + \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} + \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x} \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} P_x \\ P_y \\ P_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} + \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & T_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix} + \begin{bmatrix} G_{11} & G_{12} & 0 \\ G_{12} & G_{22} & 0 \\ 0 & 0 & G_{66} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} R_x \\ R_y \end{Bmatrix} = \begin{bmatrix} F_{44} & 0 \\ 0 & F_{55} \end{bmatrix} \begin{Bmatrix} \theta_x \\ \theta_y \end{Bmatrix}$$

where:

$$\{A_{ij}, B_{ij}, C_{ij}, D_{ij}, T_{ij}, G_{ij}\} = \int_{-h/2}^{h/2} \{1, z, f(x,y,z), z^2, zf(x,y,z), [f(x,y,z)]^2\} Q_{ij}\, dz, \ (i,j=1,2,6).$$

$$\{F_{ij}\} = \int_{-h/2}^{h/2} \left[\frac{\partial f(x,y,z)}{\partial z}\right]^2 Q_{ij}\, dz, \ (i,j=4,5)$$

The next step involves deriving the energy expressions using Hamilton's principle, which formulates the variational form of the problem:

$$\delta \int_{t_1}^{t_2} (T - U)\, dt + \delta W = 0 \tag{4}$$

where:

T – kinetic energy,

U – potential deformation energy,

δW – virtual work of external forces.

From which the differential equations of free vibrations are obtained in the form:

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=I_1\frac{\partial^2 u_0}{\partial t^2}-I_2\frac{\partial^3 w_0}{\partial t^2\partial x}+I_4\frac{\partial^2\theta_x}{\partial t^2},$$
$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=I_1\frac{\partial^2 v_0}{\partial t^2}-I_2\frac{\partial^3 w_0}{\partial t^2\partial y}+I_4\frac{\partial^2\theta_y}{\partial t^2},$$
$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial x^2}=I_1\frac{\partial^2 w_0}{\partial t^2}+I_2\left(\frac{\partial^3 u_0}{\partial t^2\partial x}+\frac{\partial^3 v_0}{\partial t^2\partial x}\right)$$
$$-I_3\left(\frac{\partial^4 w_0}{\partial t^2\partial x^2}+\frac{\partial^4 w_0}{\partial t^2\partial y^2}\right)+I_5\left(\frac{\partial^3\theta_x}{\partial t^2\partial x}+\frac{\partial^3\theta_y}{\partial t^2\partial y}\right), \quad (5)$$
$$\frac{\partial P_x}{\partial x}+\frac{\partial P_{xy}}{\partial y}-R_x=I_4\frac{\partial^2 u_0}{\partial t^2}-I_5\frac{\partial^3 w_0}{\partial t^2\partial x}+I_6\frac{\partial^2\theta_x}{\partial t^2},$$
$$\frac{\partial P_{xy}}{\partial x}+\frac{\partial P_y}{\partial y}-R_y=I_4\frac{\partial^2 v_0}{\partial t^2}-I_5\frac{\partial^3 w_0}{\partial t^2\partial y}+I_6\frac{\partial^2\theta_y}{\partial t^2}.$$

The mass inertia term $I_i (i=1,2,\ldots,6)$ is defined as:

$$\{I_1,I_2,I_3,I_4,I_5,I_6\}=\int_{-h/2}^{h/2}\{1,z,f(x,y,z),z^2,zf(x,y,z),[f(x,y,z)]^2\}\rho(x,y,z)dz.$$

3. NUMERICAL RESULTS

To obtain numerical results, both analytical and numerical methods can be used. In the implementation of new shape functions, it is simpler to use analytical methods based on assumed solution forms. The boundary conditions for the simply supported laminate plate, which can be applied to this type of laminate, are given in [22]:

$$v_0=w_0=\theta_x=N_x=M_x=P_x=0(\text{ at } x=0,\ a)$$
$$u_0=w_0=\theta_y=N_y=M_y=P_y=0(\text{ at } y=0,b)$$

In accordance with the adopted boundary conditions, the assumed forms of Navier's solutions are also given in [22] as:

$$u_0(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{i\omega t},$$
$$v_0(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}V_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t}, \quad (6)$$
$$w_0(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t},$$
$$\theta_x(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}T_{xmn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t},$$
$$\theta_y(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}T_{ymn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{i\omega t}.$$

Further, the solution of the problem is reduced to an eigenvalue problem in the form:

$$\left|\mathbf{L}-\omega^2\mathbf{I}\right|=0 \quad (7)$$

that is:

$$\underbrace{\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15}\\ L_{12} & L_{22} & L_{23} & L_{24} & L_{25}\\ L_{13} & L_{23} & L_{33} & L_{34} & L_{35}\\ L_{14} & L_{24} & L_{34} & L_{44} & L_{45}\\ L_{15} & L_{25} & L_{35} & L_{45} & L_{55}\end{bmatrix}}_{\mathbf{L}}-$$
$$\omega^2\underbrace{\begin{bmatrix} I_1 & 0 & -\alpha I_2 & I_4 & 0\\ 0 & I_4 & -\beta I_2 & 0 & I_4\\ -\alpha I_2 & -\beta I_2 & I_3\left(\alpha^2+\beta^2\right)+I_1 & -\alpha I_5 & -\beta I_5\\ I_4 & 0 & -\alpha I_5 & I_6 & 0\\ 0 & I_4 & -\beta I_5 & 0 & I_6\end{bmatrix}}_{\mathbf{I}}=\mathbf{0}$$

where the coefficients L_{ij}, $(i,j=1\div 5)$ depend on the coefficients defined by equation (3) and the parameters $\alpha=\frac{m\pi}{a}$, $\beta=\frac{n\pi}{b}$.

For the display of the obtained numerical values for a rectangular laminate plate, consisting of layers whose constitutive matrix is defined through engineering constants, it is necessary to normalize the obtained values in accordance with $\bar{\omega}=\omega\frac{ab}{h}\sqrt{\frac{\rho}{E_2}}$. In this way, a dimensionless frequency value is obtained, which for $m=1$, $n=1$, corresponds to the first mode of oscillation. The characteristics of material are:

Material 1 [22]:
$E_1/E_2=$ variable ratio, $G_{12}/E_2= \ =0.6$,
$G_{13}/E_2= \ =0.6$, $G_{23}/E_2=0.5$,
$\nu_{12}=\nu_{13}=\nu_{23}=0.25$

Material 2 [22]:
$E_1/E_2=25$, $G_{12}/E_2= \ =0.5$, $G_{13}/E_2=$ 0.5,
$G_{23}/E_2=0.2$, $\nu_{12}=\nu_{13}=\nu_{23}=0.25$.

In this paper, a new shape function is applied:

$$f(z)=z\left(\cosh\left(\frac{z}{h}\right)-1.388\right) \quad (8)$$

which is an odd function of z and satisfies the zero stress boundary conditions.

The verification of the results was performed by comparing them with the results obtained using the functions shown in the table:

Table 1. Shape Functions $f(z)$ Used in the Verification of Results

Author	Shape function f(z)
Ambartsumain [15]	$\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)$
Soldatos [16]	$h\sinh(z/h)-z\cosh(1/2)$
Mechab [18]	$\frac{z\cos\left(\frac{1}{2}\right)}{-1+\cos\left(\frac{1}{2}\right)}-\frac{h\sin\left(\frac{z}{h}\right)}{-1+\cos\left(\frac{1}{2}\right)}$

Free-supported laminate plates with 3 and 4 layers, corresponding to the group of symmetric cross-ply laminates of the form $\left[0°/90°/90°/0°\right]$ and $\left[0°/90°/0°\right]$ were considered.

Table 2. The values of the dimensionless frequency $\bar{\omega}$ of the symmetric cross-ply laminate $\left[0°/90°/90°/0°\right]$ for a variable ratio E_1/E_2, a fixed ratio $a/h=5$, and the adopted material 1.

Author	E_1/E_2 $m=1,\ n=1$				
	3	5	10	20	50
Present	6.585	7.232	8.373	9.697	11.418
Ambartsumain	6.585	7.233	8.375	9.699	11.421
Soldatos	6.585	7.233	8.374	9.698	11.419
Mechab	6.585	7.233	8.375	9.700	11.423

The results of the dimensionless frequencies obtained using the proposed method are presented in Tables 2–5, along with data from the relevant literature for mutual verification.
By comparing the values for different ratios a/h and b/a, as well as for two different materials, an exceptionally high agreement is observed between the results presented in this paper and the results obtained using reference analytical and numerical functions from previous studies.

Table 3. The values of the dimensionless frequency $\bar{\omega}$ of the symmetric cross-ply laminate $\left[0°/90°/90°/0°\right]$ for a variable ratio a/h, a fixed ratio $E_1/E_2=40$, and the adopted material 1.

Author	a/h $m=1,\ n=1$				
	2	4	10	50	100
Present	5.585	9.520	15.326	18.691	18.840
Ambartsumain	5.588	9.522	15.328	18.691	18.840
Soldatos	5.586	9.520	15.327	18.691	18.840
Mechab	5.590	9.524	15.330	18.691	18.841

The maximum deviations are negligible and in most cases do not exceed the third decimal place, indicating the exceptional precision of the developed method. It is particularly significant that, in regimes with large values of the a/ha/h and b/ab/a ratios, almost complete overlap of the results is achieved, confirming the numerical stability and accuracy of the model even in borderline cases. A similar level of precision was observed when applying different materials, demonstrating that the method is robust and independent of specific physical parameters. The agreement with the results of Ambartsumian, Soldatos, and Mechab further confirms that the new shape function does not compromise the physical consistency of the model, but rather provides a consistent response within valid limits.

Table 4. The values of the dimensionless frequency $\bar{\omega}$ of the symmetric cross-ply laminate $\left[0°/90°/0°\right]$ for a variable ratio a/h, a fixed ratio $E_1/E_2=25$ and the adopted material 2.

Author	a/h $m=1,\ n=1$				
	2	4	10	50	100
Present	4.424	7.361	12.061	15.043	15.181
Ambartsumain	4.432	7.370	12.068	15.044	15.181
Soldatos	4.426	7.364	12.064	15.043	15.181
Mechab	4.437	7.375	12.072	15.044	15.181

Based on the conducted verification, it can be concluded that the developed method is highly reliable, numerically stable, and physically consistent. Its application has achieved nearly complete agreement with established results from the literature, with minimal differences within the expected numerical deviations. The new functional form used to describe the laminate layering enables achieving the same precision as previous methods, thus confirming its validity and potential for application in further engineering analyses. Considering the results achieved, the method can be regarded as a reliable tool for analysing the vibrational behaviour of complex laminate structures in a wide range of geometric and material configurations.
In Figure 1, the dependence of the dimensionless natural frequency $\bar{\omega}$ on the aspect ratio a/h for two configurations of symmetric laminated plates was analysed: a three-layer plate with the layer arrangement [0°, 90°, 0°] [0°, 90°, 0°] and a four-layer plate with the layer arrangement [0°, 90°, 90°, 0°] [0°, 90°, 90°, 0°].

Table 5. The values of the dimensionless frequency $\bar{\omega}$ of the symmetric cross-ply laminate $[0°/90°/0°]$ for a variable ratio E_1/E_2, a fixed ratio $a/h = 2$, and the adopted material 1.

Author	E_1/E_2 $m=1,\ n=1$				
	3	5	10	20	50
Present	4.492	4.647	4.857	5.086	5.494
Ambartsumain	4.493	4.648	4.860	5.090	5.501
Soldatos	4.492	4.648	4.858	5.087	5.496
Mechab	4.493	4.650	4.862	5.092	5.507

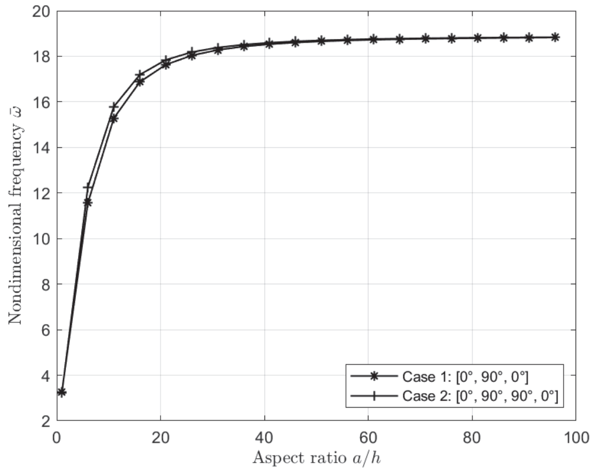

Figure 1. Dependence of the dimensionless natural frequency $\bar{\omega}$ on the aspect ratio *a/h* for two configurations of symmetric laminated plates (material 1).

A clear trend of increasing $\bar{\omega}$ with the increase in the aspect ratio a/h is observed, where the frequency rises quickly for small values of the ratio, and then the increase slows down, with the frequency asymptotically approaching a constant value. This phenomenon can be explained by the fact that at small values of a/h, i.e., for thicker plates, the effects of transverse deformation are more pronounced, which leads to a decrease in the natural frequencies. As the aspect ratio a/h increases, the plate becomes thinner, and the influence of transverse deformation decreases, while the plate's behaviour increasingly resembles the idealized model of a thin plate, which results in the stabilization of the frequency.
By comparing the two configurations, it can be observed that their frequencies are almost identical throughout the entire observed range of a/ha/h ratios. This indicates that the additional layer in the [0°, 90°, 90°, 0°] [0°, 90°, 90°, 0°] configuration does not cause a significant change in the dynamic response compared to the three-layer [*0°, 90°, 0°*] [*0°, 90°, 0°*] configuration. The explanation for this behaviour lies in the fact that both configurations are symmetric and have the same layer orientation arrangement through the thickness, resulting in similar bending stiffness. In engineering practice, this conclusion can be particularly important as it suggests that, in terms of natural frequencies, there is no significant advantage in adding an additional layer to a symmetric laminated structure. Therefore, the choice between these configurations may depend more on other factors, such as weight, cost, or manufacturing complexity, rather than on their dynamic characteristics.

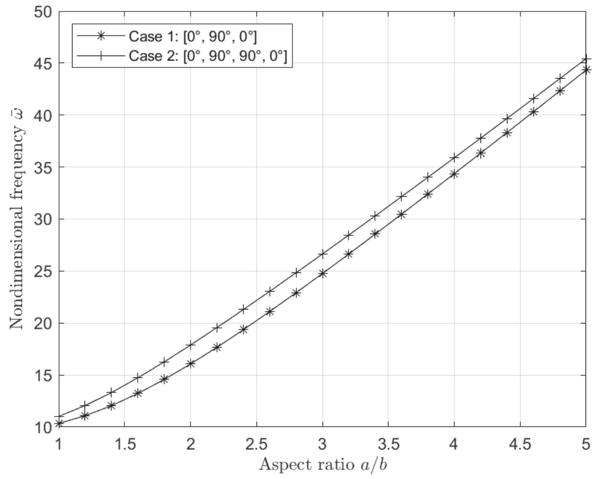

Figure 2. Influence of in-plane aspect ratio a/b on the dimensionless natural frequency $\bar{\omega}$ for two symmetric laminated plate configure-rations (material 1).

In Figure 2, the dependence of the dimensionless natural frequency $\bar{\omega}$ on the aspect ratio a/ba/b is presented for two configurations of symmetric laminated plates: a three-layer plate with the stacking sequence [0°, 90°, 0°], and a four-layer plate with the stacking sequence [0°, 90°, 90°, 0°]. A clear trend of increasing frequency with increasing *a/b* ratio is observed, where the rise is more pronounced in the lower range of *a/b*, while for higher values the relationship stabilizes and approaches a nearly linear behavior.
An increase in the *a/b* ratio indicates that the plate becomes more elongated in the xx-direction, which affects the distribution of bending stiffness and the natural mode shapes.

Higher values of *a/b* make the structure more beam-like, with vibrational behavior increasingly dominated by bending in the longer direction, leading to a rise in frequency.

When comparing the two configurations, the four-layer plate exhibits slightly higher frequency values throughout the entire examined *a/b* range, but the difference remains small and practically negligible. This suggests that the additional layer in the [*0°, 90°, 90°, 0°*] configuration does not significantly contribute to an increase in the dynamic stiffness of the plate compared to the basic three-layer configuration. Given that both plates are symmetric and share identical orientation distributions through the thickness, their effective bending stiffness remains similar, resulting in nearly identical natural frequencies.

CONCLUSION

This paper makes a significant contribution to the field of laminate composite structure analysis through the development and application of an advanced numerical method based on a new form of displacement distribution function along the thickness of the plate. The proposed approach enables highly accurate estimation of dimensionless natural frequencies for symmetric cross-ply laminate plates, without relying on correction factors or additional empirical assumptions, clearly distinguishing it from traditional higher-order models.

The validation of the results, carried out through detailed comparison with relevant reference data from the literature, demonstrates the exceptional accuracy of the developed model, with matches reaching the fourth decimal place. This level of agreement not only confirms the numerical stability and reliability of the method but also its ability to precisely reproduce the physical behaviour of complex laminate systems under various boundary and geometric conditions.

The scientific contribution lies in the fact that the proposed formulation expands the theoretical foundation for modelling multi-layer composite elements, enabling more efficient analysis of complex structures with minimal computational cost. In addition to its theoretical value, the model is highly applicable in engineering practice particularly in fields such as aerospace and automotive industries, where optimization of mass, strength, and dynamic stability are critical design requirements.

In conclusion, the developed method not only confirms its scientific validity through excellent correlation with previous works but also opens the door for further improvement of the model including the analysis of asymmetric laminates, the introduction of thermal effects, complex boundary conditions, and expansion to nonlinear behaviours. This paper thus lays a solid foundation for future research and represents a step forward in the development of comprehensive and efficient tools for the analysis of modern composite structures.

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