METRIC DIMENSION OF COMPLETE SPLIT GRAPHS

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Abstract: *In this paper, the problem of determining the metric dimension for special class of graphs, named complete split graphs* ${K^*}_{k,n-k}$ *is considered. It is stated and proved formula for the metric dimension of this graphs: for* $n-k \geq 2$ *and* $k \geq 2$, *as well as for* $k=1$ *and* $n \geq 3$, *metric dimension of* ${K^*}_{k,n-k}$ *is equal to* $n-2$, *otherwise metric dimension of* ${K^*}_{k,n-k}$ *is equal to* $n-1$.

Keywords: *metric dimension, complete split graph, graph theory, discrete mathematics*

1. INTRODUCTION

The metric dimension problem was introduced in the seventies by Slater in [1] and Harary and Melter in [2], independently of one another. This NP-hard graph invariant [3] has been very much researched the last five decades and it has applications in many diverse areas.

Let $G=(V,E)$ be a simple connected undirected graph with vertex set V and edge set E. The distance between vertices u and v, denoted $d(u,v)$, is the length of a shortest $u-v$ path in graph G. Then, it can be said that a vertex w resolves two vertices $u,v \in V$ if $d(u,w) \neq d(v,w)$. A set $S=\{w_1,\dots,w_k\}$ subset of V is named a resolving set of graph G, if every two distinct vertices from V are resolved by some vertex of S. A metric basis for graph G is a resolving set of minimal cardinality. The cardinality of a metric basis for G is called the metric dimension and is denoted by $\beta(G)$. Its applications are in several diverse areas. Applications to the direction of robots in networks are analyzed in [3] and applications to chemistry in [4] and [5], among others.

There are several other variations of metric dimension in the literature, but the motivation for this paper was an already known mixed metric dimension for complete split graph ${K^*}_{k,n-k}$. First, the distance between edge *uv* and vertex *w* is defined as *d(uv,w)=min{d(u,w), d(v,w)}*. Second, similarly as previous, mixed metric dimension is defined in [6] as minimal cardinality of mixed resolving set for graph *G*, where every two items (item is vertex or edge) are resolved by some vertex from mixed resolving set. Third, it is easy to see that $B_M(G) \geq \beta(G)$.

In [7] exact value for mixed metric dimension of the complete split graphs is found with correspnding proof, i.e. mixed metric dimension for this graphs is equal $n-1$ for $k=1$ and $n \geq 3$, otherwise it is equal to its order n, i.e. $\beta_M\left({K^*}_{k,n-k}\right) = \begin{cases} n-1, & k=1 \wedge n \geq 3 \\ n, & otherwise \end{cases}$.

Now, some theoretical properties of metric dimension known from literature will be presented, which are used in the next section.

Theorem 1. *[8] A connected graph* G *of order* n *has dimension 1 if and only if* $G=P_n$.

Theorem 2. *[8] Let* G *be a connected graph of order* $n \geq 4$. *Then* $\beta(G)=n-2$ *if and only if* $G=K_{s,t}$ $(s,t \geq 1)$, $G=K_s+\overline{K_t}$ $(s \geq 1, t \geq 2)$ *or* $G=K_s+(K_1 \cup K_t)(s,t \geq 1)$.

Let's note that the graph $G=K_s+\overline{K_t}$ is the join of a complete graph and an empty graph.

Theorem 3. *[8] A connected graph* G *of order* $n \geq 2$ *has dimension* $n-1$ *if and only if* $\mathrm{G}=K_n$.

2. METRIC DIMENSION OF COMPLETE SPLIT GRAPHS

In this paper the metric dimension will be studied for an arbitrary complete split graph. These graphs are denoted by ${K^*}_{k,n-k}$ and have n vertices and $\binom{n}{2} - \binom{n-k}{2}$ edges, where vertex set is $V({K^*}_{k,n-k}) = \{x_i | 1 \le i \le k\} \cup \{y_i | 1 \le i \le n-k\}$, while edge set is defined as $E({K^*}_{k,n-k}) = \{x_i x_j | 1 \le i < j \le k\} \cup \{x_i y_j | 1 \le i \le k; 1 \le j \le n-k\}$. It should be noted that complete split graph ${K^*}_{k,n-k}$ can be viewed as a join of a complete graph K_k and a graph without edges $\overline{K}_{n-k}$. Throughout the paper we will assume that $V_1 = \{x_i | 1 \le i \le k\}$ and $V_2 = \{y_i | 1 \le i \le n-k\}$.

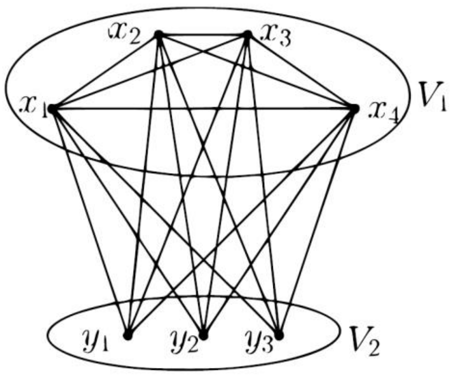

Figure 1: Graph ${K^*}_{4,3}$

In the Figure 1 is presented the complete split graph ${K^*}_{4,3}$. Its metric dimension is 5, i.e. $\beta({K^*}_{4,3}) = 5$ which is obtained by total enumeration. One metric basis of ${K^*}_{4,3}$ is set $S = \{x_1, x_2, x_3, y_1, y_2\}$.
Metric coordinate of all vertices, with respect to S, are presented in Table 1 and as it can be seen all vertices have mutually different metric coordinates.

Table 1: Metric coordinates of vertices of ${K^*}_{4,3}$ with respect to S

Type	*Item*	*Metric coordinates*
vertex	x_1	(0, 1, 1, 1, 1)
	x_2	(1, 0, 1, 1, 1)
	x_3	(1, 1, 0, 1, 1)
	x_4	(1, 1, 1, 1, 1)
	y_1	(1, 1, 1, 0, 2)
	y_2	(1, 1, 1, 2, 0)
	y_3	(1, 1, 1, 2, 2)

In the next Theorem 4 it will be obtained and proved metric dimension of any complete split graph ${K^*}_{k,n-k}$.

Theorem 4. $\beta({K^*}_{k,n-k}) = \begin{cases} n-2, & (k = 1 \wedge n \ge 3) \vee (n-k \ge 2 \wedge k \ge 2) \\ n-1, & otherwise \end{cases}$.

Proof. Several cases will be discussed:

- $n = 2 \wedge k = 1$. It is easy to see that ${K^*}_{1,1} \cong P_2$. By Theorem 1, we have the following result $\beta({K^*}_{1,1}) = \beta(P_2) = 1$;
- $n = 3 \wedge k = 1$. In a similar way to previous case, it is easy to see that ${K^*}_{1,2} \cong P_3$. By Theorem 1, we have the following result $\beta({K^*}_{1,2}) = \beta(P_3) = 1$;

- $n > 3 \wedge k = 1$. Since $K^*{}_{1,n-1} \cong K_{1,n-1}$, then by Theorem 2, the metric dimension of the complete split graph $K^*{}_{1,n-1}$ of order n is $n-2$, i.e. $\beta\left(K^*{}_{1,n-1}\right) = n-2$;
- $n-k = 1 \wedge k \geq 2$. Since $K^*{}_{n-1,1} \cong K_n$, then by Theorem 3, the metric dimension of the complete split graph $K^*{}_{n-1,1}$ of order n is $n-1$, i.e. $\beta\left(K^*{}_{n-1,1}\right) = \beta(K_n) = n-1$;
- $k \geq 2$ and $n-k \geq 2$. Since graph $K^*{}_{k,n-k}$ is a connected graph of order $n \geq 4$, then by Theorem 2 it follows $\beta\left(K^*{}_{k,n-k}\right) = n-2$.

□

It is interesting to compare metric dimension and mixed metric dimension for these graphs. As it can be seen above, mixed metric dimension and metric dimension differ by two in case of $n-k \geq 2$ and $k \geq 2$. Otherwise, mixed metric dimension and metric dimension differ by one.

3. CONCLUSIONS

In this paper, the metric dimension for complete split graphs is considered. It is proved that metric dimension is $n-2$ for $n-k \geq 2$ and $k \geq 2$, as well as for $k = 1$ and $n \geq 3$. Otherwise it is equal to $n-1$. Further work can be directed in finding edge metric dimension of complete split graphs, as well as, some other interesting classes of graphs.

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