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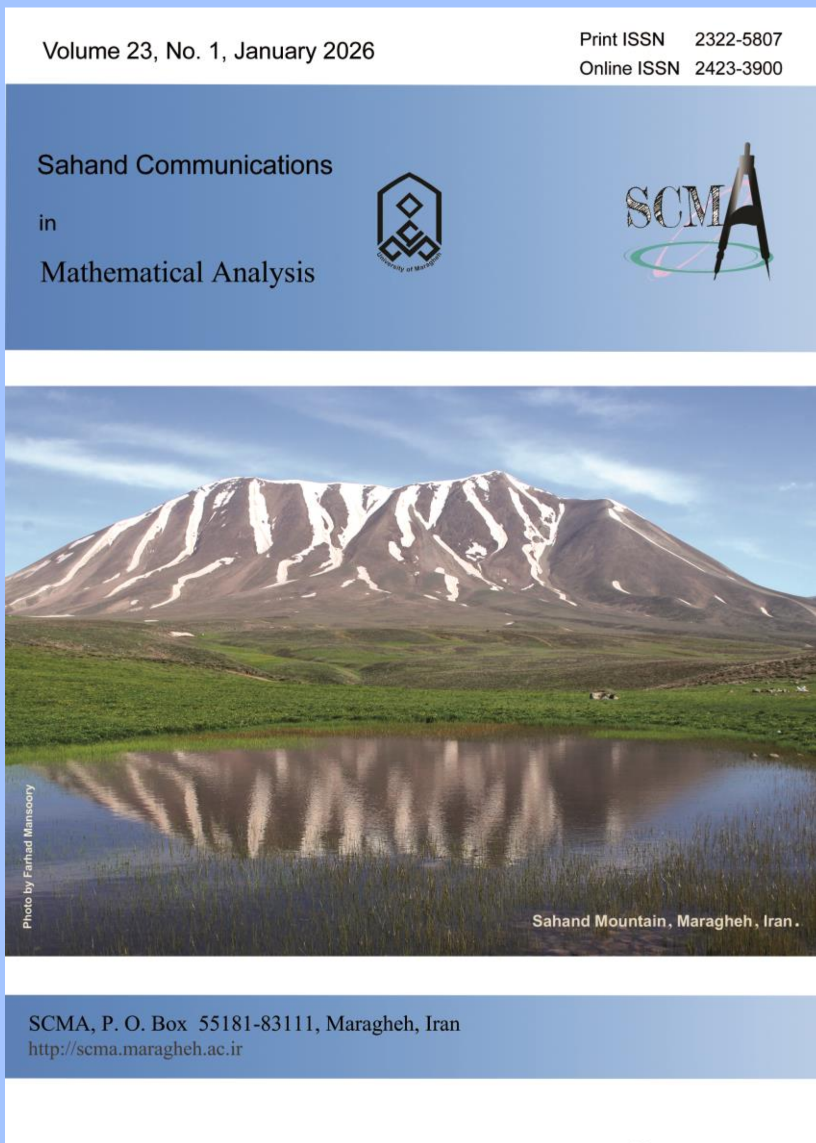

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On m-Exponential Convexity with Respect to s and Its Applications

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Abstract. The concept of convexity represents one of the fundamental notions of mathematical analysis and optimization. Various extensions of the concept of convexity have contributed to a wide range of applications, including the study of integral inequalities, approximation theory and other areas of applied mathematics. In this paper, we start from exponentially convex functions defined with respect to s and introduce a new class of m-exponentially convex functions with respect to s. Furthermore, the basic algebraic properties of this class are analyzed. In the second part, special attention is devoted to the application and the meaning of the extension of the Hermite–Hadamard inequality, where a more general framework is provided compared to the existing ones. The obtained results, in addition to their theoretical significance, also point to the potential for applications in data analysis, optimization and further generalizations.

1. Introduction

The theory of convexity nowadays occupies a very important place in both theoretical and applied mathematics. Among other things, it is one of the key tools in the development of inequalities, the construction of various numerical methods and the theoretical frameworks of optimization. For example, the Hermite–Hadamard inequality and its variants are used to characterize convex functions. Furthermore, they find applications in mathematical analysis, statistics, economics and other fields.

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In recent years, there has been a growing interest in introducing new types of convexity, both for theoretical and practical reasons. New types of convexity often generalize existing ones or are defined for specific classes of functions. For instance, very basic generalizations such as m-convex functions ([15, 18, 21, 22]), s-convex functions ([4]) and (s, m)-convex functions ([1]) have been introduced. Building on them, classes of exponential convex functions have been obtained ([9, 19, 20]), as well as exponential type P-functions [7] and exponential trigonometric convex functions [8]. In all of these works, special attention has been given to the Hermite–Hadamard inequality, which we will also analyze in this paper.

Parallel to the development of various types of convexity, fractional calculus has also gained increasing importance, with a large number of studies already exploring the application of these types of convexity in that field. The connection between convexity and fractional calculus has proven to be particularly fruitful, as it allows for the proof of new integral inequalities, through which not only theoretical extensions are obtained but also practical tools in physics, biomedicine, signal processing and so on. With deeper development and the establishment of new inequalities, these results become significantly applicable. The previously described applications and extensions can be found in [[10, 11, 13, 16, 17]]. Other works also explore the possibility of applying these results to inequalities and connections with arithmetic, geometric and other means ([2]). The paper [5] investigates generalized Humbert polynomials whose integral and series representations can serve as a basis for examining various types of convexity, including extensions, with particular emphasis on their potential applications. In the context of functions and their applications, the paper [12] demonstrates the potential use of new classes of functions, as well as their connection with convex functions.

The motivation for this paper arises precisely from these new directions of development. In [6], Kadakal introduced the concept of exponential convex functions with respect to the parameter s, thereby significantly extending the notion of classically defined exponential convex functions.

2. Preliminaries

The following definitions introduce the concept of convexity with extensions: m-convexity and (s, m)-convexity. There also exist several other variants of generalizations of the first definition.

Definition 2.1 (Convex function)**.** Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is called *convex* if for every $x, y \in I$ and for every $t \in [0, 1]$ it holds that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Definition 2.2 ([15, 18, 21])**.** Let $I \subseteq \mathbb{R}_+$ and $m \in [0,1]$. A function $f : I \to \mathbb{R}$ is called *m-convex* if for every $x, y \in I$ and for every $t \in [0,1]$ it holds that

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

Definition 2.3 ([1])**.** Let $s, m \in (0,1]$. A function $f : [0,\infty) \to \mathbb{R}$ is called *(s,m)-convex* if for every $x, y \in [0,\infty)$, $s, m \in (0,1]$ and for every $t \in [0,1]$ it holds that

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y).$$

The following three definitions introduce the concepts of exponential convexity, m-exponential convexity and exponential m-convexity. In the indicated papers, one can find properties related to such types of functions.

Definition 2.4 ([9])**.** Let $I \subseteq \mathbb{R}$. A function $f : I \to \mathbb{R}$ is called *exponentially convex* if for every $x, y \in I$ and $t \in [0,1]$ it holds that

$$f(tx + (1-t)y) \leq (e^t - 1)f(x)^t + \left(e^{1-t} - 1\right) f(y).$$

Definition 2.5 ([3])**.** A nonnegative function $f : I \to \mathbb{R}$ is called *(s,m)-exponential type convex* for some fixed $s, m \in (0,1]$, if

$$f(tx + m(1-t)y) \leq (e^{st} - 1)f(x)^t + m\left(e^{(1-t)s} - 1\right) f(y).$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 2.6 ([15])**.** Let $m \in [0,1]$ and $I \subseteq \mathbb{R}$ be a m-convex set. Then a real-valued function $f : I \to \mathbb{R}$ is said to be *exponentially m-convex* if

$$e^{f((1-t)x+mty)} \leq (1-t)e^{f(x)} + mte^{f(y)},$$

for all $x, y \in I$, $t \in [0,1]$.

As emphasized in the introduction, particular attention has been drawn to the paper that introduced the concept of exponentially convex functions with respect to the parameter s.

Definition 2.7 ([6])**.** A nonnegative function $f : I \to \mathbb{R}$ is said to be *exponentially convex with respect to s*, if for all $x, y \in I$ and all $t \in [0,1]$ it holds that

$$f\left(tx + (1-t)y\right) \leq ts^{1-t}f(x) + (1-t)s^t f(y). \tag{2.1}$$

The properties described in the following two lemmas will be used in proving some key theorems in this paper.

Lemma 2.8 ([20])**.** *Let* $0 < c \leq 1$ *and a mapping* $f : \left[a, \frac{b}{c}\right] \to \mathbb{R}$ *is differentiable on* $\left(a, \frac{b}{c}\right)$ *with* $b > a > 0$ *and* $m \in [0,1]$*. If* $f' \in L_1\left[a, \frac{b}{c}\right]$*, then*

$$\begin{aligned} &\frac{f(a) + f\left(\frac{mb}{c}\right)}{2} - \frac{c}{mb - ca}\int_a^{\frac{mb}{c}} f(t)dt \\ &= \frac{mb - ca}{2c} \cdot \int_0^1 (1 - 2t) f'\left(ta + m(1 - t)\frac{b}{c}\right) dt. \end{aligned}$$

Lemma 2.9 ([20])**.** *Let* $0 < c \leq 1$ *and a mapping* $f : [ca, b] \to \mathbb{R}$ *is differentiable on* (ca, b) *with* $b > a > 0$ *and* $m \in [0,1]$*. If* $f' \in L_1[ca, b]$*, then*

$$\begin{aligned} &\frac{f(mca) + f(b)}{2} - \frac{1}{b - ca}\int_{mca}^{b} f(t)dt \\ &= \frac{b - mca}{2} \cdot \int_0^1 (1 - 2t) f'(tb + mc(1 - t)a)\, dt. \end{aligned}$$

3. Main Results

In the following, the key contribution of this paper will be defined, namely the m-exponential convexity with respect to the parameter s. Let $I \subset \mathbb{R}$ be an interval and fix $m \in (0,1]$ and $s > 0$.

Definition 3.1 (m-exponential convexity with respect to s)**.** A non-negative function $f : I \to \mathbb{R}$ is said to be *m-exponentially convex with respect to s*, if for all $x, y \in I$ and all $t \in [0,1]$ it holds that

$$f(tx + m(1 - t)y) \leq ts^{1-t} f(x) + m(1 - t)s^t f(y). \tag{3.1}$$

Remark 3.2. The set of all m-exponentially convex functions with respect to s will be denoted by mECFs.

Remark 3.3.
1) For $s = 1$ and $m = 1$, (3.1) reduces to convexity: $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$.
2) For $s = 1$, (3.1) reduces to m-convexity:

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y).$$

3) For $m = 1$, (3.1) reduces to the exponential convexity with respect to s: $f(tx + (1 - t)y) \leq ts^{1-t} f(x) + (1 - t)s^t f(y)$.

Example 3.4. Let $f(x) = x^2$, $x \in [0, \infty)$. Show that f is m-exponentially convex with respect to s, i.e.,

$$f(tx + m(1 - t)y) \leq ts^{1-t} f(x) + m(1 - t)s^t f(y),$$

for all $x, y \geq 0$, $t \in [0,1]$, $m \in (0,1]$ and $s \geq 1$.

Proof. Since $f(x) = x^2$ is convex on $[0, \infty)$, it follows that

$$\begin{aligned}(tx + m(1-t)y)^2 &\leq tx^2 + (1-t)(my)^2 \\ &= tx^2 + m^2(1-t)y^2.\end{aligned}$$

Because $m^2 \leq m$ for $m \in (0, 1]$ and $s^{1-t} \geq 1$, $s^t \geq 1$ for $s \geq 1$, we obtain

$$tx^2 + m^2(1-t)y^2 \leq ts^{1-t}x^2 + m(1-t)s^t y^2.$$

Combining the inequalities gives

$$(tx + m(1-t)y)^2 \leq ts^{1-t}x^2 + m(1-t)s^t y^2.$$

□

In the following properties, some basic algebraic characteristics valid for the functions of the set mECFs will be studied.

Theorem 3.5. *Let $f : I \to \mathbb{R}$, $x, y \in I$ arbitrary and let $m \in (0, 1]$.*

1) *If f is m-exponentially convex with respect to s, then f is m-convex for every $s \leq 1$.*
2) *If f is m-convex, then f is m-exponentially convex with respect to s for every $s \geq 1$.*

Proof. Let $f : I \to \mathbb{R}$, $x, y \in I$ arbitrary and let $m \in (0, 1]$.

1) Let the function f be m-exponentially convex with respect to s. Then, since $s \leq 1$, it follows that $ts^{1-t} \leq t$.

   On the other hand, we have: $m(1-t)s^t \leq m(1-t)$. Hence, from the assumption that f is a function that is m-exponentially convex with respect to s, it follows that

   $$f(tx + m(1-t)y) \leq ts^{1-t}f(x) + m(1-t)s^t f(y).$$

   Therefore, f is an m-convex function.
2) Now suppose that f is an m-convex function and let $s \geq 1$. Then $s^{1-t} \geq 1$, which implies $ts^{1-t} \geq t$. Moreover, since $1 - t \geq 0$ and $m > 0$, it follows that $m(1-t)s^t \geq m(1-t)$. Thus, for every $x, y \in I$, we have:

   $$\begin{aligned}f(tx + m(1-t)y) &\leq tf(x) + m(1-t)f(y) \\ &\leq ts^{1-t}f(x) + m(1-t)s^t f(y).\end{aligned}$$

   Therefore, f is an m-exponentially convex function with respect to s.

□

Theorem 3.6. *Let $f, g : I \to \mathbb{R}$ be m-exponentially convex functions with respect to s and let $m \in (0, 1]$. Then,*

1) *$f+g$ is a function that is m-exponentially convex with respect to s.*
2) *If $c \geq 0$, then cf is an m-exponentially convex function with respect to s.*

Proof. From Definition 3.1, for fixed $m \in (0,1]$ and $s \geq 0$, the proof is obvious. □

Theorem 3.7. *Let $g : I \to J$ be an m-convex function and let $f : J \to \mathbb{R}$ be an m-exponentially convex function with respect to s that is non-decreasing, where $I, J \subset \mathbb{R}$ are intervals and $m \in (0,1]$. Then, $f \circ g : I \to \mathbb{R}$ is an m-exponentially convex function with respect to s.*

Proof. Let $x, y \in I$. Then:

$$\begin{aligned}(f \circ g)(tx + m(1-t)y) &= f(g(tx + m(1-t)y)\\ &\leq f(tg(x) + m(1-t)g(y))\\ &\leq ts^{1-t} f(g(x)) + m(1-t)s^t f(g(y)).\end{aligned}$$

Therefore, $f \circ g$ is an m-exponentially convex function with respect to s. □

Theorem 3.8. *Let $f_i : I \to \mathbb{R}$ be an arbitrary family of m-exponentially convex functions with respect to s and let $m \in (0,1]$ and define $f(x) = \sup_i f_i(x)$. If*

$$C = \{x \in I \mid f(x) < +\infty\},$$

then C is an interval and f is an m-exponentially convex function on C.

Proof. Let $x, u \in I$ be arbitrary points, then:

$$\begin{aligned}f(tx + m(1-t)y) &= \sup_i \left(f_i(tx + m(1-t)y)\right)\\ &\leq \sup_i \left(ts^{1-t} f_i(x) + m(1-t)s^t f_i(y)\right)\\ &\leq ts^{1-t} \sup_i f_i(x) + m(1-t)s^t \sup_i f_i(y)\\ &= ts^{1-t} f(x) + m(1-t)s^t f(y).\end{aligned}$$

Therefore, the previous relation proves that C is an interval and that f is an m-exponentially convex function with respect to s. □

In the continuation of this paper, the application of the new convexity will be analyzed through extensions of the Hermite–Hadamard inequality. Furthermore, the space $L_1[a,b]$ will be considered, which represents the space of integrable functions over $[a,b]$.

Theorem 3.9. *Let $f : [a, mb] \to \mathbb{R}$ be an m-exponentially convex function with respect to s, for $m \in (0, 1]$ and let $a < mb$. If $f \in L_1[a, mb]$, then*

1) *for all $s > 1$:*

$$\frac{2}{\sqrt{s}} f\left(\frac{a+mb}{2}\right) \leq \frac{1}{mb-a}\left(\int_a^{mb} f(t)dt + \int_{\frac{a}{m}}^{b} f(t)dt\right)$$
$$\leq \frac{s-\ln s-1}{\ln^2 s}\left(f(a)+f(b)+m\left(f\left(\frac{a}{m^2}\right)+f(b)\right)\right).$$

2) *for $s = 1$:*

$$2f\left(\frac{a+mb}{2}\right) \leq \frac{1}{mb-a}\left(\int_a^{mb} f(t)dt + \int_{\frac{a}{m}}^{b} f(t)dt\right)$$
$$\leq \frac{1}{2}\left(f(a)+f(b)+m\left(f\left(\frac{a}{m^2}\right)+f(b)\right)\right).$$

Proof. Let the assumptions described in the theorem hold.

1) Let

$$x_1 = ta + m(1-t)b, \text{ and } x_2 = (1-t)\frac{a}{m} + tb, t \in [0, 1].$$

Then, using the properties of an m-exponentially convex function with respect to s, we have:

$$f\left(\frac{a+mb}{2}\right) = f\left(\frac{x_1+mx_2}{2}\right)$$
$$= f\left(\frac{(ta+m(1-t)b)+((1-t)a+mtb)}{2}\right)$$
$$\leq \frac{\sqrt{s}}{2} f(ta+m(1-t)b) + \frac{\sqrt{s}}{2} f((1-t)a+mtb)$$
$$= \frac{\sqrt{s}}{2}\left(f(ta+m(1-t)b) + f((1-t)a+mtb)\right).$$

By integrating the previously obtained inequality over the interval $[0, 1]$ with respect to t, we obtain

$$f\left(\frac{a+mb}{2}\right) \leq \frac{\sqrt{s}}{2}\left(\int_0^1 f(ta+m(1-t)b)dt + \int_0^1 f\left((1-t)\frac{a}{m}+tb\right)dt\right)$$
$$= \frac{\sqrt{s}}{2}\left(\frac{1}{a-mb}\int_{mb}^{a} f(t)dt - \frac{m}{a-mb}\int_{\frac{a}{m}}^{b} f(t)dt\right)$$
$$= \frac{\sqrt{s}}{2}\cdot\frac{1}{mb-a}\left(\int_a^{mb} f(t)dt + m\int_{\frac{a}{m}}^{b} f(t)dt\right)$$

$$\begin{aligned}
&\cdot\frac{1}{mb-a}\left(\int_a^{mb} f(t)dt + m\int_{\frac{a}{m}}^{b} f(t)dt\right)\\
&= \int_0^1 f(ta+m(1-t)b)dt + \int_0^1 f\left((1-t)\frac{a}{m}+tb\right)dt\\
&\le \int_0^1 \left(ts^{1-t}f(a)+m(1-t)s^t f(b)\right)dt\\
&\quad + \int_0^1 \left(ts^{1-t}f(b)+m(1-t)s^t f\left(\frac{a}{m^2}\right)\right)dt\\
&= f(a)\frac{s-\ln s-1}{\ln^2 s} + mf(b)\frac{s-\ln s-1}{\ln^2 s} + f(b)\frac{s-\ln s-1}{\ln^2 s}\\
&\quad + mf\left(\frac{a}{m^2}\right)\frac{s-\ln s-1}{\ln^2 s}\\
&= \frac{s-\ln s-1}{\ln^2 s}\left(f(a)+f(b)+m\left(f\left(\frac{a}{m^2}\right)+f(b)\right)\right),
\end{aligned}$$

which completes the proof.

2) The inequality is obtained in a completely analogous manner to the procedure presented above.

□

Theorem 3.10. *Let $f:\left(0,\frac{b}{c}\right]\to\mathbb{R}$ be a differentiable function on $\left(0,\frac{b}{c}\right]$, such that the conditions $0<a<b$ and $0<c\le 1$ hold. Suppose that $|f'|^q$ is an m-exponentially convex function with respect to s on $\left(0,\frac{b}{c}\right]$ for every $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then the following holds:*

1)

$$\begin{aligned}
&\left|\frac{f(a)+f\left(\frac{mb}{c}\right)}{2} - \frac{c}{mb-ca}\int_a^{mb} f(t)dt\right|\\
&\le \frac{mb-ca}{2c}\cdot\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\
&\quad\cdot\left(-\frac{\ln s-s+1}{\ln^2 s}\left(|f'(a)|^q + m\left|f'\left(\frac{b}{c}\right)\right|^q\right)\right)^{\frac{1}{q}}, \quad \textit{for all } s>1.
\end{aligned}$$

2)

$$\begin{aligned}
&\left|\frac{f(a)+f\left(\frac{mb}{c}\right)}{2} - \frac{c}{mb-ca}\int_a^{mb} f(t)dt\right|\\
&\le \frac{mb-ca}{2c}\cdot\left(\frac{1}{2}\left(|f'(a)|^q + m\left|f'\left(\frac{b}{c}\right)\right|^q\right)\right)^{\frac{1}{q}}, \quad \textit{for } s=1.
\end{aligned}$$

Proof. Let us prove relation 1); relation 2) follows by an analogous argument. Starting from the left-hand side of the equality in relation 1) and by applying Lemma 2.8, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f\left(\frac{mb}{c}\right)}{2}-\frac{c}{mb-ca}\int_a^{mb} f(t)dt\right| \\
&\quad=\frac{mb-ca}{2c}\left|\int_0^1 (1-2t)\cdot f'\left(ta+m(1-t)\frac{b}{c}\right)dt\right| \\
&\quad\leq\frac{mb-ca}{2c}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\cdot\left(\int_0^1 \left|f'\left(ta+m(1-t)\frac{b}{c}\right)\right|^q dt\right)^{\frac{1}{q}},
\end{aligned}$$

where Hölder's inequality was applied to obtain the last relation. Then, using

$$\left|f'\left(ta+m(1-t)\frac{b}{c}\right)\right|^q \leq ts^{1-t}|f'(a)|^q + m(1-t)s^t\left|f'\left(\frac{b}{c}\right)\right|^q,$$

and using the theorem's assumption—namely, since f' is m-exponentially convex—we obtain the upper bound of the last relation as

$$\begin{aligned}
&\frac{mb-ca}{2c}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\cdot\left(\int_0^1 \left|f'\left(ta+m(1-t)\frac{b}{c}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\leq\frac{mb-ca}{2c}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}} \\
&\quad\cdot\left(\int_0^1 \left(ts^{1-t}|f'(a)|^q + m(1-t)s^t\left|f'\left(\frac{b}{c}\right)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\quad=\frac{mb-ca}{2c}\cdot\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(-\frac{\ln s-s+1}{\ln^2 s}\left(|f'(a)|^q + m\left|f'\left(\frac{b}{c}\right)\right|^q\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

Which was to be shown. For $s=1$, the only difference is in the step where the integration is performed and by the same procedure the desired relation 2) is also obtained. □

Theorem 3.11. *Let $f:\left(0,\frac{b}{c}\right]\to\mathbb{R}$ be a differentiable function on $\left(0,\frac{b}{c}\right]$, such that the conditions $0<a<b$ and $0<c\leq 1$ hold. Suppose that $|f'|^q$ is an m-exponentially convex function with respect to s on $\left(0,\frac{b}{c}\right]$ for every $q\geq 1$. Then the following holds:*

1)

$$\left|\frac{f(a)+f\left(\frac{mb}{c}\right)}{2}-\frac{c}{mb-ca}\int_a^{\frac{mb}{c}} f(t)dt\right|$$

$$\leq \frac{mb-ca}{2c} \cdot \left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$

$$\cdot \left(A \left(|f'(a)|^q + m \left| f'\left(\frac{b}{c}\right) \right|^q \right) \right)^{\frac{1}{q}}, \quad \textit{for all } s > 1,$$

where $A = \dfrac{-4(\sqrt{s}-1)^2 - \ln^2 s + (s + 2\sqrt{s} - 3)\ln s}{\ln^3 s}$.

2)

$$\left| \frac{f(a) + f\left(\frac{mb}{c}\right)}{2} - \frac{c}{mb-ca} \int_a^{\frac{mb}{c}} f(t)dt \right|$$

$$\leq \frac{mb-ca}{2c} \cdot \left(\frac{1}{2}\right)^{1+\frac{1}{q}} \cdot \left(|f'(a)|^q + m \left| f'\left(\frac{b}{c}\right) \right|^q \right)^{\frac{1}{q}}, \quad \textit{for } s = 1.$$

Proof. To prove the desired inequality, let us start from the left-hand side and first, as in the proof of the previous theorem, apply Lemma 2.8 and then apply the power mean. After that, we use the fact that the function f' is m-exponentially convex with respect to s.

$$\left| \frac{f(a) + f\left(\frac{mb}{c}\right)}{2} - \frac{c}{mb-ca} \int_a^{\frac{mb}{c}} f(t)dt \right|$$

$$\leq \frac{mb-ca}{2c} \int_0^1 |1-2t| \left| f'\left(ta + m(1-t)\frac{b}{c}\right) \right| dt$$

$$\leq \frac{mb-ca}{2c} \left(\int_0^1 |1-2t| dt \right)^{1-\frac{1}{q}}$$

$$\cdot \left(\int_0^1 |1-2t| \left| f'\left(ta + m(1-t)\frac{b}{c}\right) \right|^q dt \right)^{\frac{1}{q}}$$

$$\leq \frac{mb-ca}{2c} \left(\int_0^1 |1-2t| dt \right)^{1-\frac{1}{q}}$$

$$\cdot \left(\int_0^1 |1-2t| \left(ts^{1-t} |f'(a)|^q + m(1-t)s^t \left| f'\left(\frac{b}{c}\right) \right|^q \right) dt \right)^{\frac{1}{q}}$$

$$= \frac{mb-ca}{2c} \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left(\frac{-4(\sqrt{s}-1)^2 - \ln^2 s + (s + 2\sqrt{s} - 3)\ln s}{\ln^3 s} \right.$$

$$\left. \cdot \left(|f'(a)|^q + m \left| f'\left(\frac{b}{c}\right) \right|^q \right) \right)^{\frac{1}{q}},$$

which completes the proof. By substituting $s = 1$ in the preceding procedure and evaluating the corresponding integrals, relation 2) follows directly. $\square$

Theorem 3.12. *Let $f : (0, b] \to \mathbb{R}$ be a differentiable function on X, such that $0 < a < b$ and $0 < c \leq 1$ hold. Suppose that $|f'|^q$ is an m-exponentially convex function with respect to s on $(0, b]$ for every $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $m \in (0, 1]$. Then the following holds:*

1)

$$\left|\frac{f(mca) + f(b)}{2} - \frac{1}{b - ca}\int_{ca}^{b} f(t)dt\right| \leq \frac{b - mca}{2} \cdot \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \cdot \left(\frac{s - \ln s - 1}{\ln^2 s}\left(m|f'(ca)|^q + \left|f'(b)\right|^q\right)\right)^{\frac{1}{q}}, \quad \textit{for all } s > 1,$$

2)

$$\left|\frac{f(mca) + f(b)}{2} - \frac{1}{b - mca}\int_{mca}^{b} f(t)dt\right| \leq \frac{b - mca}{2} \cdot \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2}\right)^{\frac{1}{q}} \left(m|f'(ca)|^q + \left|f'(b)\right|^q\right)^{\frac{1}{q}}, \quad \textit{for } s = 1.$$

Proof. For the proof of the first part, i.e., when $s > 1$, we apply Lemma 2.9 and then apply Hölder's inequality. After that, we use the fact that the function f' is m-exponentially convex with respect to s.

$$\begin{aligned}
&\left|\frac{f(ca) + f(b)}{2} - \frac{1}{b - ca}\int_{ca}^{b} f(t)dt\right| \\
&\leq \frac{b - mca}{2}\left(\int_0^1 |2t-1|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'(tb + mc(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{b - mca}{2}\left(\int_0^1 |2t-1|^p dt\right)^{\frac{1}{p}} \\
&\quad \cdot \left(\int_0^1 \left(ts^{1-t}|f'(b)|^q + m(1-t)s^t|f'(ca)|^q\right) dt\right)^{\frac{1}{q}} \\
&= \frac{b - mca}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{s - \ln s - 1}{\ln^2 s}\left(|f'(b)|^q + m\left|f'(ca)\right|^q\right)\right)^{\frac{1}{q}},
\end{aligned}$$

which proves the first relation. For $s = 1$, using the same properties as in the first part, it follows that

$$\left|\frac{f(ca) + f(b)}{2} - \frac{1}{b - ca}\int_{ca}^{b} f(t)dt\right|$$
$$\leq \frac{b - mca}{2}\left(\int_0^1 |2t-1|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(tb + mc(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b - mca}{2}\left(\int_0^1 |2t-1|^p dt\right)^{\frac{1}{p}}$$
$$\cdot\left(\int_0^1 \left(ts^{1-t}|f'(b)|^q + m(1-t)s^t|f'(ca)|^q\right) dt\right)^{\frac{1}{q}}$$
$$= \frac{b - mca}{2}\left(\int_0^1 |2t-1|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(t|f'(b)|^q + m(1-t)|f'(ca)|^q\right) dt\right)^{\frac{1}{q}}$$
$$= \frac{b - mca}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\left(|f'(b)|^q + m\left|f'(ca)\right|^q\right)\right)^{\frac{1}{q}}.$$

□

Theorem 3.13. *Let $f : (0, b] \to \mathbb{R}$ be a differentiable function on X, such that the conditions $0 < a < b$ and $0 < c \leq 1$ hold. Let $|f'|^q$ be an m-exponentially convex function with respect to s on $(0, b]$ for every $q \geq 1$ and $m \in [0, 1]$. Then the following holds:*

1)

$$\left|\frac{f(mca) + f(b)}{2} - \frac{1}{b - mca}\int_{mca}^{b} f(t)dt\right|$$
$$\leq \frac{b - ca}{2}\cdot\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\cdot\left(\frac{-4(\sqrt{s}-1)^2 - \ln^2 s + (s + 2\sqrt{s} - 3)\ln s}{\ln^3 s}\left(m|f'(ca)|^q + \left|f'(b)\right|^q\right)\right)^{\frac{1}{q}}$$

for all $s > 1$,

2)

$$\left|\frac{f(mca) + f(b)}{2} - \frac{1}{b - mca}\int_{mca}^{b} f(t)dt\right|$$
$$\leq \frac{b - mca}{2}\cdot\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\cdot\left(m|f'(ca)|^q + \left|f'(b)\right|^q\right)^{\frac{1}{q}}, \quad \textit{for } s = 1.$$

Proof. For the proof of this theorem, we use Lemma 2.9, the power mean inequality and the fact that $|f'|^q$ is an m-exponentially convex function with respect to s.

$$\begin{aligned}
&\left|\frac{f(ca)+f(b)}{2}-\frac{1}{b-ca}\int_{ac}^{b}f(t)dt\right| \\
&\quad\leq \frac{b-mca}{2}\int_0^1 |1-2t|\left|f'(tb+mc(1-t)a)\right|dt \\
&\quad\leq \frac{b-mca}{2}\left(\int_0^1 |2t-1|dt\right)^{1-\frac{1}{q}} \\
&\quad\quad\cdot\left(\int_0^1 |1-2t|\left|f'(tb+mc(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\leq \frac{b-mca}{2}\left(\int_0^1 |2t-1|dt\right)^{1-\frac{1}{q}} \\
&\quad\quad\cdot\left(\int_0^1 |1-2t|\left(ts^{1-t}|f'(b)|^q+m(1-t)s^t\left|f'(ca)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&\quad= \frac{b-mca}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{-4(\sqrt{s}-1)^2-\ln^2 s+(s+2\sqrt{s}-3)\ln s}{\ln^3 s}\right. \\
&\quad\quad\left.\cdot\left(|f'(b)|^q+m\left|f'(ca)\right|^q\right)\right)^{\frac{1}{q}}.
\end{aligned}$$

As can be seen, we must provide a separate proof for the case $s=1$. Using the same properties as in the previous part, we obtain the following relations.

$$\begin{aligned}
&\left|\frac{f(ca)+f(b)}{2}-\frac{1}{b-ca}\int_{ac}^{b}f(t)dt\right| \\
&\quad\leq \frac{b-mca}{2}\int_0^1 |1-2t|\left|f'(tb+mc(1-t)a)\right|dt \\
&\quad\leq \frac{b-mca}{2}\left(\int_0^1 |2t-1|dt\right)^{1-\frac{1}{q}} \\
&\quad\quad\cdot\left(\int_0^1 |1-2t|\left|f'(tb+mc(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\leq \frac{b-mca}{2}\left(\int_0^1 |2t-1|dt\right)^{1-\frac{1}{q}} \\
&\quad\quad\cdot\left(\int_0^1 |1-2t|\left(t|f'(b)|^q+m(1-t)\left|f'(ca)\right|^q\right)dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$= \frac{b - mca}{2} \cdot \left(\frac{1}{2}\right)^{1+\frac{1}{q}} \cdot \left(m|f'(ca)|^q + \left|f'(b)\right|^q\right)^{\frac{1}{q}}. \qquad \square$$

4. Conclusion

In this paper, a new class of functions is defined, obtained by generalizing exponentially convex functions with respect to s, thus providing an extension of m-exponentially convex functions with respect to s. In this work, some basic algebraic properties of such defined functions are presented. Special attention is devoted to the application of these functions in the domain of extensions of the Hermite–Hadamard inequality. The obtained results indicate a great potential for the application of the introduced class of functions, both in theoretical research and especially in applications. Moreover, they point to the possibility of various generalizations in terms of other types of convexity. Recent studies in this field indicate that various forms of convexity and convex functions can be effectively applied in the analysis of fractional integral and differential inequalities (as has been done specifically for exponentially convex functions in [14]). Building upon these ideas and concepts from fractional calculus, future research could continue in this direction, especially since the parameter m introduces an additional correction to convexity that may also have significant applications.

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