SCIENTIFIC PUBLICATIONS OF THE STATE UNIVERSITY OF NOVI PAZAR
SER. A: APPL. MATH. INFORM. AND MECH. vol. 17, 1, 2025, 15–25

Hamming energy of sunlet and n-barbell graphs

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Abstract: The Hamming matrix of a graph arises from the notion of Hamming distance and provides a matrix-based framework for studying vertex dissimilarity. The corresponding Hamming energy, defined as the sum of the absolute values of the eigenvalues of the Hamming matrix, represents a natural spectral invariant that is closely related to the classical graph energy. In this paper, we investigate the Hamming matrix of two important families of graphs, namely sunlet graphs and barbell graphs. By applying the technique of equitable vertex partitions and methods from matrix spectral theory, we obtain explicit expressions for the H-spectrum and the H-energy of these graphs. Our results extend and complement existing studies on energy-like invariants for special classes of graphs.

Keywords: Hamming energy, Sunlet graph, n-Barbell graph

1 Introduction

Graphs are among the fundamental objects of discrete mathematics, and their matrix representations enable the application of spectral methods for the study of the structure and properties of networks. The classical graph energy, introduced by Gutman [2], is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix and has found numerous applications in mathematical chemistry and applied mathematics [3, 4, 5, 6, 9].

The Hamming distance originates from coding theory and measures the number of positions at which two binary strings differ. In graph theory, this concept can be naturally applied by associating each vertex with a binary string obtained from the corresponding row of the incidence matrix of the graph. Based on this idea, the Hamming index of a graph is introduced as a global measure of dissimilarity between vertices, defined as the sum of Hamming distances over all pairs of vertices. This index relates the structure of a graph to the geometry of the set of binary strings representing its vertices and appears as a natural **analogous** of classical spectral invariants [10, 13].

Manuscript received December 13, 2025; accepted December 23, 2025

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https://doi.org/10.46793/SPSUNP2501.15B

Since a graph can be described by different types of matrices, several extensions of this concept have been introduced in the literature. Among them, a recently defined invariant, known as the Hamming energy [13], has attracted particular attention. This invariant is based on the so-called Hamming matrix and has been shown to be comparable with classical graph energy, as well as to possess chemical relevance [12]. Further fundamental properties of the Hamming energy were established in [11].

In this paper, we study the Hamming matrix of two classes of graphs, namely sunlet graphs and barbell graphs. For these families, we derive explicit formulas for the H-spectrum and the corresponding H-energy by employing the technique of equitable vertex partitions combined with tools from matrix spectral theory. This approach yields closed-form expressions for the eigenvalues and allows us to analyze their behavior as the number of vertices increases. The graph classes considered here have been the subject of numerous studies, particularly in connection with graph energy and various topological indices [7, 8]. The results obtained in this paper complement existing research on different notions of graph energy for special classes of graphs.

2 Preliminaries

Let $G = (V(G), E(G))$ be a simple graph with the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. Two vertices v_i and v_j are adjacent if they are joined by an edge, which we denote by $v_i \sim v_j$; otherwise, $v_i \not\sim v_j$. An edge is incident to each of its end–vertices, and the degree of a vertex v in G, written as $d_G(v)$ or simply $d(v)$, is the number of edges incident to v.

The adjacency matrix of G is the $n \times n$ matrix $A(G) = (a_{ij})$ whose entries record the adjacency of vertices, while the degree matrix $D(G)$ is the diagonal matrix with diagonal entries equal to the degrees of the vertices of G.

We write $I = I_n$ for the identity matrix of order n and $J_{m\times n}$ for the all–ones matrix of size $m \times n$; when $m = n$, we simply write J_n or J if the order is clear from the context.

Remark 2.1. *The all–ones matrix J_n has one non–zero eigenvalue, namely n, with eigenvector $\mathbf{1} = (\underbrace{1, 1, \ldots, 1}_{n})^T$. All remaining eigenvalues are equal to 0 (with multiplicity $n-1$), and their eigenvectors are orthogonal to $\mathbf{1}$.*

In addition to the adjacency matrix $A(G)$ and the degree matrix $D(G)$, we consider the incidence matrix $B(G) = (b_{ij})_{n\times m}$ of G, defined by

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is incident with } e_j, \\ 0, & \text{otherwise.} \end{cases}$$

Interpreting each row of $B(G)$ as a binary string, we denote by $s(v)$ the string corresponding to a vertex v. Following [10], the Hamming index (or H-index) of G is

$$H_B(G) = \sum_{i<j} H_d\big(s(v_i), s(v_j)\big),$$

where $H_d(s(v_i), s(v_j))$ is the usual Hamming distance between the binary strings $s(v_i)$ and $s(v_j)$.

In [13] the Hamming matrix $H(G) = (h_{ij})_{n\times n}$ of a graph G was introduced by

$$h_{ij} = H_d\big(s(v_i), s(v_j)\big), \qquad i, j = 1, \dots, n. \tag{1}$$

To distinguish spectra of different graph matrices, the eigenvalues of $A(G)$ and $H(G)$ will be called A-eigenvalues and H-eigenvalues, respectively; their multisets are the A-spectrum and the H-spectrum of G. The Hamming energy of G (or H-energy) is defined as the sum of absolute values of the H-eigenvalues.

Since $H(G)$ is symmetric, all H-eigenvalues are real. We order them as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and write $Spec(H(G))$ for the multiset of these eigenvalues. For repeated eigenvalues we use the shorthand $\lambda^{[k]}$ to indicate that λ has multiplicity k.

A useful tool for computing $H(G)$ is a result from [10], which will be employed several times in what follows.

Theorem 2.2. *[10] Let u and v be vertices of a graph G. Then*

$$H_d(G)(s(u), s(v)) = \begin{cases} d(u) + d(v) - 2, & \text{if } u \sim v, \\ d(u) + d(v), & \text{if } u \nsim v, \\ 0, & \text{if } u = v. \end{cases}$$

Its trace satisfies $\mathrm{tr}(H(G)) = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} h_{ii} = 0$. Furthermore, all properties that hold for Hermitian or non-negative matrices also apply to $H(G)$.

Having in mind Theorem 2.2 we conclude that all off-diagonal entries of $H(G)$ are strictly positive except in the graphs K_2 and nK_1. Consequently, $H(G)$ is a nonnegative irreducible matrix for every graph other than K_2 and nK_1.

In what follows, we present fundamental properties of irreducible matrices that will be utilized in the subsequent analysis.

Theorem 2.3. *[1] Let M be an irreducible symmetric matrix with non-negative entries. Then the largest eigenvalue λ_1 of M is simple, with a corresponding eigenvector whose entries are all positive (known as the Perron vector). Moreover, $|\lambda| \leq \lambda_1$ for all eigenvalues λ of M.*

Theorem 2.4. *[1] Let A be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Given a partition $\{1, 2, \dots, n\} = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$, with $|\Delta_i| = n_i > 0$, $\sum\limits_{i=1}^{m} n_i = n$, consider the corresponding blocking $A = (A_{ij})$ such that A_{ij} is an $n_i \times n_j$ block. Let e_{ij} be the sum of the entries in A_{ij} and set $Q = \big(e_{ij}/n_i\big)$ (so e_{ij}/n_i is the average row sum in A_{ij}). Then the spectrum of Q is contained in the segment $[\lambda_n, \lambda_1]$.*

Theorem 2.5. *[1] Let A be any non-negative symmetric matrix partitioned into blocks as in Theorem 2.4. Let the blocks A_{ij} have constant row sums q_{ij} and set $Q = (q_{ij})$. Then the spectrum of Q is contained in the spectrum of A (taking into account the multiplicities of the eigenvalues). Furthermore, the largest eigenvalue (index) of Q is equal to the largest eigenvalue of A.*

Remark 2.6. *The matrix Q defined in Theorem 2.4 is called the quotient matrix of A. In the case of constant row sums of A_{ij} for each pair i and j, Q is called an equitable quotient matrix of A, and the corresponding partition of the matrix A into blocks A_{ij} is called an equitable partition.*

3 Main Results

In this section, we concentrate on two notable classes of graphs, namely the sunlet graphs S_n and the n-barbell graphs. For each of these families, we exploit the symmetry of their structure to introduce a suitable equitable partition of the vertex set, which yields a low-dimensional quotient matrix B. The eigenvalues of B are contained in the H-spectrum of the Hamming matrix $H(G)$, and the remaining H-eigenvalues can then be determined from the block structure. In this way, we obtain closed expressions for the H-spectrum and, in particular, for the Hamming energy of the sunlet and n-barbell graphs.

3.1 The H-spectrum of the sunlet graph S_n

The sunlet graph S_n is obtained by taking a simple n-gon, that is, a cycle with vertices $v_1,\dots,v_n$, and attaching to each vertex v_i a new pendant vertex u_i. In this way, we obtain a graph with $2n$ vertices: the "inner" cycle, whose vertices all have degree 3 (two neighbours on the cycle and one pendant neighbour), and the "outer" set of n leaves of degree 1. Because of this structure, the graph is often called a sunlet or n–sun in the literature, with the inner cycle playing the role of the "sun" and the pendants corresponding to its "rays". This symmetric, circular organization is well suited for spectral analysis: the adjacency matrix and the corresponding Hamming matrix admit a natural 2-block structure (cycle versus pendants), so that their spectra can be described by combining the quotient matrix of this equitable partition with the Fourier diagonalization of the cycle.

Figure 1 shows a sunlet graph with $2n$ vertices.

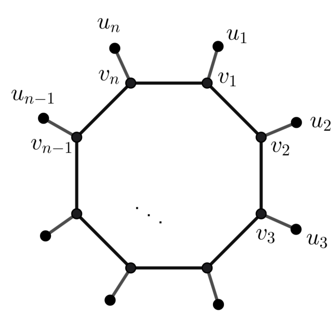

Fig. 1. Sunlet S_n graph with $2n$ vertices.

In the next theorem we determine the H-spectrum of the sunlet graph S_n and derive an explicit formula for its H-energy.

Theorem 3.1. *Let S_n be the sunlet graph on $2n$ vertices obtained from the cycle C_n by attaching one pendant vertex to each vertex of C_n. Then the H-spectrum of S_n is*

$$Spec\big(H(S_n)\big) = \Big\{ 4n-6 \pm 2\sqrt{5n^2-8n+5} \Big\} \cup \Big\{ \mu_k^{\pm} \,:\, k = 1,2,\ldots,n-1 \Big\},$$

where for each $k = 1,\ldots,n-1$,

$$\mu_k^{\pm} = -4 - 2\cos\frac{2k\pi}{n} \pm \sqrt{4\cos^2\frac{2k\pi}{n} + 8\cos\frac{2k\pi}{n} + 8}\,.$$

Furthermore, the H-energy of S_n is

$$HE(S_n) = 4\sqrt{5n^2-8n+5} + 8n - 12.$$

Proof. Label the vertices on the cycle C_n of S_n by $v_1, v_2, \ldots, v_n$, and by $u_1, \ldots, u_n$ the remaining vertices, so that u_i is pendant at v_i. Thus, the natural equitable partition of $V(S_n)$ is

$$C_1 = \{v_1, \ldots, v_n\}, \qquad C_2 = \{u_1, \ldots, u_n\}.$$

Every vertex $v_i \in C_1$ has degree 3, and every vertex $u_i \in C_2$ has degree 1. By Theorem 2.2, the entries of the Hamming matrix $H(S_n)$ follow directly from these degrees:

- two adjacent cycle vertices contribute $d(v_i) + d(v_j) - 2 = 4$,
- the remaining $n-3$ cycle vertices contribute $d(v_i) + d(v_j) = 6$,
- u_i contributes $d(v_i) + d(u_i) - 2 = 2$,
- each u_j $(j \neq i)$ contributes $d(v_i) + d(u_j) = 4$,
- each pair u_i, u_j $(i \neq j)$ contributes $d(u_i) + d(u_j) = 2$.

With respect to the ordering $(v_1, \ldots, v_n, u_1, \ldots, u_n)$, the matrix $H(S_n)$ therefore has the block form

$$H(S_n) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

where

$$H_{11} = 6(J_n - I_n) - 2A(C_n), \qquad H_{22} = 2(J_n - I_n), \qquad H_{12} = H_{21}^T = 4J_n - 2I_n.$$

Since the partition $\{C_1, C_2\}$ is equitable, the corresponding quotient matrix is

$$B = \begin{bmatrix} 2\cdot 4 + (n-3)\cdot 6 & (n-1)\cdot 4 + 2 \\ (n-1)\cdot 4 + 2 & (n-1)\cdot 2 \end{bmatrix} = \begin{bmatrix} 6n-10 & 4n-2 \\ 4n-2 & 2n-2 \end{bmatrix}.$$

The eigenvalues of B satisfy $\det(B-\lambda I_2)=0$, yielding

$$\lambda_{1,2}=4n-6\pm 2\sqrt{5n^2-8n+5}.$$

By Theorem 2.5, these two eigenvalues belong to the spectrum of $H(S_n)$, and the index of $H(S_n)$ equals $\lambda_1=4n-6+2\sqrt{5n^2-8n+5}$.

For $n\geq 3$ it is easy to check that $\lambda_1>0$ and $\lambda_2<0$.

To obtain the remaining $2n-2$ eigenvalues, we consider eigenvectors whose coordinates in each n-vertex part sum to zero (i.e., they are orthogonal to $\mathbf{1}$ in both parts). For any such vector x we have $J_n x=(\mathbf{1}^T x)\mathbf{1}=0$, hence every term containing J_n vanishes. Therefore, on this set of vectors the matrix $H(S_n)$ acts as

$$\widetilde{H}=\begin{bmatrix}-6I_n-2A(C_n) & -2I_n\\ -2I_n & -2I_n\end{bmatrix}.$$

Recall that the adjacency matrix of C_n is diagonalised by the Fourier vectors

$$x^{(k)}=\left(1,\omega^k,\omega^{2k},\dots,\omega^{(n-1)k}\right)^T,\quad \omega=e^{2\pi i/n},\quad k=0,1,\dots,n-1,$$

and

$$A(C_n)\,x^{(k)}=\lambda_k x^{(k)},\qquad \lambda_k=2\cos\frac{2k\pi}{n}.$$

For $k\geq 1$ the vectors $x^{(k)}$ are orthogonal to $\mathbf{1}$, so $J_n x^{(k)}=0$. Hence, for each $k=1,\dots,n-1$ we may look for eigenvectors of $\widetilde{H}$ of the form

$$\begin{bmatrix}\alpha x^{(k)}\\ \beta x^{(k)}\end{bmatrix},$$

which leads to a 2×2 eigenvalue problem

$$\begin{bmatrix}-6-2\lambda_k & -2\\ -2 & -2\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\mu\begin{bmatrix}\alpha\\ \beta\end{bmatrix}.$$

Therefore, for each $k=1,\dots,n-1$, the remaining eigenvalues are the roots of

$$\det\begin{bmatrix}-6-2\lambda_k-\mu & -2\\ -2 & -2-\mu\end{bmatrix}=0,$$

that is,

$$\mu_k^{\pm}=-4-\lambda_k\pm\sqrt{\lambda_k^2+4\lambda_k+8}$$

with $\lambda_k=2\cos\frac{2k\pi}{n}$, which gives the stated form

$$\mu_k^{\pm}=-4-2\cos\frac{2k\pi}{n}\pm\sqrt{4\cos^2\frac{2k\pi}{n}+8\cos\frac{2k\pi}{n}+8}\,.$$

This yields all $2n$ eigenvalues of $H(S_n)$.

To compute the H-energy, first note that for $\lambda_k \in [-2,2]$ we have

$$\mu_k^{\pm} \le 0, \qquad k = 1,\dots,n-1,$$

and for even n one of them is 0 (when $k = n/2$ and $\lambda_k = -2$). Moreover

$$\mu_k^+ + \mu_k^- = -8 - 2\lambda_k.$$

Since both eigenvalues are non–positive,

$$|\mu_k^+| + |\mu_k^-| = -(\mu_k^+ + \mu_k^-) = 8 + 2\lambda_k.$$

Hence the contribution of these $2n-2$ eigenvalues to $HE(S_n)$ is

$$\sum_{k=1}^{n-1} \left(|\mu_k^+| + |\mu_k^-|\right) = \sum_{k=1}^{n-1} \left(8 + 2\lambda_k\right) = 8(n-1) + 2\sum_{k=1}^{n-1} \lambda_k.$$

Since

$$\sum_{k=0}^{n-1} \lambda_k = 2\sum_{k=0}^{n-1} \cos\frac{2k\pi}{n} = 0,$$

we obtain

$$\sum_{k=1}^{n-1} \lambda_k = -\lambda_0 = -2,$$

and therefore

$$\sum_{k=1}^{n-1} \left(|\mu_k^+| + |\mu_k^-|\right) = 8(n-1) + 2(-2) = 8n - 12.$$

Finally, since $\lambda_1 > 0$ and $\lambda_2 < 0$,

$$|\lambda_1| + |\lambda_2| = \lambda_1 - \lambda_2 = 4\sqrt{5n^2 - 8n + 5}.$$

Thus

$$HE(S_n) = \left(|\lambda_1| + |\lambda_2|\right) + \sum_{k=1}^{n-1} \left(|\mu_k^+| + |\mu_k^-|\right) = 4\sqrt{5n^2 - 8n + 5} + 8n - 12,$$

which completes the proof. □

3.2 The H-spectrum of the n-barbell graph BB_n

A class of graphs for which the full spectrum of the Hamming matrix, and thus the Hamming energy, can be determined is the class of n-barbell graphs. These graphs are constructed by connecting two complete graphs K_n with a single edge. In this way, we obtain a graph in which two vertices have degrees greater by one than that of the other vertices.

Note that the 3-barbell graph is isomorphic to the kayak paddle graph $KP(3,3,1)$. For illustration purposes, a 5-barbell graph is shown in Figure 2.

By applying the equitable partition technique, we can determine the H-spectrum as well as the H-energy of the n-barbell graph BB_n.

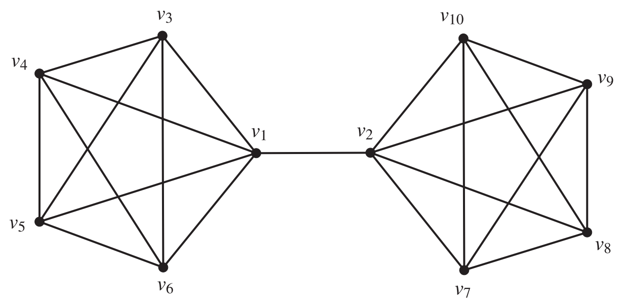

Fig. 2. A representation of the 5-barbell graph

Theorem 3.2. *The H-spectrum of the n-barbell graph BB_n is given by*

$$Spec(H(BB_n)) = \left\{\lambda_1, \lambda_2, -4(n-1), -2(n-2)^{[2n-3]}\right\},$$

where the set $\{\lambda_1, \lambda_2\}$ is the spectrum of the equitable quotient matrix B corresponding to the Hamming matrix $H(BB_n)$.

Theorem 3.3. *Let BB_n be the n-barbell graph obtained by joining two copies of K_n with a single edge. Let v_1 and v_2 be the two vertices incident with the bridging edge, and consider the equitable partition*

$$C_1 = \{v_1, v_2\}, \qquad C_2 = V(BB_n) \setminus C_1.$$

The corresponding (equitable) quotient matrix of the Hamming matrix $H(BB_n)$ is

$$B = \begin{bmatrix} 2n-2 & 4(n-1)^2 \\ 4(n-1) & 2(n-1)^2 + 2(n-2)^2 \end{bmatrix}.$$

Let λ_1, λ_2 be the eigenvalues of B. Then the H-spectrum of BB_n is

$$Spec(H(BB_n)) = \left\{\lambda_1, \lambda_2,\ -4(n-1),\ \big(-2(n-2)\big)^{[2n-3]}\right\}.$$

Proof. Let $V(BB_n)$ denote the vertex set of the graph BB_n. Since exactly two vertices have degree greater than all others, we collect them into the first partition class C_1. The remaining vertices all share the same degree and thus form the second partition class C_2.

Without loss of generality, label the vertices so that the first complete subgraph K_n has vertex set $\{v_1, v_3, \ldots, v_{n+1}\}$ and the second complete subgraph K_n has vertex set $\{v_2, v_{n+2}, \ldots, v_{2n}\}$, with the unique bridging edge being $\{v_1, v_2\}$. Then, v_1 and v_2 are the only vertices of degree n in BB_n. Accordingly, we define the partition classes

$$C_1 = \{v_1, v_2\}, \qquad C_2 = V(BB_n) \setminus C_1.$$

Based on the definition of the Hamming matrix of a graph, we can explicitly construct the matrix $H(BB_n)$, which takes the form

$$H(BB_n)=\begin{bmatrix} 0 & 2n-2 & 2n-3 & 2n-3 & \cdots & 2n-3 & 2n-1 & 2n-1 & \cdots & 2n-1\\ 2n-2 & 0 & 2n-1 & 2n-1 & \cdots & 2n-1 & 2n-3 & 2n-3 & \cdots & 2n-3\\ 2n-3 & 2n-1 & 0 & 2n-4 & \cdots & 2n-4 & 2n-2 & 2n-2 & \cdots & 2n-2\\ 2n-3 & 2n-1 & 2n-4 & 0 & \cdots & 2n-4 & 2n-2 & 2n-2 & \cdots & 2n-2\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 2n-3 & 2n-1 & 2n-4 & 2n-4 & \cdots & 0 & 2n-2 & 2n-2 & \cdots & 2n-2\\ 2n-1 & 2n-3 & 2n-2 & 2n-2 & \cdots & 2n-2 & 0 & 2n-4 & \cdots & 2n-4\\ 2n-1 & 2n-3 & 2n-2 & 2n-2 & \cdots & 2n-2 & 2n-4 & 0 & \cdots & 2n-4\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 2n-1 & 2n-3 & 2n-2 & 2n-2 & \cdots & 2n-2 & 2n-4 & 2n-4 & \cdots & 0 \end{bmatrix},$$

and can be expressed as a block matrix

$$H=H(BB_n)=\begin{bmatrix} H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix},$$

where H_{11} is the 2×2 block corresponding to the two highest-degree vertices from the class $C_1=\{v_1,v_2\}$. The block H_{12} is a $2\times(2n-2)$ matrix representing the connections between the vertices $\{v_1,v_2\}$ and all the remaining vertices. Accordingly, $H_{21}=H_{12}^T$. Finally, H_{22} is the $(2n-2)\times(2n-2)$ block corresponding to the vertices from the class C_2.

The quotient matrix $B=(b_{ij})_{2\times 2}$ of this equitable partition has the following elements: $b_{11}=2n-2$, $b_{12}=4(n-1)^2$, $b_{21}=4n-4$, and $b_{22}=2(n-1)^2+2(n-2)^2$, so we have

$$det(B-\lambda I)=\begin{vmatrix} 2(n-1)-\lambda & 4(n-1)^2\\ 4(n-1) & 2(n-1)^2+2(n-2)^2-\lambda\end{vmatrix}=0,$$

from where we determine λ_1 and λ_2

$$\lambda_{1,2}=2n^2-5n+4\pm\sqrt{4n^4-12n^3+25n^2-36n+20}.$$

By Theorem 2.5, the obtained eigenvalues λ_1 and λ_2 belong to the spectrum of $H(BB_n)$, and the index (i.e., the largest eigenvalue) of $H(BB_n)$ is equal to λ_1. We will show that $H(BB_n)$ also has the eigenvalue $-2(n-2)$ with multiplicity $2n-3$ and the eigenvalue $-4(n-1)$ with multiplicity 1. To prove this, we will construct a set of $2n-2$ linearly independent eigenvectors corresponding to these eigenvalues.

Let us consider the space $\mathbb{R}^{2n}$ and observe the standard basis $S=\{e_1,e_2,\dots,e_{2n}\}$ of $\mathbb{R}^{2n}$. We construct the vectors $y_1^{(1)}$, $y_i^{(2)}$ for $i=1,\dots,n-1$, and $y_j^{(3)}$ for $j=1,\dots,n-2$, in the following way.

Set

$$y_1^{(1)}=e_2-e_1+e_{n+2}+e_{n+3}+\cdots+e_{2n}-e_3-e_4-\cdots-e_{n+1}.$$

Next, let

$$y_i^{(2)} = e_1 - e_2 - e_3 + e_{2n+1-i},$$

where $i = 1,\ldots,n-1$, and

$$y_j^{(3)} = -e_3 + e_{n+2-i},$$

where $j = 1,\ldots,n-2$.
Set $w_1^{(1)} = \left(y_1^{(1)}\right)^T$, $w_i^{(2)} = \left(y_i^{(2)}\right)^T$ for $i = 1,\ldots,n-1$, and $w_j^{(3)} = \left(y_j^{(3)}\right)^T$ for $j = 1,\ldots,n-2$. The vectors $w_1^{(1)}, w_1^{(2)},\ldots,w_{n-1}^{(2)}, w_1^{(3)}, w_2^{(3)},\ldots,w_{n-2}^{(3)}$ are linearly independent and satisfy the relations

$$H(BB_n)\cdot w_1^{(1)} = -4(n-1)\cdot w_1^{(1)},$$

$$H(BB_n)\cdot w_i^{(2)} = -2(n-2)\cdot w_i^{(2)}, \quad i = 1,2,\ldots,n-1,$$

and

$$H(BB_n)\cdot w_j^{(3)} = -2(n-2)\cdot w_j^{(3)}, \quad j = 1,2,\ldots,n-2.$$

Thus, the spectrum of $H(BB_n)$ is given by

$$Spec(H(BB_n)) = \left\{\lambda_1, \lambda_2, -4(n-1), (-2(n-2))^{[2n-3]}\right\}.$$

□

Now that we have determined the spectrum of $H(BB_n)$, we can easily compute the Hamming energy of the graph BB_n.

Corollary 3.4. *The Hamming energy of the n-barbell graph BB_n is given by*

$$HE(BB_n) = |\lambda_1| + |\lambda_2| + 4(n-1) + 2(n-2)(2n-3),$$

where λ_1 and λ_2 are the eigenvalues of the equitable quotient matrix B corresponding to the matrix $H(BB_n)$.

4 Conclusions

In this paper we determined the Hamming spectra and Hamming energies of two notable graph families: sunlet graphs and n-barbell graphs. By exploiting the symmetry of these graphs and using equitable vertex partitions, we derived closed–form expressions for the eigenvalues of the Hamming matrix $H(G)$ and, in particular, for the corresponding Hamming energies, thus adding new examples to the list of graphs with explicitly known H-spectra.

Further research may focus on extending these techniques to other structured graph classes and to various graph products, as well as on clarifying how the Hamming energy of a composite graph (for instance, a product or corona) relates to the Hamming energies of its factor graphs.

Acknowledgement

This work was supported by the Serbian Ministry of Science, Technological Development and Innovation (Agreement No. 451-03-137/2025-03/ 200122).

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