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Algorithms for Decision-Making Based on Energies of Probabilistic and Dual Probabilistic Soft Set

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Abstract: After introducing the concepts of soft sets and graph energy as independent terms from different areas of mathematics, there has been significant application of both concepts. The theory of soft sets has been combined with other theories, such as fuzzy set theory and probability theory. From this combination with probability theory, a structure known as probabilistic soft set has emerged, which has been discussed relatively little in the context of decision-making. In this paper, we propose new decision-making algorithms based on numerical characteristics, which we call energies of probabilistic soft sets and dual probabilistic soft sets. The introduced concept of energy for probabilistic soft sets and dual probabilistic soft sets originated from integrating the idea of graph energy into probabilistic soft sets. The paper also presents a comparison of the obtained results using energy with results obtained by other algorithms.

Keywords and phrases: energy of probabilistic soft set, dual probabilistic soft set, decision-making.

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1 Introduction

Today, we have a large number of researchers who continue to develop the theory of fuzzy sets, originally described by Zadeh in 1965 in the paper [21], thus initiating the development of this theory and its many limitations and extensions. After Zadeh laid the foundations, it is worth mentioning the work of Atanassov [4], where he defined intuitionistic fuzzy sets, which are highly applicable in many areas. Following numerous extensions of fuzzy sets with certain drawbacks in practical applications, a new approach emerged towards the end of the last century, introducing the concept of soft sets by Molodtsov [17], as a

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new mathematical tool for dealing with uncertainties and ambiguities. From 1999 to the present day, the development, improvement, and research of soft set theory have become frequent and have progressed rapidly in many directions (see [3], [5], [14], [15], [20], and the references cited there).

The hybridization of the concepts of soft sets and fuzzy sets gave rise to the structure of fuzzy soft sets, as an indispensable tool dealing with vague and incomplete data (see the paper [16]). Until recently, there were relatively few researchers who connected soft set theory with probability theory. Although there have been relatively few works so far, there is certainly interest in hybridizing the concepts of soft sets and probability. The foundations of this structure were laid by Zhu and Wen in the paper [23], and operations with such sets were defined in the paper [9]. Until recently, probabilistic soft sets had not been applied to decision-making problems, and the paper [8] aimed to fill that gap. For this purpose, two new algorithms for probabilistic soft sets in decision-making problems were developed, illustrated with relevant examples.

The concept of graph energy, as a numerical value characterizing the graph, was defined by Gutman in 1978 [10], and since then, graph energy has been studied and applied in chemistry and many other areas of computer science. The study of matrix properties, including eigenvalues and singular values, belongs to linear algebra (see [1]). These notions are, however, indispensable in the definition of graph energy and in many other fundamental concepts. The graph energy studied in papers such as [7], [11], [12], [13], [18], [19], [22], and many others, is the idea for defining the energy of the concept of probabilistic soft sets as a numerical characteristic, as well as its potential applications in decision-making.

In Section 2 of this paper, basic concepts of soft set theory, fuzzy set theory, as well as probabilistic soft sets, are listed. This section is, in a way, a list of concepts introduced by many researchers in the previous period. In Section 3 of this paper, the central concept is introduced, which is the energy of a probabilistic soft set, where an example is given to show how we arrive at this numerical characteristic of a probabilistic soft set. This section also mentions one upper limitation of the introduced concept of energy. In Section 4 of this paper, an algorithm for decision-making based on the introduced concept of energy is presented, and a comparative analysis with other methods dealing with the same problems is conducted. It is also significant that this section contains corrected tables that contained errors in the calculations in the paper [8]. Certainly, the method based on energies helped significantly in finding such computational errors. Section 5 contains concepts such as the energy of a dual probabilistic set, a structure introduced in the paper [8], while Section 6 contains one application of the energy of a dual probabilistic soft set, namely for forming an algorithm for decision-making. The mentioned algorithms were tested on a large number of examples, and the results were compared with the results obtained in the papers [2], [8], [14], where the comparison of all observed algorithms is an integral part of Sections 4 and 6. In the concluding remarks, we summarize the obtained results and also mention plans for further research on the introduced energies of probabilistic soft sets and dual probabilistic soft sets and their potential applications.

2 Preliminaries

In this section, we list the basic concepts of soft set theory, as well as the concept of a fuzzy set. Next, we provide the definition of a probabilistic soft set, while other properties and operations can be found in the papers [8] and [9].

Let us assume that U is a non-empty finite universal set, and let E be the set of parameters, which we call the set of parameters. As is known, we denote the power set of U as $P(U)$. Let $A \subset E$.

Definition 1. *[5] A soft set, denoted as F_A, over the universe U is a set defined by the mapping f_A such that $f_A : E \to P(U)$, where $f_A(x) = \emptyset$ if $x \notin A$.*

As is common and widely accepted, the mapping f_A is called the approximating function of the soft set F_A for each $x \in E$. The soft set F_A over the universe U can also be represented using ordered pairs. Such representation is more straightforward, as the soft set F_A can be written as

$$F_A = \{(x, f_A(x)) \mid x \in E, f_A(x) \in P(U)\}.$$

The set of all soft sets over the universe U is commonly denoted as $S(U)$.

Definition 2. *[21] Let U be a universal set. A fuzzy set X over the universe U is a set defined by the function μ_X which represents the mapping $\mu_X : U \to [0,1]$, where μ_X is called the membership function of X, and the value $\mu_X(u)$ is called the degree of membership of element $u \in U$ in the fuzzy set X.*

So, a fuzzy set X over U can be represented as follows: $X = \{(\mu_X(u)/u) \mid u \in U, \mu_X(u) \in [0,1]\}$. The set of all fuzzy sets over the universe U is commonly denoted as $F(U)$.

By combining the concept of a soft set and a fuzzy set, we obtain the concept of a fuzzy soft set. Below is the definition of a fuzzy soft set, as done in [6].

Definition 3. *[6] A fuzzy soft set, denoted as Γ_A, over the universe U is a set defined by the function γ_A, which represents the mapping $\gamma_A : E \to F(U)$, such that $\gamma_A(x) = \emptyset$ if $x \notin A$.*

Similar to soft sets, γ_A is called the fuzzy approximating function of the fuzzy soft set Γ_A, and the value $\gamma_A(x)$ is the set called the x-element of the fuzzy soft set for all $x \in E$. Therefore, the fuzzy soft set Γ_A over the universe U can be represented by a set of ordered pairs as follows:

$$\Gamma_A = \{(x, \gamma_A(x)) \mid x \in E, \gamma_A(x) \in F(U)\}.$$

The set of all fuzzy soft sets over the universe U is commonly denoted as $FS(U)$. In the following, we will use the following notations $\Gamma_A, \Gamma_B, \Gamma_C, \ldots$ for fuzzy soft sets, and $\gamma_A, \gamma_B, \gamma_C, \ldots$ for their fuzzy approximating functions, respectively.

In order to apply a probabilistic soft set to real problems, the authors of the paper [8] revised the definition in the following way.

Definition 4. *[8] Let $D(U)$ be the set of all probability distributions subsets of U. A pair (F,A) is a probabilistic soft set over U, if $F : A \to D(U)$. Equivalently, $F(e_j) \in D(U)$, for any j.*

In other words, a probabilistic soft set is a parameterized family of probability distributions subsets of U. Suppose that (F,A) is a probabilistic soft set over U. Then for each parameter e_j, one has $F(e_j)(u_i) \in [0,1]$ for each option u_i Therefore, for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$, we can define $P(a_{ij}) = F(e_j)(u_i)$, a number that verifies $0 \leqslant P(a_{ij}) \leqslant 1$ where $\sum_{i=1}^{m} P(a_{ij}) = 1$.

3 Energy of a probabilistic soft set

The central concept of this paper is the energy of a probabilistic soft set as a numerical value that characterizes that structure. We will illustrate all the concepts defined in this section with a corresponding example. To define the central concept of this work, it is necessary to represent a probabilistic soft set tabularly, as done in the [8].

	e_1	e_2	$\cdots$	e_n
u_1	$P(a_{11})$	$P(a_{12})$	$\cdots$	$P(a_{1n})$
u_2	$P(a_{21})$	$P(a_{22})$	$\cdots$	$P(a_{2n})$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$P(a_{m1})$	$P(a_{m2})$	$\cdots$	$P(a_{mn})$
Total	$\sum_{i=1}^{m} P(a_{i1}) = 1$	$\sum_{i=1}^{m} P(a_{i2}) = 1$	$\cdots$	$\sum_{i=1}^{m} P(a_{in}) = 1$

In the following example, we demonstrate how a probabilistic soft set can be represented in tabular form and thus indicate further research in that direction. The example is illustrative and is taken from the paper [8].

Example 1. *[8] Let $U = \{u_i \mid i = 1,2,3,4,5,6\}$ be a set of six houses, and $E = \{expensive, beautiful, wooden, cheap, in\,the\,green\,surroundings, modern, in\,good\,repair, in\,bad\,repair\}$ be a set of parameters. Suppose that Mr. X is interested in buying a house on the basis of his choice parameters that are $A = \{e_i \mid i = 1,2,3,4,5,6\}$ where $e_1 =$"beautiful," $e_2 =$"wooden," $e_3 =$"cheap," $e_4 =$"in the green surroundings," $e_5 =$"modern," and $e_6 =$"in good repair." This situation is represented by the following probabilistic soft set (F,A):*

$F(e_1) = \{P(a_{11})/u_1, P(a_{21})/u_2, P(a_{31})/u_3, P(a_{41})/u_4, P(a_{51})/u_5, P(a_{61})/u_6\}$
$\quad = \{0.3/u_1, 0.2/u_2, 0.1/u_3, 0.05/u_4, 0.05/u_5, 0.3/u_6\}$,
$F(e_2) = \{P(a_{12})/u_1, P(a_{22})/u_2, P(a_{62})/u_6\} = \{0.4/u_1, 0.2/u_2, 0.4/u_6\}$,
$F(e_3) = \{P(a_{13})/u_1, P(a_{23})/u_2, P(a_{33})/u_3, P(a_{43})/u_4, P(a_{53})/u_5, P(a_{63})/u_6\}$
$\quad = \{0.3/u_1, 0.1/u_2, 0.1/u_3, 0.1/u_4, 0.2/u_5, 0.2/u_6\}$,
$F(e_4) = \{P(a_{14})/u_1, P(a_{24})/u_2, P(a_{64})/u_6\} = \{0.5/u_1, 0.1/u_2, 0.4/u_6\}$,
$F(e_5) = \{P(a_{15})/u_1, P(a_{35})/u_3, P(a_{65})/u_6\} = \{0.4/u_1, 0.1/u_3, 0.5/u_6\}$,
$F(e_6) = \{P(a_{16})/u_1, P(a_{26})/u_2, P(a_{36})/u_3, P(a_{46})/u_4, P(a_{56})/u_5, P(a_{66})/u_6\}$
$\quad = \{0.2/u_1, 0.2/u_2, 0.1/u_3, 0.1/u_4, 0.1/u_5, 0.3/u_6\}$.

We can also represent the observed probabilistic soft set tabularly as described above, resulting in

	e_1	e_2	e_3	e_4	e_5	e_6
u_1	0.3	0.4	0.3	0.5	0.4	0.2
u_2	0.2	0.2	0.1	0.1	0	0.2
u_3	0.1	0	0.1	0	0.1	0.1
u_4	0.05	0	0.1	0	0	0.1
u_5	0.05	0	0.2	0	0	0.1
u_6	0.3	0.4	0.2	0.4	0.5	0.3
Total	1	1	1	1	1	1

Since each probabilistic soft set can be represented tabularly, this leads us to the idea of defining an appropriate matrix representation of a probabilistic soft set.

Definition 5. *Let (F,A) be a probabilistic soft set. Suppose $U = \{u_1,u_2,\ldots,u_m\}$, $E = \{e_1,e_2,\ldots,e_n\}$ and $A \subseteq E$. If $x_{ij} = P(a_{ij}) = F(u_i)(e_j) \in [0,1]$ for every $i = 1,2,\ldots,m$ and every $j = 1,2,\ldots,n$, then the probabilistic soft set (F,A) is uniquely determined by the matrix*

$$Q = \left[x_{ij}\right]_{m\times n} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$

which we call the matrix of a probabilistic soft set (F,A) over the universe U, of size $m \times n$.

The matrix of a probabilistic soft set (F,A) observed in Example 3.1 is of the form

$$Q = \begin{bmatrix} 0.3 & 0.4 & 0.3 & 0.5 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0 & 0.1 & 0 & 0.1 & 0.1 \\ 0.05 & 0 & 0.1 & 0 & 0 & 0.1 \\ 0.05 & 0 & 0.2 & 0 & 0 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.4 & 0.5 & 0.3 \end{bmatrix}.$$

Now that we have introduced the matrix representation of a probabilistic soft set, we note that this matrix Q is, in general, rectangular. Consequently, the product QQ^T is a square matrix, and its eigenvalues are therefore well-defined. Moreover, QQ^T is symmetric and positive semidefinite; indeed, for any x one has

$$x^T QQ^T x = (Q^T x)^T (Q^T x) = \|Q^T x\|^2 \geq 0,$$

which implies that all eigenvalues of QQ^T are real and nonnegative. We will use the singular values of Q to define the central concept of this work.

Definition 6. *The energy of a probabilistic soft set* (F,A)*, denoted as* $\mathbb{E}_{(F,A)}$*, is defined as* $\mathbb{E}_{(F,A)} = \sum_{i=1}^{m} \sigma_i$*, where* $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_m$ *are the singular values of the matrix* Q *corresponding to the probabilistic soft set* (F,A)*.*

Finally, we can determine the energy of a probabilistic soft set (F,A) as mentioned in Example 3.1. Since the singular values are

$$\sigma_1 = 1.2994,\ \ \sigma_2 = 0.2920,\ \ \sigma_3 = 0.1830,\ \ \sigma_4 = 0.1589,\ \ \sigma_5 = 0.0464,\ \ \sigma_6 = 0.0195,$$

the energy is equal to $\mathbb{E}_{(F,A)} = \sum_{i=1}^{5} \sigma_i = 1.9992$.

The concept of energy will be useful for forming the decision-making algorithm that will be discussed in the next section, and what is important is that the concept of energy is analogously defined for an incomplete probabilistic soft set as well, i.e., a probabilistic soft set where one row is missing in the corresponding matrix representation. Regarding problems where parameters have different weights or importance, this is realized in a similar way. As shown in [8], these structures can be encoded in a table, which defines an associated matrix. We compute the singular values of this matrix and define the energy as the resulting numerical quantity (that is, the sum of the singular values).

By using the constraint of probability, as well as the inequality between the arithmetic and quadratic means, we can easily conclude what value is the upper bound of the introduced energy.

Theorem 3.1. *Let* (F,A) *be a probabilistic soft set,* $U = \{u_1, u_2, \ldots, u_m\}$*,* $E = \{e_1, e_2, \ldots, e_n\}$ *and* $A \subseteq E$*. Let* $\sigma_1, \sigma_2, \ldots, \sigma_m$ *be the singular values of the matrix* Q *corresponding to the probabilistic soft set* (F,A)*. Then the following inequality holds:*

$$\mathbb{E}_{(F,A)} \leqslant m\sqrt{n}.$$

Proof. Using the inequality between the arithmetic mean and the quadratic mean applied to the singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$, we obtain

$$\mathbb{E}_{(F,A)} = \sum_{i=1}^{m} \sigma_i \leqslant \sqrt{m \sum_{i=1}^{m} \sigma_i^2}.$$

From fundamental characteristics of matrices, eigenvalues and singular values and, also, knowing that $P(a_{ij}) \leqslant 1$, for $i = 1, \ldots, m$, and $j = 1, \ldots, n$ we can derive

$$\sum_{i=1}^{m} \sigma_i^2 = \operatorname{tr}\left(Q \cdot Q^T\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} (P(a_{ij}))^2 \leqslant mn.$$

Finally, $\mathbb{E}_{(F,A)} \leqslant \sqrt{nm^2} = m\sqrt{n}$. □

4 Probabilistic soft sets in decision-making

There are several proposed algorithms used for decision-making, each of which is somewhat successful but with certain drawbacks. In this section, we will analyze several algorithms and propose a new algorithm based on the energy of a probabilistic soft set. Let us start by formulating the algorithm in 4 steps to arrive at a decision.

Step 1: Construct a probabilistic soft set (F,A) over U;

Step 2: Form incomplete probabilistic soft sets $(F,A)_i$ over $U \setminus u_i$ for each $u_i \in U$;

Step 3: Determine the energies $\mathbb{E}_{(F,A)_i}$ for each incomplete probabilistic soft set $(F,A)_i$;

Step 4: Determine the minimum energy among all the energies of incomplete probabilistic soft sets obtained in Step 3 and interpret the obtained result.

We can apply the introduced algorithm to Example 3.1. in order to make decisions. It is clear that the energy of the probabilistic soft set formed in Example 3.1. is approximately equal to 2. For space saving, we will show how we arrive at the energy for $(F,A)_1$, i.e., the energy of the system where values for u_1 are missing. Based on the matrix

$$Q_1 = \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0.1 & 0 & 0.2 \\ 0.1 & 0 & 0.1 & 0 & 0.1 & 0.1 \\ 0.05 & 0 & 0.1 & 0 & 0 & 0.1 \\ 0.05 & 0 & 0.2 & 0 & 0 & 0.1 \\ 0.3 & 0.4 & 0.2 & 0.4 & 0.5 & 0.3 \end{bmatrix},$$

by applying linear algebra tools, we find that the singular values are

$$\sigma_1 = 0.9606,\ \ \sigma_2 = 0.2870,\ \ \sigma_3 = 0.1825,\ \ \sigma_4 = 0.0772,\ \ \sigma_5 = 0.0237,$$

and the first of the requested energies in Step 3 is $\mathbb{E}_{(F,A)_1} = 1.5310$.

Similarly, we can obtain the other 5 energies, and we have
$\mathbb{E}_{(F,A)_2} = 1.7668$,
$\mathbb{E}_{(F,A)_3} = 1.9019$,
$\mathbb{E}_{(F,A)_4} = 1.9473$,
$\mathbb{E}_{(F,A)_5} = 1.8719$,
$\mathbb{E}_{(F,A)_6} = 1.5405$.

Now, it is necessary to interpret the obtained result. Namely, by omitting element u_1 from the observed system, we conclude that the most energy is lost, while omitting element u_4 results in the least energy loss of the system. In other words, element u_1 contributes the most to the entire system, so it is the one we should choose. Based on the results and the analysis conducted, we can establish a linear ranking of the given elements in this specific example. Therefore, it holds that

$$u_1 > u_6 > u_2 > u_5 > u_3 > u_4.$$

By applying the algorithm presented in [14], which is based on the value of choices, we find that in the same example, we have the ranking $u_1 = u_6 > u_2 > u_3 > u_5 > u_4$, as seen in Table 6 of [8]. As for the algorithm described in [2], which is based on opportunity cost (OC) values, it gives the ranking $u_1 = u_6 > u_2 > u_3 > u_5 > u_4$. Even the algorithm using positive comparison matrices does not provide a unique decision in the observed example in [8]. Based on the considered methods, we can conclude that

Procedure	Obtained ranking
Method based on choice values [14]	$u_1 = u_6 > u_2 > u_3 > u_5 > u_4$
OC [2]	$u_1 = u_6 > u_2 > u_3 > u_5 > u_4$
Method based on $\mathbb{E}$	$u_1 > u_6 > u_2 > u_5 > u_3 > u_4$

The energy-based algorithm provides a unique decision, unlike the three observed algorithms mentioned earlier. Namely, in all three observed algorithms, we get equality between elements u_1 and u_6, while this algorithm still gives preference to element u_1. By summing probability distributions and values when using the method from [2], a rough estimate is made, because one value (characteristic) can prioritize an element that other characteristics have little or no impact on. Based on all that has been said, we can create a table showing clear characteristics of all observed methods

Procedure	Ranking methodology	Unique solution	Fine assessment during selection
[14]	Scores based on choice values	No	No
[2]	Opportunity cost (OC) values	No	No
[8]	Positive comparison matrices	No	Yes
Algorithms based on the energy	Scores based on comparison of energies	Yes	Yes

Now, we can apply this method to Example 3.1 and use of weighted parameters. Here, we need to delve deeper into the problem. Namely, during testing this method, we encountered quite different results compared to the results obtained with two methods presented in the paper [8]. After a thorough analysis of this method as well as the methods presented in the mentioned paper, we discovered a computational error during multiplication in tables 8 and 9 with weighted parameters in the paper [8]. Here, we will first present the corrected tables 7 and 8 of the mentioned paper, and then determine the results using the methods from [8] on the corrected tables. After that, we will compare the results with the results obtained by this energy-based method. Namely, table 8 in [8] should have the following form

	e_1 $w_1 = 0.6$	e_2 $w_2 = 0.3$	e_3 $w_3 = 0.4$	e_4 $w_4 = 0.5$	e_5 $w_5 = 0.6$	e_6 $w_6 = 0.7$	*Choice values*
u_1	0.18	0.12	0.12	0.25	0.24	0.14	1.05
u_2	0.12	0.06	0.04	0.05	0	0.14	0.41
u_3	0.06	0	0.04	0	0.06	0.07	0.25
u_4	0.03	0	0.04	0	0	0.07	0.14
u_5	0.03	0	0.08	0	0	0.07	0.18
u_6	0.18	0.12	0.08	0.2	0.3	0.21	1.09

,

and the linear ranking based on choice values is $u_6 > u_1 > u_2 > u_3 > u_5 > u_4$. Table 9 in [8] should have the following form

	e_1 $w_1 = 0.6$	e_2 $w_2 = 0.3$	e_3 $w_3 = 0.4$	e_4 $w_4 = 0.5$	e_5 $w_5 = 0.6$	e_6 $w_6 = 0.7$	*OC*
u_1	0	0	0	0	0.06	0.07	0.13
u_2	0.06	0.06	0.08	0.2	0.3	0.07	0.77
u_3	0.12	0.12	0.08	0.25	0.24	0.14	0.95
u_4	0.15	0.12	0.08	0.25	0.3	0.14	1.04
u_5	0.15	0.12	0.04	0.25	0.3	0.14	1
u_6	0	0	0.04	0.05	0	0	0.09

,

so we get the same ranking as above, i.e., $u_6 > u_1 > u_2 > u_3 > u_5 > u_4$.

If we apply the energy-based algorithm to the same example, we get in order
$\mathbb{E}_{(F,A)_1} = 0.8081$,
$\mathbb{E}_{(F,A)_2} = 0.9159$,
$\mathbb{E}_{(F,A)_3} = 0.9918$,
$\mathbb{E}_{(F,A)_4} = 1.0153$,
$\mathbb{E}_{(F,A)_5} = 0.9862$,
$\mathbb{E}_{(F,A)_6} = 0.7912$.
By analyzing the obtained energies, we can conclude that $u_6 > u_1 > u_2 > u_5 > u_3 > u_4$.

As we can see, the results obtained by all three methods are similar, i.e., all these methods make a decision in the first iteration. Namely, in the table below, we see only one difference in the decision-making system.

Procedure	Obtained ranking
Method based on choice values [8]	$u_6 > u_1 > u_2 > u_3 > u_5 > u_4$
OC [8]	$u_6 > u_1 > u_2 > u_3 > u_5 > u_4$
Method based on $\mathbb{E}$	$u_6 > u_1 > u_2 > u_5 > u_3 > u_4$

It is clear that the proposed energy-based approach made it possible to detect the inconsistency reported in [8]; however, this is not its only advantage. In comparison with the other two considered methods, which rely on cumulative aggregation of values and therefore tend to blur structural differences among alternatives, the energy-based criterion captures the global spectral structure of the associated matrix. Nevertheless, the uniqueness of the obtained decision cannot be guaranteed in all situations. In particular, if two or more alternatives induce identical spectral contributions, or if the underlying matrix possesses certain symmetries, multiple optimal solutions may occur. Hence, while the method provides a more structurally sensitive ranking mechanism, the issue of uniqueness must be treated with appropriate mathematical caution.

5 Energy of a dual probabilistic soft set

In paper [8], a new concept is introduced, namely the concept of dual probabilistic soft sets. The rationale for introducing this concept is illustrated by an example on which the authors of the mentioned paper demonstrated their decision-making method. To begin with, Let us recall the formal definition of the introduced concept.

Definition 7. *[8] Let U be a non-empty finite universe set, which is denoted by $U = \{u_i \mid i = 1,2,\dots,m\}$. Let $E = \{e_j \mid j = 1,2,\dots,n\}$ be a set of parameters and A be a subset of E. Let $\mathfrak{D}(A)$ be the set of all probability distributions on the set A. A pair (F,A) is called a dual probabilistic soft set over A when $F(u_i) \in \mathfrak{D}(A)$ or equivalently, $F(u_i) : A \to [0,1]$.*

For the observed dual probabilistic soft set defined over the universe U, for simplicity and clarity of notation, we often abbreviate the notation, and for all $u_i \in U$, we use the notation $P(a_{ij}) = F(u_i)(e_j)$. Based on the definition, the fact that $F(u_i) \in \mathfrak{D}(A)$ implies that $0 \leqslant P(a_{ij}) \leqslant 1$ for all i, j, and the equality $\sum_{j=1}^{n} P(a_{ij}) = 1$ holds.

As mentioned in the paper [8], dual probabilistic soft sets can be represented in tabular form as follows.

	e_1	e_2	$\cdots$	e_n	$Total$
u_1	$P(a_{11})$	$P(a_{12})$	$\cdots$	$P(a_{1n})$	$\sum_{j=1}^{n} P(a_{1j}) = 1$
u_2	$P(a_{21})$	$P(a_{22})$	$\cdots$	$P(a_{2n})$	$\sum_{j=1}^{n} P(a_{2j}) = 1$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
u_m	$P(a_{m1})$	$P(a_{m2})$	$\cdots$	$P(a_{mn})$	$\sum_{j=1}^{n} P(a_{mj}) = 1$

In the paper [8], the authors provided a good motivational example why this introduced concept is needed. Namely, among other things, it was mentioned that with five attributes or behavior phenotypes, each subject can be distributed in any population, while their personality is distributed among these five phenotypes. The example on which the authors of the mentioned paper demonstrated their decision-making method will be the subject of Section 6 of our paper, where we will also present an algorithm for decision-making based on the energy of a dual probabilistic soft set.

Definition 8. *Let (F,A) be a dual probabilistic soft set. Suppose $U = \{u_1,u_2,\dots,u_m\}$, $E = \{e_1,e_2,\dots,e_n\}$ and $A \subseteq E$. If $x_{ij} = P(a_{ij}) = F(u_i)(e_j)$ for every $i = 1,2,\dots,m$ and every $j = 1,2,\dots,n$, then the dual probabilistic soft set (F,A) is uniquely determined by the matrix*

$$M = \left[x_{ij}\right]_{mxn} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$

which we call the matrix of a dual probabilistic soft set (F,A) over the universe U, of size $m \times n$.

Now that we have introduced the concept of the matrix of a dual probabilistic soft set, we can, similar to Section 3, introduce the concept of the energy of a dual probabilistic soft set.

Definition 9. *The energy of a dual probabilistic soft set (F,A), denoted as $\mathbb{E}^{DP}_{(F,A)}$, is defined as $\mathbb{E}^{DP}_{(F,A)} = \sum_{i=1}^{m} \sigma_i$, where $\sigma_1 \geqslant \sigma_2 \geqslant \ldots \geqslant \sigma_m \geqslant 0$ are the singular values of the matrix M corresponding to the dual probabilistic soft set (F,A).*

In the following section, we will present an algorithm for decision-making based on the introduced energy and conduct a comparative analysis with the method described in the paper [8].

6 Dual probabilistic soft sets in decision-making

At the beginning of this section, Let us consider an example described in the paper [8].

Example 2. *Let $U = \{u_i \mid i = 1,2,3,4\}$ be a universe of participants. Suppose E contains an individualfs expertise. For simplicity, we use $A \subseteq E$ and $A = \{e_1,e_2,e_3\}$ where e_1 mathematician, e_2 statistician and e_3 computer scientist. A dual probabilistic soft set is given by the following table*

	e_1	e_2	e_3	$Total$
u_1	0.5	0.5	0	1
u_2	0.3	0.7	0	1
u_3	0.2	0.4	0.4	1
u_4	0.6	0.2	0.2	1

.

The decision-making problem in this example is characteristic because it involves a dual probabilistic soft set. Namely, the sum of all values in each row is 1. For this reason, the authors of paper [8] defined an algorithm based on Herrero's Comparison matrix, and using that algorithm, they obtained the following linear ranking $u_4 > u_1 = u_2 > u_3$, thus concluding that the decision is participant u_4.

The method described in the mentioned paper is good, but it has some shortcomings and needs refinement and improvement. Namely, if we were to consider a similar example with the same number of participants and the same characteristics but with different probabilities, we would get the same result. Let us consider the following example.

Example 3. *Let a dual probabilistic soft set be given by the following table*

	e_1	e_2	e_3	$Total$
u_1	0.40	0.41	0.19	1
u_2	0.21	0.60	0.19	1
u_3	0.2	0.4	0.4	1
u_4	0.6	0.2	0.2	1

.

As we can see, this example differs significantly from the previous one in terms of probability distribution. However, by applying the same algorithm, we arrive at the same decisions, as the same Herrero's Comparison matrix is obtained during the calculation.

These observed examples have motivated us to define a new decision-making algorithm based on the energy of a dual probabilistic soft set.

Step 1: Construct a dual probabilistic soft set (F,A) over U;

Step 2: Form dual probabilistic soft sets $(F,A)_i$ over $U \setminus u_i$ for each $u_i \in U$;

Step 3: Determine the energies $\mathbb{E}^{DP}_{(F,A)_i}$ for each dual probabilistic soft set $(F,A)_i$;

Step 4: Determine the minimum energy among all the energies of dual probabilistic soft sets obtained in Step 3 and interpret the obtained result.

Applying the newly defined algorithm to Example 6.1, we find that

$$\mathbb{E}^{DP}_{(F,A)_1} = 1.7884,\ \mathbb{E}^{DP}_{(F,A)_2} = 1.6742,\ \mathbb{E}^{DP}_{(F,A)_3} = 1.6620,\ \mathbb{E}^{DP}_{(F,A)_4} = 1.6865.$$

Based on these energies, we find that

$$u_3 > u_2 > u_4 > u_1,$$

indicating that element u_1 contributes the least to the system's energy, as $\mathbb{E}^{DP}_{(F,A)_1}$ is the highest among the obtained energies (which represents the system's energy without member u_1). The decision made by this algorithm is thus member u_3, contrary to the previous decisions.

When we apply the energy-based decision-making algorithm to Example 6.2, we obtain

$$\mathbb{E}^{DP}_{(F,A)_1} = 1.6247,\ \mathbb{E}^{DP}_{(F,A)_2} = 1.4844,\ \mathbb{E}^{DP}_{(F,A)_3} = 1.4430,\ \mathbb{E}^{DP}_{(F,A)_4} = 1.4347,$$

resulting in the linear order

$$u_4 > u_3 > u_2 > u_1,$$

consistent with the previous analysis.

Thus, one naturally expects that substantially different probability distributions should, in principle, lead to different decision outcomes. The energy-based decision-making algorithm is designed to provide structurally consistent guidance regarding which alternative can be removed while preserving the overall functionality of the system. This follows from the fact that the criterion is grounded in the singular values of the associated matrices, which encode global dependencies and mutual conditioning among all components of the model.

In contrast, the algorithm presented in [8] produces identical decisions in both analyzed examples, despite the significant differences in the underlying probability distributions. This suggests a certain insensitivity of that method to structural variation. At the same time, it should be emphasized that the uniqueness of the solution obtained by the energy-based approach is not guaranteed in all cases. For instance, in the presence of spectral symmetries or equivalent alternatives, multiple optimal outcomes may arise. The table below presents a comparative analysis of selected aspects of the considered decision-making methods.

Table 1. Comparative analysis of decision-making procedures

Procedure	Ranking methodology	Unique solution	Sensitive to changes in distributions	Main remarks
[8]	Eigenvector of Herrero's comparison matrix	Not guaranteed	No	Insensitive to structural changes; coarse aggregation
Energy-based decision algorithm	Spectral scores based on singular values (energy)	Not guaranteed (depends on spectrum)	Yes	Reflects global dependencies; may admit ties in symmetric cases

The comparison presented in Table 1 highlights the fundamental conceptual difference between the two approaches. While the method from [8] is based on cumulative aggregation and therefore may remain insensitive to substantial structural variations in the underlying probability distributions, the energy-based procedure directly reflects global spectral characteristics of the system. In particular, its sensitivity to changes in distributions follows from the dependence of singular values on the entire matrix structure.

At the same time, neither method guarantees uniqueness in all possible configurations. However, in the energy-based framework, non-uniqueness can be precisely interpreted in terms of spectral symmetries or equivalent alternatives, which provides a clearer mathematical explanation of such cases. This structural transparency represents an additional advantage of the proposed approach.

7 Conclusions

In this paper, we have explored probabilistic soft sets as well as dual probabilistic soft sets with the aim of improving decision-making algorithms. To achieve this, we defined numerical characteristics that we called energies of probabilistic soft sets and dual probabilistic soft sets. We utilized the introduced concept of energy to formulate a decision-making algorithm that has proven to be very efficient compared to other existing algorithms. Furthermore, the energy-based algorithm allowed us to identify a discrepancy in the calculations of other authors. Regarding the introduced concept of energy, it could be investigated what probabilities are needed for certain characteristics to have a predefined energy or desired energy in a probabilistic soft set. We believe that the introduced concept of energy indeed contributes to decision-making and is worthy of attention and further development.

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